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Power r-cyclicity

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Abstract: In this paper, we prove that any bounded linear operator on a separable Banach space is circle- cyclic if and only if it is hypercyclic. As a continuation of studying cyclic phenomena we define and study a new concept called a power r-cyclic operator. We show that any power r-cyclic operator is supercyclic, but the converse need not be true in general. We give an example of a power r-cyclic operator which is not hypercyclic. Also we give necessary and sufficient conditions for the power r-cyclic operator to be hypercyclic and we give necessary and sufficient conditions for the operator to be power r-cyclic. Finally, we get some results concerning some spectral properties of power r-cyclic operators.

Keywords: Power *r*-cyclic, hypercyclic, and *G*-cyclic operators.
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المؤثرات ذات القوة r الدائرية

ملخص: لقد تم إثبات أن المؤثرات الخطية المحدودة على فراغ بناخ المجزأ "دائرة – دائرية" إذا وفقط إذا "فوق دائرية". استمراراً لدراسة الظاهرة الدائرية للمؤثرات تم تعريف ودراسة مفهوم جديد سُمي "المؤثرات ذات القوة r الدائرية". كذلك تم إثبات أن أي مؤثر من نوع "ذات القوة r الدائرية" هو من النوع "الدائري الأفضل" ولكن العكس ليس صحيحاً بصورةٍ عامة وتم إعطاء مثال يوضح ذلك. لقد تم إيجاد شروط كافية وضرورية ليكون "المؤثر ذو القوة r الدائرية" من "المؤثرات الفوق دائرية". إضافةً لذلك لقد تم إيجاد شروط كافية وضرورية ليكون "المؤثر فوق الدائري" من "المؤثرات ذات القوة r الدائرية". في النهاية تم الحصول على بعض النتائج حول بعض الخواص الطيفية للمؤثرات ذات القوة r الدائرية.

1. Introduction.

Cyclic phenomena is considered as important phenomena in operator theory, where cyclic, hypercyclic and supercyclic operators have been studied by many mathematicians. The study of the cyclic phenomena originated in paper by Birkhoff, 1929 [1]. He showed essentially the hypercyclicity of the translation operator, while Mac Lane proved the hypercyclicity of the differentiation operator in 1952 in [2]. However, an example of a hypercyclic operator on Hilbert space was constructed by Rolewicz in 1969 in [3]. Later on, starting from the eighties these phenomena have been widely explored. They were studied by C. Kitai, J. Shapiro, S. Fledman and others. In 2002 and 2006 , A. Naoum and Z. Jamil investigated cyclic phenomena of operators and introduced new concepts, see [4] and [5]. In this paper we give a positive answer to a question (conjecture) that was given by Jamil Z. in [5]. Moreover, we define and study a new concept on linear operators namely, a power r-cyclic operator.

Let X be an infinite dimensional separable complex Banach space and $B(X)$ be the complex Banach algebra of bounded linear operators on X.

Following [6], we call an operator $T\in B(X)$ *hypercyclic* if there exists, $x \in H$ such that $\{T^n x : n \geq 0\}$ is dense in X. In this case x is said to be *a hypercyclic vector* for T.

Throughout this paper all linear spaces and algebras are assumed to be defined over $\mathbb{C}$, the field of complex numbers, and H will denote an infinite dimensional separable complex Hilbert space. For $T\in B(H)$, we define T^* to be the Hilbert adjoint operator of T, and $s\,p(T)$ to be the set of all eigenvalues of T.

Following [7], we call an operator $T\in B(H)$ *supercyclic* if there exists $x \in H$ such that $\{\alpha T^n x : n \geq 0, \alpha\in \mathbb{C}\}$ is dense in H. In this case x is said to be *a supercyclic vector* for T.

In [5], and [4], A. Naoum and Z. Jamil introduced the following definitions:

1) An operator $T\in B(H)$ is said to be *circle-cyclic* if there exists $x \in H$ such that $\{\alpha T^n x : n \geq 0, \alpha\in \mathbb{C}, |\alpha| = 1\}$ is dense in H. In this case x is said to be *a circle-cyclic vector* for T, [5].

