IUG Journal of Natural Studies
Peer-reviewed Journal of Islamic University-Gaza

ISSN 2409-4587

IUGNS
Vol 29, No 2, 2021, pp 70-83

Received on (18-12-2020) Accepted on (05-03-2021)

Ayman Y. Mizyed,[1]
As'ad Y. As'ad,[2]

The Spectrum of Elements of a Complex Normed Hyperalgebra

1 Department of Mathematics, Islamic University of Gaza,

* Corresponding author
e-mail address: ayman20201@gmail.com

https://doi.org/10.33976/IUGNS.29.2/2021/5

Abstract:
In this paper, we present new results of the spectrum of elements of a complex normed hyper-algebra. Also, we study the unitization and the quasi-inverse of a normed hyperalgebra without a unit element. Moreover, we prove that if X is a complex normed normal hyperalgebra without a unit element and Y is the unitization X + C of X. Then $\sigma Y((x, 0)) \subseteq \sigma X(x)$.

Keywords:
Banach hyperalgbera, multiplicative hyperalgebra, spectrum, unitization, inverse, quasi-inverse.

1. Preliminaries and Introduction

The beginning of algebraic hyperstructures theory due to a France mathematician F. Marty in [2] in 1934 who define the hypergroups in his paper at the eight congress of Scandinavian mathematician in Stockholm. He introduced the concept of hypergroups as a generalization of groups and used it in different contexts like al-gebraic functions, rational fractions and noncommutative groups. After few years, many of hyper algebraic cocepts appear such that, canonical hypergroups, regular hypergroups, Krasner hyperrings, multiplicative hy-perrings, hyperfields and many other concepts. Despite the abundance of content of algebraic hyperstructure but the papers in the analytic hyperstructure are very poor. The concept of hypervector spaces was first intro-duced by M. S. Tallini in [4]. Also, she defined the normed hypervector space in 1990 in [4] which forms a fundamental base of this paper. Recently, A. Taghavi and R. Parviniazadeh define the Banach hyperalgebra in [7]. They study the spectrum of elements in a unital Banach hyperalgebra, and they consider the w-linear functionals on this hyperalgebra. Also, in 2016 they prove the Gelfand theorem for Banach hyperalgebras in [3]. In this paper, we are interesting in a normed hyperalgebra, specially for a Banach hyperalgebra which is a complete normed hyperalgebra. We focus our attention for studying the inverse and the spectrum of elements on a Banach hyperalgebra which have a unit element. For hyperalgebra without a unit element, we adjunct the unit element by the unitization and by using the concept of quasi-inverse in order to study the inverse and the spectrum of elements on it. Let us begin with the definition of a normed hypervector space which introduced by M. S. Tallini [4] where the symbol $P^*(X)$ is referred to the family of all nonempty subsets of X.

Definition 1.1. *[8] Let F be a field and $(X,+)$ is an abelian group. A hypervector space over F is a quadruplet $(X,+,\circ,F)$ where $\circ : F\times X \to P^*(X)$ such that for all $x,y \in X$ and for all $a,b \in F$ the following conditions hold:*

1. $(a+b)\circ x \subseteq (a\circ x)+(b\circ x)$
2. $a\circ(x+y) \subseteq (a\circ x)+(a\circ y)$
3. $a\circ(b\circ x) = (ab)\circ x$
4. $(-a)\circ x = a\circ(-x) = -(a\circ x)$
5. $x \in 1\circ x$, *where* 1 *is the identity element of* F.

Remark 1.2. *[8]*

1. *The sum of $(a\circ x)+(b\circ x)$ is meant in the sense of Frobenius, that is,*
$$(a\circ x)+(b\circ x) = \{s+t : s\in(a\circ x), t\in(b\circ x)\}$$
2. *A hypervector space is called anti-left distributive if the inverse inclusion in Definition 1.1 condition 1 holds, that is,*
$$(a+b)\circ x \supseteq (a\circ x)+(b\circ x)$$
and strongly left distributive if equality in Definition 1.1 condition 1 hold, that is,
$$(a+b)\circ x = (a\circ x)+(b\circ x)$$
Similarly, a hypervector space is called an anti-right distributive and strongly right distributive hypervector spaces if the inverse inclusion and equality in Definition 1.1 condition 2 hold, respectively. Finally, a hypervector space is called strongly distributive if it is both strongly left and strongly right distributive.
3. *In Definition 1.1 condition 3,* $a\circ(b\circ x) = \{t : t\in(a\circ y), \quad \text{such that} \quad y\in(b\circ x)\}$.

