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A CONSTRUCTION OF A SEMITOPOLOGICAL SEMIGROUP OF SOFT ULTRAFILTERS

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Abstract:
Abstract. In this paper, we construct a semitopological semigroup consisting entirely of soft ultrafilters.

Keywords:

soft sets; soft topological space; soft ultrafilters; compact semitopological

1. Introduction

Soft sets was introduced by the Russian Demetry Molodtsove 1999 [Molodtsov, 1999] as a general mathematical tool for dealing with uncertain objects. Operations on soft sets was introduced by P.K. Maji , R . Biswas and A. R. Roy 2003 [Maji and Roy, 2003]. Shabir and Nas 2011 [Shabir and Naz, 2011] introduced and studied the concept of soft topological spaces over soft sets and some related concepts. In [Aygunoglu and Aygun, 2011] Aygunoglu , Aygun introduced the soft product topology, E. Peygh and B. Samadi , A.Tayebi 2013 [Peyghan and Tayebi, 2012] introduced soft locally connected of a soft point and soft connected spaces depending on soft disjoint non-null soft open sets. Let (X, τ, A) be a soft topological space, let

$$\beta(X, \tau, A) = \{\, u \;:\; u \text{ is a soft ultrafilter on } X\}.$$

A EL-Mabhouh and W. Mousa 2018 [EL-Mabhouh and Mousa, 2018] have shown that $\beta(X, \tau, A)$ is a weakly soft compactification of (X, τ, A) which is Hausdorff. In this paper, We show that if $(X, *)$ in addition is a semigroup, then $\beta(X, \tau, A)$ can be given the structure of a semitopological semigroup where the operation is an extension of $(*)$.

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Key words and phrases. soft sets, soft topological space, soft ultrafilters, semitopological semi-group .

2. Preliminaries

Definition 2.1. [Georgiou and Megaritis, 2014] *Let X be an initial universe set and A a set of parameters. A pair (F, A), where F is a map from A to $\mathcal{P}(X)$, is called a soft set over X. In what follows by $SS(X, A)$, we denote the family of all soft sets (F, A) over X.*

Definition 2.2. [Georgiou and Megaritis, 2014] *The soft set (F, A), where $F(a) = \emptyset$, for every $a \in A$ is called the $A-$ null soft set of $SS(X, A)$ and denoted by 0_A. The soft set (F, A), where $F(a) = X$, for every $a \in A$ is called the $A-$absolute soft set of $SS(X, A)$ and denoted by 1_A.*

Definition 2.3. [Georgiou and Megaritis, 2014] *Let $(F, A), (G, A) \in SS(X, A)$. We say that (F, A) is a soft subset of (G, A) if $F(a) \subseteq G(a)$ for every $a \in A$. Symbolically, we write $(F, A) \sqsubseteq (G, A)$. Also we say that the pairs $(F, A), (G, A)$ are soft equal if $(F, A) \sqsubseteq (G, A)$ and $(G, A) \sqsubseteq (F, A)$. Symbolically, we write $(F, A) = (G, A)$.*

Definition 2.4. [Georgiou and Megaritis, 2014] *Let I be an arbitrary index set and $\{(F_i, A) : i \in I\} \subseteq SS(X, A)$.*

(1) *The soft union of these sets is the soft set $(F, A) \in SS(X, A)$, where $F : A \to \mathcal{P}(X)$ defined by $F(a) = \cup\{(F_i(a)) : i \in I\}$, for all $a \in A$ and we write $(F, A) = \sqcup\{(F_i, A) : i \in I\}$.*

(2) *The soft intersection of these sets is the soft set $(F, A) \in SS(X, A)$, where $F : A \to \mathcal{P}(X)$ defined by $F(a) = \cap\{(F_i(a)) : i \in I\}$, for every $a \in A$ and we write $(F, A) = \sqcap\{(F_i, A) : i \in I\}$.*

Proposition 2.1. [Georgiou and Megaritis, 2014] *If $(G, A), (H, A), (F_1, A), (F_2, A) \in SS(X, A)$ such that*

$(F_1, A) \sqsubseteq (G, A)$ and $(F_2, A) \sqsubseteq (H, A)$, then $(F_1, A) \sqcap (F_2, A) \sqsubseteq (G, A) \sqcap (H, A)$.