2) An operator $T\in B(H)$ is said to be *G-cyclic* over a multiplication semigroup S of $\mathbb{C}$ with identity if there exists $x \in H$ such that $\{\alpha T^n x : n \geq 0, \alpha\in S\}$ is dense in H. In this case x is said to be *a G-cyclic vector* for T over S, [4].

In [5], Z. Jamil proved that a weighted shift operator is hypercyclic if and only if it is circle-cyclic. Also, she proved that the closure of the set of all hypercyclic operators on H coincides with the closure of the set of all circle-cyclic operators on H. She posed the following question Does the set of all hypercyclic operators on H coincides with the set of all circle-cyclic operators on H?. In this paper we give a positive answer to this question. In fact, we show that this is true for any infinite dimensional separable complex Banach space X.

Power r-cyclicity

As a continuation of studying cyclic phenomena we define and study a new concept called a power *r*-cyclic operator, and we study some of its properties and its relation with the other concepts of the cyclic phenomena.

In order to prove our results we need the following known results:

Theorem 1.1. [7] *Suppose* $T : l^2[\mathbb{Z}] \to l^2[\mathbb{Z}]$ *is a backward weighted shift with weight sequence* $< w_n >_{n\in \mathbb{Z}}$, *and let* $m > 0$ *be fixed, and either* $w_n \geq m > 0$ *for all* $n > 0$ *or* $w_n \leq m$ *for all* $n < 0$. *Then* T *is a hypercyclic operator if and only if there exists a sequence* $< n_r >$ *in* $\mathbb{Z}$; $n_r \to \infty$ *such that* $\lim\limits_{r\to\infty}\prod\limits_{k=1}^{n_r} w_k = 0$ *and* $\lim\limits_{r\to\infty}\prod\limits_{k=1}^{n_r} \frac{1}{w_{-k}} = 0$.

Theorem 1.2. [7] *Suppose* $T : l^2[\mathbb{Z}]\to l^2[\mathbb{Z}]$ *is a backward weighted shift with weight sequence* $< w_n >_{n\in \mathbb{Z}}$, *and let* $m > 0$ *be fixed, and either* $w_n \geq m > 0$ *for all* $n > 0$ *or* $w_n \leq m$ *for all* $n < 0$. *Then* T *is a supercyclic operator if and only if there exists a sequence* $< n_r >$ *in* $\mathbb{Z}$; $n_r \to \infty$ *such that* $\lim\limits_{r\to\infty}\prod\limits_{k=1}^{n_r} w_{-k} \prod\limits_{k=1}^{n_r}\frac{1}{w_k} = 0$.

Theorem 1.3. [5]

(i) *Let x be a hypercyclic vector for T∈ B(H), then inf* $\{\| T^n x \|:$ *n ³ 0}* $= 0$ *and sup*$\{\| T^n x \|:$ *n ³ 0}* $= \infty$.

(ii) *The range of a hypercyclic operator is dense in a Hilbert space H.*

(iii) *If T∈ B(H) and || T || £ 1, then T Ï HC(H).*

(iv) *Let {Hi} be a family of Hilbert spaces, let Ti ∈ B(H i) for all i. If T = Å Ti ∈ B(ÅH i)), then Ti ∈ HC(H i) for al i.*

(v) *T is hypercyclic if and only if for any two open sets U and V in H there exists an* $n \geq 0$ *such that* T^n *(U) Ç V ¹ j .*

(vi) *Let* $T \in$ *HC(H). Then* T^* *does not have any eigenvalue.*

2. The Relation Between Hypercyclic and Circle - Cyclic Operators

In this section we extend the definitions of circle-cyclic vector and circle- cyclic operator for *T* in *B(X)*, where *X* is again as above, a separable Banach space. To prove our result of this section we need the following corollary of [6].