Definition 1.3. *[9] Let X be a hypervector space over a field F and Y be a nonempty subset of X. Then Y is called a subhypervector space if the following two conditions hold for all $x,y\in Y$ and $a\in F$:*

1. $x-y\in Y$
2. $a\circ x\subseteq Y$

Definition 1.4. *[4] Let X be a hypervector space over a hyperfield F. A pseudonorm on X is a mapping $\|.\|$: $X\to\mathbb{R}$ such that for all $x,y\in X$ and $a\in F$ the following peoperties hold:.*

1. $\|0\|=0$,
2. $\|x+y\|\le\|x\|+\|y\|$,

3. $\sup \|a \circ x\| = |a|.\ \|x\|$, *where* $\sup \|a \circ x\| = \sup\{\|t\| : t \in a \circ x\}$.

A pseudonorm on X is called a norm if the following condition satisfied, $\|x\| = 0$ if and only if $x = 0$.

Remark 1.5. *In order to the third condition of a pseudonorm be well defined, we must assume that $a \circ x$ is a closed and bounded subset of X.*

By [13], for x in a normed hypervector space $(X, +, \circ, \|.\|, F)$ and $\epsilon > 0$, the open ball $B_\epsilon(x)$ is defined by $B_\epsilon(x) = \{y \in X : \|x - y\| < \epsilon\}$. The set of all open balls $\{B_\epsilon(x) : x \in X, \epsilon > 0\}$ form a basis for the topology on X which induced by this norm.

Definition 1.6. *[11] Let (x_n) be a sequence in a normed hypervector space $(X, +, \circ, \|.\|, F)$. This sequence converge to a point $x \in X$, if for every $\epsilon > 0$, there exists a positive number m such that $\|x_n - x\| < \epsilon$, for every $n \geq m$ and we write $\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} \|x_n - x\| = 0$.*

Definition 1.7. *[11] Let $(X, +, \circ, \|.\|, F)$ be a normed hypervector space. A sequence (x_n) in X is said to be a Cauchy sequence if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \epsilon$, for every $m, n \geq N$.*

Definition 1.8. *[10] Let $(X, +, \circ, F), (Y, \acute{+}, \acute{\circ}, F)$ be two hypervector spaces over a field F. A homomorphism between X and Y is a mapping $f : X \to Y$ such that for all $a \in F$ and $x, y \in X$ f satisfies, $f(x + y) = f(x)\acute{+}f(y)$ and $f(a \circ x) \subseteq a\acute{\circ}f(x)$.*

Also, in [10] a strongly homomorphism is a homomorphism such that the equality in the second condition holds, that is: $f(a \circ x) = a\acute{\circ}f(x)$.

Definition 1.9. *[11] Let $(X, +, \circ, \|.\|, F), (Y, \acute{+}, \acute{\circ}, \acute{\|.\|}, F)$ be two normed hypervector spaces over F. A homomorphism or a strongly homomorphism $f : X \to Y$ is said to be bounded if there exists $K \geq 0$ such that $\acute{\|f(x)\|} \leq K.\ \|x\|$, for every $x \in X$.*

Theorem 1.10. *[11] Let $(X, +_1, \circ_1, \|.\|_1, F)$, $(Y, +_2, \circ_2, \|.\|_2, F)$ be two normed hypervector spaces and $f : X \to Y$ a strongly homomorphism. Then the following are equivalent:*

1. *f is continuous,*
2. *f is continuous at $x_0 \in X$,*
3. *f is bounded.*

Definition 1.11. *[11] Let $(X, +, \circ, \|.\|, F), (Y, \acute{+}, \acute{\circ}, \acute{\|.\|}, F)$ be two normed hypervector spaces over F. For a bounded strongly homomorphism $f : X \to Y$, we define the norm on f by: $\|\acute{f}\| = \sup\{\sup \|f(\frac{1}{\|x\|}' \circ x)\| : 0 \neq x \in X\}$*

Remark 1.12. *[11] In the previous Definition 1.11,*

$$\|\acute{f}\| = \sup\{\sup \|f(\frac{1}{\|x\|}' \circ x)\| : 0 \neq x \in X\}$$

$$= \inf\{K : \acute{\|f(x)\|} \leq K\|x\|, \forall x \in X\}$$

Theorem 1.13. *[11] Let $(X, +_1, \circ_1, \|.\|_1, F)$, $(Y, +_2, \circ_2, \|.\|_2, F)$ be two normed hypervector spaces and $f : X \to Y$ a strongly homomorphism. Then the following are equivalent:*