Definition 2.5. [Pei and Miao, 2005] *Let $x \in X$. The soft set (F, A) over X, where $F(a) = \{x\}$ for all $a \in A$, is called the singleton soft point and denoted by x_A or (x, A).*

Definition 2.6. [Georgiou and Megaritis, 2014] *Let $(F, A) \in SS(X, A)$. The soft complement of (F, A) is the soft set $(H, A) \in SS(X, A)$, where $H : A \to \mathcal{P}(X)$ defined by, $H(a) = X \setminus F(a)$, for every $a \in A$ and we write $(H, A) = (F, A)^c$.*

Definition 2.7. [P. Wang and He, 2015] *Let X be a nonempty set. A soft filter on X is a non empty subset $\mathcal{U} \subseteq SS(X, A)$ such that :*

(1) *If $(G, A), (H, A) \in \mathcal{U}$, then $(G, A) \sqcap (H, A) \in \mathcal{U}$.*
(2) *If $(G, A) \in \mathcal{U}$ and $(G, A) \sqsubseteq (H, A) \in SS(X, A)$, then $(H, A) \in \mathcal{U}$.*
(3) *$0_A \notin \mathcal{U}$.*

A soft ultrafilter on X is a soft filter which is not properly contained in any other soft filter on X.

Proposition 2.2. [EL-Mabhouh and Mousa, 2018] *Let $x \in X$, and $a \in A$ be fixed. Let*

$$e_a(x) = \{(G, A) : x \in G(a)\}$$

Then $e_a(x)$ is a soft ultrafilter on X which is called the soft Principal ultrafilter on X generated by x and a.

Proposition 2.3. [EL-Mabhouh and Mousa, 2018] *Let $x \in X$, and $a \in A$ be fixed, then $e_a(x)$ is the only soft principal ultrafilter containing (x, A).*

Proposition 2.4. [EL-Mabhouh and Mousa, 2018] *Let $x \in X$, $a \in A$ be fixed, and $(G, A) \in SS(X, A)$ where $\forall b \in A$,*

$$G(b) = \begin{cases} \{x\} & , b = a\,, \\ \\ \emptyset & , \textit{otherwise}\,. \end{cases}$$

Then the only soft ultrafilter containing (G, A) *is* $e_a(x)$*.*

Definition 2.8. [Georgiou and Megaritis, 2014] *Let* X *be an initial universe set and* A *be a set of parameters, and* $\tau \subseteq SS(X, A)$*. We say that the family* τ *defines a soft topology on* X *if the following axioms are true:*

(1) $0_A, 1_A \in \tau$.

(2) *If* $(G, A), (H, A) \in \tau$*, then* $(G, A) \sqcap (H, A) \in \tau$*.*

(3) *If* $(G_i, A) \in \tau$ *for every* $i \in I$*, then* $\sqcup\{(G_i, A) : i \in I\} \in \tau$*.*

The triple (X, τ, A) *is called a soft topological space or soft space. The members of* τ *are called soft open sets on* X*. Also, a soft set* (F, A) *is called soft closed if the complement* $(F, A)^c \in \tau$*.*

Theorem 2.1. [EL-Mabhouh and Mousa, 2018] *Let* (X, τ, A) *be a soft topological space and let* $\mathcal{U} \subseteq SS(X, A)$*. Then the following statements are equivalent:*

(1) $\mathcal{U}$ *is a soft ultrafilter on* (X, τ, A)

(2) $\mathcal{U}$ *has the finite intersection property and for each* $(G, A) \in SS(X, A) \backslash \mathcal{U}$ *there is some* $(H, A) \in \mathcal{U}$ *such that* $(G, A) \sqcap (H, A) = 0_A$

(3) $\mathcal{U}$ *is maximal with respect to finite intersection property, that is ;* $\mathcal{U}$ *is maximal member of* $\{\mathcal{V} \subseteq SS(X, A) : \mathcal{V}$ *has the finite intersection property* $\}$

(4) $\mathcal{U}$ *is a soft filter on* (X, τ, A) *and for all* $\mathcal{F} \in \mathcal{P}_f(SS(X, A))$*, if* $\sqcup\mathcal{F} \in \mathcal{U}$ *then* $\mathcal{F} \cap \mathcal{U} \neq \phi$

(where $\mathcal{P}_f(S)$ *is the collection of all finite subsets of S.)*

(5) *$\mathcal{U}$ is a soft filter on (X, τ, A) and for all $(G, A) \in \mathrm{SS}(X, A)$ either $(G, A) \in \mathcal{U}$ or $(G, A)^c \in \mathcal{U}$*

Remark 2.9. *Two soft ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on X are equal if and only if $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$.*

Definition 2.10. [EL-Mabhouh and Mousa, 2018] *Let (X, τ, A) be a soft topological space, then*

(1) $\mathcal{B}(X, \tau, A) = \{\mathcal{U} : \mathcal{U}$ *is a soft ultrafilter on* $X\}$.
(2) *Given $(G, A) \in SS(X, A)$, $\widehat{(G, A)} = \{\mathcal{U} \in \mathcal{B}(X, \tau, A) : (G, A) \in \mathcal{U}\}$.*

Theorem 2.2. [EL-Mabhouh and Mousa, 2018] *Let (X, τ, A) be a soft topological space. Then*

(a) $\mathcal{B}(X, \tau, A)$ is a Hausdorff space .
(b) The sets of the form $\widehat{(G, A)}$ are the clopen subsets of $\mathcal{B}(X, \tau, A)$.