Lemma 2.1 [6]

Let X be an infinite dimensional separable complex Banach space and let $T \in B(X)$. *Then* $x \in X$ *is hypercyclic for* T *if and only if the set* $\{\alpha T^k x : k \geq 0,\ \alpha \in \mathbb{C},\ |\alpha| = 1\}$ *is dense in* X.

The following Theorem is our result:

Theorem 2.2.

Let X be an infinite dimensional separable complex Banach space and let $T \in B(X)$. *Then* T *is hypercyclic if and only if* T *is circle-cyclic.*

Proof.

Use the definition of circle-cyclic and Lemma 2.1 to get the result

3. Power r - Cyclic Operators

In this section we define the new concept of a power r-cyclic operator and we study some of its properties and its relation with the other concepts of cyclic phenomena.

Lemma 3.1. *Let* $Sr = \{\alpha \in \mathbb{C} : |\alpha| = r\}$ *where* r *is a positive real number.Then* $S = \{\alpha^n : \alpha \in Sr,\ n \geq 0\}$ *is a semigroup with identity 1 under usual multiplication.*

Proof.

Let α^n, $\beta^m \in S$. Then $|\alpha^n| = r^n$ and $|\beta^m| = r^m$ and hence $|\alpha^n \beta^m| = r^{n+m}$. Let $\gamma \in \mathbb{C}$ (one can take $\gamma = r$) be such that $|\gamma| = r^{n+m}$. Now clearly one can take $\alpha^n \beta^m = \gamma^{n+m}$ and hence $\alpha^n \beta^m \in S$. Finally, since $\alpha^0 = 1$, then $1 \in S$. Therefore, S is a semigroup with identity 1

Definition 3.2. Let $Sr = \{\alpha \in \mathbb{C} : |\alpha| = r\}$ where r is a positive real number. An operator $T \in B(H)$ is said to be *a power r-cyclic operator* if there exists $x \in H$ such that $\{\alpha^n T^n x : n \geq 0,\ \alpha \in Sr\}$ is dense in H. In this case x is said to be *a power r-cyclic vector* for T.

Notation

1- $PCr(H) = \{T \in B(H) :$ T is a power r – cyclic operator$\}$.
2- $PCr(T) = \{x \in H :$ x is a power r - cyclic vector for $T\}$.
3- $PSr\ orbt(T; x) = \{\alpha^n T^n x : n \geq 0,\ \alpha \in Sr\}$.
4- $HC(H) = \{T \in B(H) :$ T is a hypercyclic operator$\}$.
5- $SC(H) = \{T \in B(H) :$ T is a supercyclic operator$\}$.
6- $GCS(H) = \{T \in B(H) :$ T is a G-cyclic operator over a semigroup $S\}$.

Proposition 3.3. ***(i)*** *For any* $r > 0$, $PCr(H) \subseteq SC(H)$.

(ii) *Let* S *be as in Lemma 3.1. Then* $PCr(H) \subseteq GCS(H)$.

Proof.

(i) Since $\{\alpha^n T^n x : n \geq 0,\ \alpha \in Sr\} \subseteq \{\beta T^n x : n \geq 0,\ \beta \in \mathbb{C}\}$, we have $PCr(H) \subseteq SC(H)$

(ii) Since $\{\alpha^n T^n x : n \geq 0, \alpha \in Sr\} \subseteq \{\beta T^n x : n \geq 0, \beta \in S\}$, and by Lemma 3.1 S is a semigrop under multiplication with identity 1, we get $PCr(H) \subseteq GCS(H)$

Theorem 3.4. *For any* $r > 0$, $T \in PCr(H)$ *if and only if* $rT \in HC(H)$.

Proof.