1. *f is continuous*
2. *f sends Cauchy sequences in X to Cauchy sequences in Y*
3. *f sends convergent sequences in X to convergent sequences in Y*

Proposition 1.14. *Let $B_h(X, X)$ be the set of all bounded strongly homomorphisms from a hypervector space X into itself. Then $B_h(X, X)$ is a hypervector space with the usual operation of addition defined by,*

$$(T + S)(x) = T(x) + S(x)$$

, and the hyperscalar multiplication defined by,

$$(a \circ T)(x) = \{S \in B_h(X, X) : S(x) \in a \circ T(x), \forall x \in X\}.$$

Proof. We will prove the conditions of a hypervector spaces as follows, let $x \in X$, $a, b \in F$ and $T, T_1, T_2 \in B_h(X, X)$. Then,

1. $((a+b) \circ T(x)) = T((a+b) \circ x) \subseteq T(a \circ x + b \circ x) = T(a \circ x) + T(b \circ x) = (a \circ T(x)) + (b \circ T(x))$,
2. $a \circ (T_1 + T_2)(x) = (T_1 + T_2)(a \circ x) = T_1(a \circ x) + T_2(a \circ x) = a \circ T_1(x) + a \circ T_2(x)$
3. $a \circ (b \circ T(x)) = a \circ T(b \circ x) = T(a \circ (b \circ x)) = T((a \circ b) \circ x) = T((ab) \circ x) = (ab) \circ T(x)$,
4. $-a \circ T(x) = T(-a \circ x) = T(a \circ -x) = a \circ T(-x) = a \circ -T(x)$,
5. Since $x \in 1 \circ x$, we have, $T(x) \in T(1 \circ x)$.

■

Definition 1.15. *[12] A hyperBanach space X is a complete normed hypervector space. That is, every Cauchy sequence in X is convergent.*

Theorem 1.16. *Let X be a hyperBanach space. Then, $B_h(X, X)$ the hypervector space consisting of all bounded strongly homomorphisms from X into itself, is a hyperBanach space with a norm defined by,*

$$\|T\| = \sup\{\sup \|T(\frac{1}{\|x\|} \circ x)\| : 0 \neq x \in X\}$$

.

Proof. Let $(T_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $B_h(X, X)$. Then for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that $\|T_n - T_m\| < \epsilon$, for all $n, m > K$. Since T_n, T_m are bounded strongly homomorphisms,

$$\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\| \leq \|T_n - T_m\|\|x\| < \epsilon\|x\|.$$

Now, for any fixed $x \in X$ and given $\acute{\epsilon} > 0$, choose $\epsilon = \epsilon_x$ such that $\epsilon_x\|x\| < \acute{\epsilon}$ and so we have,

$$\|T_n(x) - T_m(x)\| \leq \epsilon_x\|x\| < \acute{\epsilon}.$$

Thus, $(T_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in a hyperBanach space X and so $(T_n(x))_{n \in \mathbb{N}}$ converge to $y \in X$. Define $T : X \to X$ by $T(x) = y = \lim_{n \to \infty} T_n(x)$. Now for any scalar $a \in F$ and $x, z \in X$ we have,

1. $T(x+z) = \lim_{n\to\infty} T_n(x+z) = \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(z) = T(x) + T(z)$
2. $T(a \circ x) = \lim_{n\to\infty} T_n(a \circ x) = \lim_{n\to\infty} a \circ T_n(x) = a \circ \lim_{n\to\infty} T_n(x) = a \circ T(x)$.

Thus, T is a strongly homomorphism form X into X. Now by the first equation we have, $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\|x\| \leq \epsilon\|x\|$ and by letting $m \to \infty$ we have $\|T_n - T(x)\| = \|T_n(x) - T(x)\| \leq \epsilon\|x\|$ and so, $T_n - T$ is bounded for $n \geq K$ and T_n is bounded which implies, $T = T_n - (T_n - T)$ is bounded. Thus, T ia a bounded strongly homomorphism form X into X and hence, $T \in B_h(X, X)$. Also, by the second equation, we have, for all $n \geq K$,

$$\|T_n - T\| = \sup\{\frac{\|(T_n - T)(x)\|}{\|x\|}, 0 \neq x \in X\} \leq \epsilon.$$

Hence, $(T_n)_{n\in\mathbb{N}}$ converge to $T \in B_h(X, X)$ and therefore, $B_h(X, X)$ is a hyperBanach space. ■

2 Banach Hyperalgebra

Depending on the definition of a hypervector space which introduced by M. S. Tallini [8], R. Parvinianzadeh and A. Taghavi [7] study the hyperstructure on algebra by giving the definition of a hyperalgebra in [7]. Moreover, using the definition of normed hypervector space which studied by M. S. Tallini [4] , they give the definition of Banach hyperalgebra in [7] and [3]. In this section we present the definition of hyperalgebra and Banach hyperalgebra and give a new example of a Banach hyperalgebra.