Theorem 2.3. [EL-Mabhouh and Mousa, 2018] *Let (X, τ, A) be a soft topological space, and $a \in A$ be fixed. Define $e_a : X \to \mathcal{B}(X, \tau, A)$ such that for each $x \in X$, $e_a(x)$ is the soft principal ultrafilter defined in Proposition (2.2). Then*

(a) Let $(G, A) \in SS(X, A)$ where $\forall b \in A$,

$$G(b) \neq \emptyset \text{ if } b = a \text{ and } G(b) = \emptyset \text{ otherwise.}$$

Then $\overline{e_a(G(a))} = \widehat{(G, A)}$.

(b) If $x \in X$, and $(F, A) \in SS(X, A)$ where $\forall b \in A$,

$$F(b) = \begin{cases} \{x\} & , b = a\,, \\ \\ \emptyset & , \textit{otherwise}\,. \end{cases}$$

Then $\widehat{(F,A)} = \{e_a(x)\}$.

(c) *The mapping* e_a *is injective and*

$$e[X] = \cup\{e_a(X) : a \in A\} = \{e_a(x) : a \in A \text{ and } x \in X\}$$

is a dense subset of $\mathcal{B}(X,\tau,A)$ *whose points are precisely the isolated points of* $\mathcal{B}(X,\tau,A)$.

(d) *If* (X,τ,A) *is soft discrete topology, then* e_a *is softly embedding.*

Definition 2.11. [Clifford and Preston, 1961] *A semigroup is a pair* $(S,*)$ *where* S *is a nonempty set and* $*$ *is a binary associative operation on* S.

Definition 2.12. [Clifford and Preston, 1961] *A group is a pair* $(S,*)$ *such that*

(1) $(S,*)$ *is a semigroup and*

(2) *there is an element* $e \in S$, *called the identity, such that* $ex = xe = x$ *for all* $x \in S$.

(3) *for each* $x \in S$ *there exists* $y \in S$ *such that* $xy = yx = e$. y *is called the inverse of* x.

Definition 2.13. [Clifford and Preston, 1961] *Let* S *be a semigroup. Then*

(a) S *is commutative if and only if* $xy = yx$ *for all* $x, y \in S$.

(b) *The center of* S *is* $\{x \in S : \text{ for all } y \in S, xy = yx\}$.

(c) *Given* $x \in S$, $\lambda_x : S \to S$ *is defined by* $\lambda_x(y) = xy$.

(d) *Given* $x \in S$, $\rho_x : S \to S$ *is defined by* $\rho_x(y) = yx$.

(e) $L(S) = \{\lambda_x : x \in S\}$.

(f) $\mathcal{R}(S) = \{\rho_x : x \in S\}$.

Definition 2.14. [Clifford and Preston, 1961] *Let* S *be a semigroup.*

(a) *An element* $x \in S$ *is an idempotent if and only if* $xx = x$.

(b) *$E(S) = \{x \in S : x$ is an idempotent$\}$.*

(c) *T is a subsemigroup of S if and only if $T \subseteq S$, and T is a semigroup under the restriction of the operation of S.*

(d) *T is a subgroup of S if and only if $T \subseteq S$, and T is a group under the restriction of the operation of S.*

(e) *Let $t \in E(S)$. Then $H(t) = \cup\{G : G$ is a subgroup of S and $t \in G\}$.*

Note that if G be a group with identity e, then $E(G) = \{e\}$.

Definition 2.15. [Clifford and Preston, 1961] *Let $(S, .)$ be a semigroup with a topology τ defined on S. Then*

(a) *S is a right topological semigroup if $\rho_x : S \to S$ is continuous for all $x \in S$.*

(b) *S is a left topological semigroup $\lambda_x : S \to S$ is continuous for all $x \in S$.*

(c) *S is a semitopological semigroup if it is a right topological semigroup which is also a left topological semigroup.*

3. Extending the operation to $\mathcal{B}(X, \tau, A)$

All semigroups in this section are assumed to have an identity. If $(X, \cdot)$ is a semigroup without identity, we adjoint one element say $1 \notin X$. Then $(X \cup \{1\}, *)$ is a semigroup with identity 1 where

(1) $1 * 1 = 1$.

(2) $1 * x = x * 1 = x$ for all $x \in X$.

(3) $x * y = x \cdot y$ for all $x, y \in X$.

Definition 3.1. *Let $(X, *)$ be a semigroup. Let $M, N \subseteq X$. We define $M * N = \{m * n : m \in M, n \in N\}$. If $M = \emptyset$ or $N = \emptyset$, then $M * N = \emptyset$*

Definition 3.2. *Let $(X, *)$ be a semigroup. We define $\acute{*}$ on $SS(X, A)$ by: for any $(F, A), (G, A) \in SS(X, A)$, $(F, A)\acute{*}(G, A) = (H, A)$ where $H(a) = F(a) * G(a) \forall a \in A$*

Proposition 3.1. *Let* $(X,*)$ *be a semigroup then* $(SS(X,A),\acute{*})$ *is a semigroup with identity* $(1,A)$.