First note that for any $\alpha \in Sr$, $|\alpha^n| = r^n$. Then we can assume that $\alpha^n = r^n \beta$ for some $\beta \in S1 = \{\gamma \in \mathbb{C} : |\gamma| = 1\}$. Hence $\{\alpha^n T^n x : n \geq 0, \alpha \in Sr\} = \{\beta r^n T^n x : n \geq 0, \beta \in S1\} = \{\beta (rT)^n x : n \geq 0, \beta \in S1\}$. Finally, by using Theorem 2.2 and this equality we have, $rT \in HC(H)$ if and only if rT is circle - cyclic if and only if $\{\beta (rT)^n x : n \geq 0, \beta \in S1\}$ is dense in H if and only if $\{\alpha^n T^n x : n \geq 0, \alpha \in Sr\}$ is dense in H if and only if $T \in PCr(H)$

Corollary 3.5. *Let* $r > 0$. *Then*

- ***(i)*** $PC1(H) = HC(H)$.
- ***(ii)*** *For any* $r > 0$, $T \in PCr(H)$, *if and only if* $\{r^n T^n x : n \geq 0\}$ *is dense in* H.
- ***(iii)*** *For any* $x \in PCr(T)$, $\inf\{r^n \| T^n x \| : n \geq 0\} = 0$ *and* $\sup\{r^n \| T^n x \| : n \geq 0\} = \infty$.
- ***(iv)*** *The range of* $T \in PCr(H)$ *is dense in* H.
- ***(v)*** *If* $T \in B(H)$ *and* $\| T \| \leq 1/r$, *then* $T \notin PCr(H)$.
- ***(vi)*** *Let* $\{Hi\}$ *be a family of Hilbert spaces, let* $Ti \in B(H i)$ *for all* i. *If* $T = \oplus Ti \in PCr(\oplus H i))$, *then* $Ti \in PCr(H i)$ *for all* i.
- ***(vii)*** $T \in PCr(H)$, *if and only if for any two open sets* U *and* V *in* H *there exists an* $n \geq 0$ *such that* $(rT)^n (U) \cap V \neq \emptyset$.

Proof.

(i) Follows directly from Theorem 3.4.

(ii) Follows from Theorem 3.4 and the definitions of hypercyclic and power r-cyclic operators.

(iii) Use Theorem 3.4 and Theorem 1.3 (i).

(iv) By Theorem 3.4, $rT \in HC(H)$ and by Theorem 1.3 (ii), $\{(rT)x : x \in H\}$ is dense in H. Let $y \in H$, then $ry \in H$ and it follows that there is a sequence $< rTxn >$ converges to ry. Hence the sequence $< Txn >$ converges to y. Therefore, the range of T, $\{Tx : x \in H\}$ is dense in H.

(v) Suppose that $T \in PCr(H)$. Then by Theorem 3.4, $rT \in HC(H)$. By Theorem 1.3 (iii), $\| rT \| > 1$ and so $\| T \| > 1/r$. Therefore, we have (v).

(vi) Use Theorem 3.4 and Theorem 1.3 (iv) to get the result.
(vii) Use Theorem 3.4 and Theorem 1.3 (iiv) to get the result

Proposition 3.6 *For any* $r > 0$ *and any* $\beta \in S1$, $T \in PCr(H)$ *if and only if* $\beta T \in PCr(H)$.

Proof.

To prove the proposition it is enough to show that $\{\alpha^n T^n x : n \geq 0, \alpha \in Sr\} = \{\gamma^n (\beta T)^n x : n \geq 0, \gamma \in Sr\}$. Let $y \in \{\alpha^n T^n x : n \geq 0, \alpha \in Sr\}$. Then $y = \alpha^n T^n x$ for some $n \geq 0$ and some $\alpha \in Sr$. Then $|\alpha/\beta| = r$ and $\alpha/\beta \in Sr$. Hence $y = (\alpha/\beta)^n (\beta T)^n x$ and so $y \in \{\gamma^n (\beta T)^n x : n \geq 0, \gamma \in Sr\}$. The converse follows from $\gamma^n (\beta T)^n x = (\gamma\beta)^n T^n x$, $|\gamma\beta| = r$ and $\gamma\beta \in Sr$

The following is an example of a power r-cyclic operator:

Example 3.7. Let B: $l^2(\mathbb{N}) \rightarrow l^2(\mathbb{N})$ be a backward weighted shift with weight $wn = 1$ for all $n \in \mathbb{N}$. Then $B \in PCr(l^2(\mathbb{N}))$ for all $r > 1$.