Definition 2.1. *[7] Let $(X, +, \circ, F)$ be a hypervector space over a field F. Then X is called a hyperalgebra over the field F if there exists a mapping $. : X \times X \to X$ of (x, y) into $x.y \in X$ such that:*

1. *$(x.y).z = x.(y.z)$*
2. *$(x + y).z = x.z + y.z$,*
3. *$x.(y + z) = x.y + x.z$*
4. *$(c \circ x).y = c \circ (x.y) = x.(c \circ y)$ $\forall x, y, z \in X$, $\forall c \in F$.*

Let X be a hyperalgebra and Y be a nonempty subset of X. Then Y is said to be a subhyperalgebra of X if Y is a subhypervector space such that $x.y \in Y$ whenever $x, y \in Y$.

Definition 2.2. *[7] A normed hyperalgebra is a hyperalgebra which is normed as a hypervector space and in which $\|x.y\| \leq \|x\| \; \|y\|$.*

Definition 2.3. *[7] A Banach hyperalgebra is a complete normed hyperalgebra.*

A Banach hyperalgebra is called real or complex when the underlying hyperalgebra is real or complex.

Proposition 2.4. *Let X be a hypervector space and $B_h(X, X)$ be the set of all bounded strongly homomorphisms operators on X into itself. Then $B_h(X, X)$ is a hyperalgebra with the usual operation of addition defined by, (T+ S)(x)= T(x)+ S(x), the hyperscalar multiplication defined by, $(a \circ T)(x) = \{S \in B_h(X, X) : S(x) \in a \circ T(x), \forall x \in X\}$. and the multiplication operation defined by $T.S(x) = T(S(x))$.*

Proof. By the Proposition 1.14, $B_h(X,X)$ is a hypervector space and so it remains to prove the axioms of hyperalgebra. For any $S,T,U \in B_h(X,X)$ and $a \in F$ as follows,

1. $(S.T).U(x) = (S.T)(U(x)) = S(T(U(x))) = S.(T(U(x))) = S.(T.U)(x)$,
2. $(S+T).U(x) = (S+T)(U(x)) = S(U(x)) + T(U(x)) = ((S.U) + (T.U))(x)$, this hold by definition of addition on $B_h(X,X)$.
3. $S.(T+U)(x) = S.(T(x)+U(x)) = S(T(x)+U(x)) = S(T(x)) + S(U(x)) = ((S.T) + (S.U))(x)$, this hold because the operators in $B_h(X,X)$ ere strongly homomorphisms.
4. $(a \circ S).T(x) = (a \circ S)(T(x)) = S(a \circ T(x)) = S.(a \circ T)(x)$, holds by the properties of strongly homomorphisms. Also, $S(a \circ T(x)) = a \circ (S(T(x))) = a \circ (S.T)(x)$. Thus, $(a \circ S).T(x) = a \circ (S.T)(x) = S.(a \circ T)(x)$

■

Corollary 2.5. *Let X be a Banach hypervector space. Then $B_h(X,X)$ is a Banach hyperalgebra with the operation mentioned in Proposition 2.4.*

Proof. By Proposition 2.4, $B_h(X,X)$ is a hyperalgebra and by Theorem 1.16 , $B_h(X,X)$ is a hyperBanach space. Also, for any $S,T \in B_h(X,X)$, we have, $\|(ST)(x)\| = \|S(T(x))\| \leq \|S\|\|T(x)\| \leq \|S\|\|T\|\|x\|$ and so, $\|ST\| \leq \|S\|\|T\|$. Thus, $B_h(X,X)$ is a Banach hyperalgebra. ■

3 Inverses

The aim of this section is to study the behavior of inverses in Banach hyperalgebra containing the identity element. Let us firstly begin with the definition of the essential point which introduced in [5] for the first time on weak hypervector space and we use the same definition on hyperalgebra.