Proof. The operation $\acute{*}$ is binary. Also it is associative, since
$[(F,A)\acute{*}(G,A)]\acute{*}(C,A) = (H,A)\acute{*}(C,A) = (D,A)$ where $\forall a \in A$
$D(a) = H(a) * C(a) = [F(a) * G(a)] * C(a) = F(a) * [G(a) * C(a)]$.
This implies that $(D,A) = (F,A)\acute{*}[(G,A)]\acute{*}(C,A)]$. Also for each $(F,A) \in SS(X,A)$, we have $(F,A)\acute{*}(1,A) = (F,A)$ and $(1,A)\acute{*}(F,A) = (F,A)$. Hence $(1,A)$ is the identity of $(SS(X,A),\acute{*})$. □

Proposition 3.2. *Let* $(X,*)$ *be a semigroup, let* $M = \{(x,A) : x \in X\}$*, then* M *is a subsemigroup of* $(SS(X,A),\acute{*})$*. When* $(X,*)$ *is a group, then* M *is a subgroup of* $(SS(X,A),\acute{*})$.

Proof. Let $(x,A),(y,A) \in M$. Then $(x,A)\acute{*}(y,A) = (G,A)$ where
$G(a) = \{x\} * \{y\} = \{x * y\}$ for all $a \in A$.
This implies that $(G,A) = (x * y, A) \in M$. When $(X,*)$ is a group, then For each $(x,A) \in M$, there exists $(x^{-1},A) \in M$ such that
$(x,A)\acute{*}(x^{-1},A) = (x^{-1},A)\acute{*}(x,A) = (1,A)$. □

Lemma 3.1. *Let* $(X,*)$ *be a semigroup, let* $(G,A),(H,A),(F,A) \in SS(X,A)$*. The following statements are true:*

(1) $[(G,A) \sqcap (H,A)]\acute{*}(F,A) \sqsubseteq [(G,A)\acute{*}(F,A)] \sqcap [(H,A)\acute{*}(F,A)]$.
(2) $(F,A)\acute{*}[(G,A) \sqcap (H,A)] \sqsubseteq [(F,A)\acute{*}(G,A)] \sqcap [(F,A)\acute{*}(H,A)]$.
(3) $[(G,A) \sqcup (H,A)]\acute{*}(F,A) = [(G,A)\acute{*}(F,A)] \sqcup [(H,A)\acute{*}(F,A)]$.
(4) $(F,A)\acute{*}[(G,A) \sqcup (H,A)] = [(F,A)\acute{*}(G,A)] \sqcup [(F,A)\acute{*}(H,A)]$.
(5) *If* $(G,A) \sqsubseteq (H,A)$*, then* $(G,A)\acute{*}(F,A) \sqsubseteq (H,A)\acute{*}(F,A)$.

Proof. (1) Let $a \in A$, then $[G(a) \cap H(a)] * F(a) \subseteq [G(a) * F(a)] \cap [H(a) * F(a)]$.
Hence
$[(G, A) \sqcap (H, A)]\acute{*}(F, A) \sqsubseteq [(G, A)\acute{*}(F, A)] \sqcap [(H, A)\acute{*}(F, A)]$.

(2) Similar to (1).

(3) Let $a \in A$, $[G(a) \cup H(a)] * F(a) = [G(a) * F(a)] \cup [H(a) * F(a)]$. Hence
$[(G, A) \sqcup (H, A)]\acute{*}(F, A) = [(G, A)\acute{*}(F, A)] \sqcup [(H, A)\acute{*}(F, A)]$.

(4) Similar to (3).

(5) Let $a \in A$, then $[G(a) * F(a)] \subseteq H(a) * F(a)$. Hence, $(G, A)\acute{*}(F, A) \sqsubseteq (H, A)\acute{*}(F, A)$.

□

Note that the equality in parts 1,2 of Lemma (3.1) do not hold as shown in the following example.
Consider the semigroup $(\mathbb{N} \cup \{0\}, +)$ and the soft sets $(G, A), (H, A), (F, A) \in SS(\mathbb{N} \cup \{0\}, A)$, where $A = \{a, b\}$, and
$G(a) = \{0, 1, 2, 3, 4\}, H(a) = \{3, 4, 5\}, F(a) = \{0, 1, 5\}$.
Now $[G(a) \cap H(a)] + F(a) = \{3, 4\} + \{0, 1, 5\} = \{3, 4, 5, 8, 9\}$ but
$[G(a) + F(a)] \cap [H(a) + F(a)] = [\{0, 1, 2, 3, 4\} + \{0, 1, 5\}] \cap [\{3, 4, 5\} + \{0, 1, 5\}]$
$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \cap \{3, 4, 5, 6, 8, 9, 10\} = \{3, 4, 5, 6, 8, 9\}$.
Hence $[G(a) \cap H(a)] + F(a) \neq [G(a) + F(a)] \cap [H(a) + F(a)]$. Consequently
$[(G, A) \sqcap (H, A)]\acute{+}(F, A) \neq [(G, A)\acute{+}(F, A)] \sqcap [(H, A)\acute{+}(F, A)]$.