Proof.

By [3], $rB \in HC(l^2(\mathbb{N}))$ for all $r > 1$. Hence by Theorem 3.4, $B \in PCr(l^2(\mathbb{N}))$

Theorem 3.8. Suppose $T : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ *is a backward weighted shift with weight sequence* $< wn >_{n \in \mathbb{Z}}$, *and let* $m > 0$ *be fixed, and either* $wn \geq m > 0$ *for all* $n > 0$ *or* $wn \leq m$ *for all* $n < 0$. *Then* $T \in PCr((l^2(\mathbb{Z}))$ *if and only if there exists a sequence* $< nr >$ *in* $\mathbb{Z}$; $nr \rightarrow \infty$ *such that* $\lim_{r\to\infty} \prod_{k=1}^{n_r} rw_k = 0$ *and* $\lim_{r\to\infty} \prod_{k=1}^{n_r} \frac{1}{rw_{-k}} = 0$.

Proof.

Let $T \in PCr((l^2(\mathbb{Z}))$ then by Theorem 3.4, $rT \in HC(l^2(\mathbb{Z}))$. But rT is a backward weighted shift with weight sequence $< rwn >_{n \in \mathbb{Z}}$. Hence by Theorem 1.1 there exists a sequence $< nr >$ in $\mathbb{Z}$; $nr \rightarrow \infty$ such that $\lim_{r\to\infty} \prod_{k=1}^{n_r} rw_k = 0$ and $\lim_{r\to\infty} \prod_{k=1}^{n_r} \frac{1}{rw_{-k}} = 0$.

Conversely, let $T : l^2(\mathbb{Z}) \rightarrow l^2(\mathbb{Z})$ be a backward weighted shift with weight sequence $< wn >_{n \in \mathbb{Z}}$, and for some positive real number r there exists a sequence $< nr >$ in $\mathbb{Z}$; nr $\rightarrow \infty$ such that $\lim_{r\to\infty} \prod_{k=1}^{n_r} rw_k$

$= 0$ and $\lim\limits_{r\to\infty}\prod\limits_{k=1}^{n_r}\frac{1}{rw_{-k}} = 0$. Hence by Theorem 1.1, $rT \in HC(l^2(\mathbb{Z}))$. Thus there exists $x\in l^2(\mathbb{Z})$ such that $\{r^nT^nx : n \geq 0\}$ is dense in $l^2(\mathbb{Z})$. Hence by Corollary 3.5, $T\in PCr((l^2(\mathbb{Z}))$

Example 3.9. *Let* $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ *be a backward weighted shift with weight sequence* $w_n = \begin{cases} \frac{1}{4} & n < 0 \\ \frac{1}{2} & n \geq 0 \end{cases}$. *Then* T *is supercyclic but* T *is not hypercyclic and* T Ï $PCr((l^2(\mathbb{Z}))$ *for all* $r\in (\frac{1}{4},\frac{1}{2})$.

Proof.

Since $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n} w_{-k}\prod\limits_{k=1}^{n}\frac{1}{w_k} = \lim\limits_{n\to\infty}(\frac{1}{4^n})(2^n) = 0$, then by Theorem 1.2, T is supercyclic. Also, $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\frac{1}{w_{-k}} = \lim\limits_{n\to\infty}(4^n) = \infty$, then by Theorem 1.1, T is not hypercyclic. Now, for all $r\in (\frac{1}{4},\frac{1}{2})$, rT is a backward weighted shift with weight sequence $b_n = \begin{cases} \frac{r}{4} & n < 0 \\ \frac{r}{2} & n \geq 0 \end{cases}$.