Definition 3.3. *Let $(X, *)$ be a semigroup, let $(G, A), (F, A) \in SS(X, A)$. Define*

$$(G, A)(F, A)^{-1} = \{(M, A) : (M, A)\acute{*}(F, A) \sqsubseteq (G, A)\} \cup \{(G, A)\}.$$

Definition 3.4. *Let (X, τ, A) be a soft topological space. Given two soft ultrafilters $\mathcal{U}, \mathcal{V}$ on (X, τ, A), define $\mathcal{U} \otimes \mathcal{V}$ to be the collection of all soft sets (G, A) such that*

$(G,A)(F,A)^{-1} \cap \mathcal{U} \neq \emptyset$ *for some* $(F,A) \in \mathcal{V}$. *That is*

$$\mathcal{U} \otimes \mathcal{V} = \{(G,A) : (G,A)(F,A)^{-1} \cap \mathcal{U} \neq \emptyset \ for\ some\ (F,A) \in \mathcal{V}\}$$

By Theorem (2.3), X can be considered as a subset of $\mathcal{B}(X,\tau,A)$ where each $x \in X$ can be identified with $e_a(x)$.

Theorem 3.1. *Let* $(X,*)$ *be a semigroup, let* $a \in A$ *be fixed, and* $\mathcal{U}, \mathcal{V} \in \mathcal{B}(X,\tau,A)$ *then*

(a) $\mathcal{U} \otimes \mathcal{V} \in \mathcal{B}(X,\tau,A)$.

(b) *The operation* $\otimes$ *is associative.*

(c) $\otimes$ *extends the operation* $*$ *on* X; *that is, for any two soft principal ultrafilters* $e_a(x), e_a(y)$, *we have* $e_a(x) \otimes e_a(y) = e_a(x * y)$.

Proof. (a) We show $\mathcal{U} \otimes \mathcal{V}$ satisfies all properties of soft ultrafilters.

(1) For all $(F,A) \in \mathcal{V}$, we have

$0_A(F,A)^{-1} = \{(M,A) : (M,A)\acute{*}(F,A) \sqsubseteq 0_A\} = \{0_A\}$. Hence

$0_A(F,A)^{-1} \cap \mathcal{U} = \emptyset$. So

$0_A \notin \mathcal{U} \otimes \mathcal{V}$. Also

$1_A \in 1_A 1_A^{-1}$ and $1_A \in \mathcal{V}$. So

$1_A 1_A^{-1} \cap \mathcal{U} \neq \emptyset$. Hence

$1_A \in \mathcal{U} \otimes \mathcal{V}$.

(2) If $(G_1,A),(G_2,A) \in \mathcal{U} \otimes \mathcal{V}$, then $(G_1,A)(S_1,A)^{-1} \cap \mathcal{U} \neq \emptyset$ and $(G_2,A)(S_2,A)^{-1} \cap \mathcal{U} \neq \emptyset$ for some $(S_1,A),(S_2,A) \in \mathcal{V}$.

Let $(M_1,A),(M_2,A) \in \mathcal{U}$ such that $(M_1,A)\acute{*}(S_1,A) \sqsubseteq (G_1,A)$ and $(M_2,A)\acute{*}(S_2,A) \sqsubseteq (G_2,A)$, so by parts 1,5 of Lemma (3.1) we get

$$\begin{aligned}
[(M_1,A)\sqcap(M_2,A)]\acute{*}[(S_1,A)\sqcap(S_2,A)] &\sqsubseteq (M_1,A)\acute{*}[(S_1,A)\sqcap(S_2,A)]\sqcap \\
&\quad (M_2,A)\acute{*}[(S_1,A)\sqcap(S_2,A)] \\
&\sqsubseteq [(M_1,A)\acute{*}(S_1,A)]\sqcap[(M_2,A)\acute{*}(S_2,A)] \\
&\sqsubseteq (G_1,A)\sqcap(G_2,A).
\end{aligned}$$

So, $[(G_1,A)\sqcap(G_2,A)][(S_1,A)\sqcap(S_2,A)]^{-1}\cap\mathcal{U}\neq\emptyset$ and $(S_1,A)\sqcap(S_2,A)\in\mathcal{V}$. Hence $(G_1,A)\sqcap(G_2,A)\in\mathcal{U}\otimes\mathcal{V}$.

(3) If $(G,A)\sqsubseteq(H,A)$ where $(G,A)\in\mathcal{U}\otimes\mathcal{V}$, then $(G,A)(S,A)^{-1}\cap\mathcal{U}\neq\emptyset$ for some $(S,A)\in\mathcal{V}$. Let $(M,A)\in\mathcal{U}$ such that $(M,A)\acute{*}(S,A)\sqsubseteq(G,A)\sqsubseteq(H,A)$. Hence $(M,A)\in(H,A)(S,A)^{-1}$. So $(H,A)(S,A)^{-1}\cap\mathcal{U}\neq\emptyset$. Consequently $(H,A)\in\mathcal{U}\otimes\mathcal{V}$.

(4) Let $(G,A),(G,A)^c\notin\mathcal{U}\otimes\mathcal{V}$. Since $(G,A)\notin\mathcal{U}\otimes\mathcal{V}$, then $(G,A)(S,A)^{-1}\cap\mathcal{U}=\emptyset$ for all $(S,A)\in\mathcal{V}$. Since $(G,A)^c\notin\mathcal{U}\otimes\mathcal{V}$, then $(G,A)^c(S,A)^{-1}\cap\mathcal{U}=\emptyset$ for all $(S,A)\in\mathcal{V}$. Hence (G,A) and $(G,A)^c\notin\mathcal{U}$ which is a contradiction to Theorem (2.1(5)). Thus (G,A) or $(G,A)^c\in\mathcal{U}\otimes\mathcal{V}$.

This shows that $\mathcal{U}\otimes\mathcal{V}\in\mathcal{B}(X,\tau,A)$ and so $\otimes$ defines a binary operation on $\mathcal{B}(X,\tau,A)$.

(b) Let $(G,A)\in(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W}$. Then for some $(F,A)\in\mathcal{W}$

$$(G,A)(F,A)^{-1}\cap(\mathcal{U}\otimes\mathcal{V})\neq\emptyset \tag{3.1}$$

Let $(T,A)\in\mathcal{U}\otimes\mathcal{V}$ and $(T,A)\acute{*}(F,A)\sqsubseteq(G,A)$. This implies that $(T,A)(S,A)^{-1}\cap\mathcal{U}\neq\emptyset$ for some $(S,A)\in\mathcal{V}$ and $(T,A)\acute{*}(F,A)\sqsubseteq(G,A)$ for some $(F,A)\in\mathcal{W}$). Let $(N,A)\in\mathcal{U}$ such that

$$(N,A)\acute{*}(S,A)\sqsubseteq(T,A) \tag{3.2}$$

Now $(S,A)\in[(S,A)\acute{*}(F,A)](F,A)^{-1}$, and so
$[(S,A)\acute{*}(F,A)](F,A)^{-1}\cap\mathcal{V}\neq\emptyset$, $(F,A)\in\mathcal{W}$. Hence

$$(S,A)\acute{*}(F,A)\in\mathcal{V}\otimes\mathcal{W} \tag{3.3}$$

Also $(N,A)\in\mathcal{U}$ and from (3.1) and (3.2)

$$\begin{aligned}(N,A)\acute{*}[(S,A)\acute{*}(F,A)] &= [(N,A)\acute{*}(S,A)]\acute{*}(F,A)\\ &\sqsubseteq (T,A)\acute{*}(F,A)\\ &\sqsubseteq (G,A)\end{aligned}$$

Hence

$$(G,A)[(S,A)\acute{*}(F,A)]^{-1}\cap\mathcal{U}\neq\emptyset \tag{3.4}$$

Now combine (3.3) and (3.4) to obtain that $(G,A)\in\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})$. Hence, $(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W}\subseteq\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})$.

So by Remark (2.8), we get that $(\mathcal{U}\otimes\mathcal{V})\otimes\mathcal{W}=\mathcal{U}\otimes(\mathcal{V}\otimes\mathcal{W})$.

(c) Let $x,y\in X, a\in A$ be fixed. Take $(G,A)\in SS(X,A)$ where $\forall b\in A$,

$$G(b)=\begin{cases}\{x\} & , b=a\,,\\ \\ \emptyset & , \text{otherwise}\,.\end{cases}$$

Let $(F,A)\in SS(X,A)$ where $\forall b\in A$

$$F(b)=\begin{cases}\{x*y\} & , b=a\,,\\ \\ \emptyset & , \text{otherwise}\,.\end{cases}$$

Now $(G, A) \in (F, A)(y, A)^{-1}$, since $(G, A)\acute{\ast}(y, A) = (F, A)$ and $(y, A) \in e_a(y)$, $(G, A) \in e_a(x)$. So $(F, A) \in e_a(x) \otimes e_a(y)$.

Since the only soft ultrafilter contains (F, A) is $e_a(x * y)$, the final conclusion follows.

□

Corollary 3.1. $(\mathcal{B}(X, \tau, A), \otimes)$ *is a semigroup.*

Proof. Follows from Theorem (3.1). □

Proposition 3.3. *Let* $a \in A$ *be fixed, and* $e_a(1)$ *is the identity of* $((\mathcal{B}(X, \tau, A), \otimes)$ *where* 1 *is the identity of* $(X, *)$.