Hence $bn > \frac{1}{8}$. For all $n > 0$, and $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\frac{1}{b_{-k}} > \lim\limits_{n\to\infty}(8^n) = \infty$. Hence by theorem 1.1 T Ï $PCr((l^2(\mathbb{Z}))$ for all $r\in (\frac{1}{4},\frac{1}{2})$

Example 3.10. *Let* $T : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ *be a backward weighted shift with weight sequence* $w_n = \begin{cases} \frac{1}{2} & n < 0 \\ \frac{1}{n} & n \geq 0 \end{cases}$. *Then* $T\in PCr((l^2(\mathbb{Z}))$ *for all* $r > 2$ *but* T *is not hypercyclic.*

Proof.

Let $U = rT$ for any fixed $r > 2$, then U is a backward weighted shift with weight sequence $rw_n = \begin{cases} \frac{r}{2} & n < 0 \\ \frac{r}{n} & n \geq 0 \end{cases}$. Hence $rwn > 1$ for all $n < 0$ and any fixed $r > 2$, then $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n} rw_k = \lim\limits_{n\to\infty}(r/n)^n = 0$, and $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\frac{1}{rw_{-k}} = \lim\limits_{n\to\infty}(\frac{2}{r})^n = 0$, then by Theorem 3.8, $T \in PCr((l^2(\mathbb{Z}))$ for all $r > 2$.

Since $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\frac{1}{w_{-k}} = \lim\limits_{n\to\infty}(2^n) = \infty$, then by Theorem 1.1, T is not hypercyclic

In the following theorem we study some of the spectral properties of power r-cyclic operators, where we find that these properties are similar to those of hypercyclic operators and their proofs are similar to those for hypercyclic operators, [5].

Theorem 3.11.

(i) *Let $T \in PCr(H)$, then T^* does not have any eigenvalue.*

(ii) *If $T \in B(H)$ is a finite rank operator, then $I + T$ Ï $PCr(H)$.*

(iii) *Let $T \in B(H)$.*

- ***a)*** *If $T \in PCr(H)$, then for any non-constant polynomial P in T, $s\,p\,(P(T^*)) = j$.*
- ***b)*** *If there is a non-constant polynomial P in T such that $P(T) \in PCr(H)$, then for $s\,p\,(T^*) = j$.*

Proof.

(i) Suppose that l is an eigenvalue of T^*, then there exists $x \in H$ such that $T^*x = lx$. Hence $(rT)^*x = (rl)x$. This means that rl is an eigenvalue of $(rT)^*$. However, $T \in PCr\,(H)$, then by Theorem 3.4, $rT \in HC\,(H)$ and by Theorem 1.3. (vi), $(rT)^*$ does not have any eigenvalue. Therefore, we have a contradiction. Hence T^* does not have any eigenvalue.

(ii) Since T is of a finite rank, so is T^*, but H is infinite dimensional, then the kernel of T is a nontrivial space. Hence 1 is an eigenvalue of $(I + T)^*$. Then by (i), $I + T$ Ï $PCr\,(H)$.

(iii) Let Q be a non-zero polynomial of complex coefficients and let $Q(z) = P(z^*)^*$. By By Ref. [8], $s\,p(P(T^*)) = Q(s\,p(T^*))$.

(a) If $T \in PCr\,(H)$, then by (i), $s\,p(T^*) = j$. Therefore, $s\,p(P(T^*)) = j$.

(b) If there is a non-constant polynomial P in T such that $P(T) \in$ PCr(H), then by (i), $s\ p(P(T^*)) = j$, and so $Q(s\ p(T^*)) = j$. *Hence* $s\ p(T^*) = j$.

3. Conclusion.

In this paper we define the concept of a power *r*-cyclic operator, and we find that this family of operators is different from the families of hypercyclic and supercyclic operators. Also we show that this family of operators has properties that are similar to those of the other families. In fact the proofs of our results are similar to that for the hypercyclic and *G*-cyclic operators. We think that more properties (for example spectral properties) can be studied and new results can be obtained.

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