Proof. Let $\mathcal{U} \in (\mathcal{B}(X, \tau, A)$. We show that $\mathcal{U} = \mathcal{U} \otimes e_a(1)$. So let $(G, A) \in \mathcal{U}$, then

$$\begin{aligned} (G, A)\acute{\ast}(1, A) = (G, A) \sqsubseteq (G, A) \quad &\Rightarrow \quad (G, A)(1, A)^{-1} \cap \mathcal{U} \neq \emptyset, \ (1, A) \in e_a(1) \\ &\Rightarrow \quad (G, A) \in \mathcal{U} \otimes e_a(1) \\ &\Rightarrow \quad \mathcal{U} \subseteq \mathcal{U} \otimes e_a(1) \\ &\Rightarrow \quad \mathcal{U} = \mathcal{U} \otimes e_a(1), \ \text{by Remark (2.8)} \end{aligned}$$

Next, we show $\mathcal{U} = e_a(1) \otimes \mathcal{U}$. So let $(G, A) \in \mathcal{U}$, then

$$\begin{aligned} ((1, A)\acute{\ast}(G, A) = (G, A) \sqsubseteq (G, A) \quad &\Rightarrow \quad ((1, A) \in (G, A)\acute{\ast}(G, A)^{-1} \\ &\Rightarrow \quad (G, A)\acute{\ast}(G, A)^{-1} \cap e_a(1) \neq \emptyset, (G, A) \in \mathcal{U} \\ &\Rightarrow \quad (G, A) \in e_a(1) \otimes \mathcal{U} \\ &\Rightarrow \quad \mathcal{U} \subseteq e_a(1) \otimes \mathcal{U} \\ &\Rightarrow \quad \mathcal{U} = e_a(1) \otimes \mathcal{U}, \ \text{by Remark (2.8)} \end{aligned}$$

Proposition 3.4. *Let $a \in A$ be fixed, and $(X, *)$ be a semigroup with identity 1, then $M = \{e_a(x) : x \in X\}$ is a subsemigroup of $((\mathcal{B}(X, \tau, A), \otimes)$. When $(X, *)$ is a group, then M is a subgroup of $((\mathcal{B}(X, \tau, A), \otimes)$.*

Proof. Let $e_a(x), e_a(y) \in M$, then $e_a(x) \otimes e_a(y) = e_a(x * y) \in M$. When $(X, *)$ is a group, then for each $e_a(x) \in S$, there exists $(e_a(x))^{-1} = e_a(x^{-1}) \in M$ such that
$e_a(x) \otimes e_a(x^{-1}) = e_a(x * x^{-1}) = e_a(1)$ and
$e_a(x^{-1}) \otimes e_a(x) = e_a(x^{-1} * x) = e_a(1)$. □

Proposition 3.5. *For all $\mathcal{U} \in \mathcal{B}(X, \tau, A)$, the map*
$\lambda_{\mathcal{U}} : \mathcal{B}(X, \tau, A) \to \mathcal{B}(X, \tau, A)$ such that $\lambda_{\mathcal{U}}(\mathcal{V}) = \mathcal{U} \otimes \mathcal{V}$ is continuous.

Proof. Let $\widehat{(G, A)}$ be a basic open set in $\mathcal{B}(X, \tau, A)$. We show $\lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)})$ is open. So let $\mathcal{V} \in \lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)})$. Then

$$\begin{aligned} \lambda_{\mathcal{U}}(\mathcal{V}) \in \widehat{(G, A)} &\Rightarrow \mathcal{U} \otimes \mathcal{V} \in \widehat{(G, A)} \\ &\Rightarrow (G, A) \in \mathcal{U} \otimes \mathcal{V} \\ &\Rightarrow (G, A)(F, A)^{-1} \cap \mathcal{U} \neq \emptyset \text{ for some } (F, A) \in \mathcal{V} \end{aligned}$$

We claim that $\widehat{(F, A)} \subseteq \lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)})$. To show this let $\mathcal{W} \in \widehat{(F, A)}$. Then

$$\begin{aligned} (F, A) \in \mathcal{W} &\Rightarrow (G, A) \in \mathcal{U} \otimes \mathcal{W} \\ &\Rightarrow \mathcal{U} \otimes \mathcal{W} \in \widehat{(G, A)} \\ &\Rightarrow \lambda_{\mathcal{U}}(\mathcal{W}) \in \widehat{(G, A)} \\ &\Rightarrow \mathcal{W} \in \lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)}) \end{aligned}$$

Thus $\widehat{(F, A)} \subseteq \lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)})$.
So for each element $\mathcal{V} \in \lambda_{\mathcal{U}}^{-1}(\widehat{(G, A)})$, there exists a basic open set $\widehat{(F, A)}$ such that

$\mathcal{V} \in \widehat{(F,A)} \subseteq \lambda_{\mathcal{U}}^{-1}(\widehat{(G,A)})$. This implies that $\lambda_{\mathcal{U}}^{-1}(\widehat{(G,A)})$ is open and consequently we get $\lambda_{\mathcal{U}}$ is continuous. □

Proposition 3.6. *For all* $\mathcal{U} \in \mathcal{B}(X,\tau,A)$*, the map* $\rho_{\mathcal{U}} : \mathcal{B}(X,\tau,A) \to \mathcal{B}(X,\tau,A)$ *such that* $\rho_{\mathcal{U}}(\mathcal{V}) = \mathcal{V} \otimes \mathcal{U}$ *is continuous.*

Proof. Let $\widehat{(G,A)}$ be a basic open set in $\mathcal{B}(X,\tau,A)$. We show $\rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$ is open. So, let $\mathcal{V} \in \rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$. Then

$$\begin{aligned} \rho_{\mathcal{U}}(\mathcal{V}) \in \widehat{(G,A)} &\Rightarrow \mathcal{V} \otimes \mathcal{U} \in \widehat{(G,A)} \\ &\Rightarrow (G,A) \in \mathcal{V} \otimes \mathcal{U} \\ &\Rightarrow (G,A)(F,A)^{-1} \cap \mathcal{V} \neq \emptyset \; for\ some\ (F,A) \in \mathcal{U}. \end{aligned}$$

Let $(S,A) \in \mathcal{V}$ such that $(S,A)\acute{*}(F,A) \sqsubseteq (G,A)$.
We claim that $\widehat{(S,A)} \subseteq \rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$. To show this, let $\mathcal{W} \in \widehat{(S,A)}$, then

$$\begin{aligned} (S,A) \in \mathcal{W} &\Rightarrow (G,A)(F,A)^{-1} \cap \mathcal{W} \neq \emptyset, \; (F,A) \in \mathcal{U} \\ &\Rightarrow (G,A) \in \mathcal{W} \otimes \mathcal{U} \\ &\Rightarrow \mathcal{W} \otimes \mathcal{U} \in \widehat{(G,A)} \\ &\Rightarrow \rho_{\mathcal{U}}(\mathcal{W}) \in \widehat{(G,A)} \end{aligned}$$

Thus $\widehat{(S,A)} \subseteq \rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$.
So for each element $\mathcal{V} \in \rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$, there exists a basic open set $\widehat{(S,A)}$ such that: $\mathcal{V} \in \widehat{(S,A)} \subseteq \rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$.

This implies that $\rho_{\mathcal{U}}^{-1}(\widehat{(G,A)})$ is open and cosequently, we get $\rho_{\mathcal{U}}$ is continuous. □

Proposition 3.7. *Let* $a \in A$ *be fixed, and* $x \in (X,*)$*, then* $e_a(x)$ *is an idempotent of* $(\mathcal{B}(X,\tau,A),\otimes)$ *if and only if* x *is an idempotent in* $(X,*)$*.*

Proof. Let $e_a(x) \in (\mathcal{B}(X,\tau,A),\otimes)$. Then

$$\begin{aligned} e_a(x) \otimes e_a(x) = e_a(x) \quad &\Leftrightarrow \quad e_a(x * x) = e_a(x) \\ &\Leftrightarrow \quad x * x = x \\ &\Leftrightarrow \quad x \text{ is an idempotent in } (X,*). \end{aligned}$$

□

Theorem 3.2. *$(\mathcal{B}(X,\tau,A),\otimes)$ is a semitopological semigroup.*

Proof. Using Theorem (3.1) and Propositions (3.5), (3.6). □

Theorem 3.3. *Let $(X,*)$ be a semigroup. Given two soft ultrafilters $\mathcal{U},\mathcal{V}$ on (X,τ,A),*

$$\mathcal{U} \ominus \mathcal{V} = \{(G,A) : (G,A)\acute{*}(F,A) \in \mathcal{U} \text{ for some } (F,A) \in \mathcal{V}\}$$

Then $((\mathcal{B}(X,\tau,A),\ominus)$ is a semitopological semigroup. Moreover, if $(X,*)$ is a group, then $\ominus$ extends the operation $*$ on X.

Theorem 3.4. *Let $(X,*)$ be a semigroup and Y be a subgroup of X. Then $\mathcal{B}(Y,\tau_Y,A)$ is a subsemigroup of $\mathcal{B}(X,\tau,A)$*

Theorem 3.5. *Let $(X,*)$ be a semigroup. Given two soft ultrafilters $\mathcal{U},\mathcal{V}$ on (X,τ,A), define $\mathcal{U} \odot \mathcal{V}$ collection of all*

$$\mathcal{U} \odot \mathcal{V} = \{(G,A) : (G,A) \sqcap (F,A) \in \mathcal{U} \text{ for some } (F,A) \in \mathcal{V}\}.$$

Then $((\mathcal{B}(X,\tau,A),\odot)$ is a semitopological semigroup.

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