Definition 3.1. *Let X be a hyperalgebra over a field F such that $a \in F$ and $x \in X$. Then, an element $z_{a\circ x} \in a \circ x$ is called an essential point of $a \circ x$ if and only if $x \in a^{-1} \circ z_{a\circ x}$ when $a \neq 0$ and for $a = 0$, $z_{a\circ x} = 0$.*

Remark 3.2. *The essential point $z_{a\circ x}$ for $a \circ x$ in a hyperalgebra is not unique. The set of all essential points for $a \circ x$ is denoted by $Z_{a\circ x}$.*

Definition 3.3. *[7] A hyperalgebra X is called a normal hyperalgebra if the following two conditions holds for $a,b \in F$ and $x,y \in X$*

1. $z_{(a+b)\circ x} = z_{a\circ x} + z_{b\circ x}$,
2. $z_{a\circ(x+y)} = z_{a\circ x} + z_{a\circ y}$.

The following Lemma used in [5] for the normal weak hypervector space and we will use it for normal hyperalgebra. The proof of this Lemma is similar for the two cases so we delete it.

Lemma 3.4. *Let X be a hyperalgebra over a field F such that $x \in X$ and $a \in F$. Then, if X is normal, then $Z_{a \circ x}$ is a singleton.*

Proposition 3.5. *[7] Let X be a normed hyperalgebra and $x \in X$, $a \in F$. Then there exists an essential point $z_{a \circ x} \in a \circ x$ has the property $\|z_{a \circ x}\| = \sup \|a \circ x\|$.*

Definition 3.6. *[7] Let X be a hyperalgebra. An element $e \in X$ is called an identity or a unit if for every $x \in X$, $e.x = x.e = x$. In this case, we say that X is a unital hyperalgebra.*

Definition 3.7. *[7] Let X be a unital hyperalgebra. An element $x \in X$ is said to be invertible if it has an inverse in X, that is, there exists an element $x^{-1} \in X$ such that, $x.x^{-1} = x^{-1}.x = e$, where e is the identity element in X.*

Remark 3.8. *[7] In a unital hyperalgebra X,*

1. *Any nonzero element x in X, has at most one inverse,*
2. *The set of all invertible elements is denoted by $Inv(X)$. The complement of $Inv(X)$ in X is the set of all non invertible elements in X and it's denoted by $Sing(X)$.*

Lemma 3.9. *[7] Let $(X, \|.\|)$ be a Banach hyperalgebra. If $x \in X$ with $\|x\| < 1$, then $(e - x) \in Inv(X)$.*

Lemma 3.10. *[7] Let $(X, \|.\|)$ be a Banach hyperalgebra. Then, $Inv(X)$ the set of all invertible elements in X, is an open set in X.*

Lemma 3.11. *[7] Let $(X, \|.\|)$ be a Banach hyperalgebra. Then, the inversion $x \to x^{-1}$ is continuous on $Inv(X)$.*

Definition 3.12. *[7] Let X be a hyperalgebra and $x \in X$. The spectrum of x, is denoted by $\sigma_X(x)$ or simply $\sigma(x)$, is the set of all complex numbers λ such that $(z_{\lambda \circ e} - x) \in Sing(X)$. That is,*

$$\sigma(x) = \{\lambda \in \mathbb{C} : (z_{\lambda \circ e} - x) \in Sing(X)\}$$

The complement of $\sigma(x)$ in $\mathbb{C}$ is called the resolvent of x and it's denoted by $\rho(x)$.

Theorem 3.13. *[7] Let $(X, \|.\|)$ be a Banach normal hyperalgebra and $x \in X$. Then $\sigma(x)$ is nonempty.*

Theorem 3.14. *[7] Let $(X, \|.\|)$ be a Banach normal hyperalgebra and $x \in X$. Then $\sigma(x)$ is bounded in $\mathbb{C}$ and is contained in the closed disk $\{\lambda \in \mathbb{C} : |\lambda| \leq \|x\|\}$.*

Definition 3.15. *Let $(X, \|.\|)$ be a normed hyperalgebra with $x \in X$. The spectral radius of x, denoted by $r(x)$, is defined by*

$$r(x) = \inf\{\|x^n\|^{\frac{1}{n}} : n = 1, 2, ...\}.$$

The following Theorems which discuss the spectral radius and the formula of the inverse of elements in a normad hyperalgebra and it's proof similar to the Theorems in a normed algebra.

Theorem 3.16. *Let x be an element of a normed hyperalgebra $(X, \|.\|)$. Then*

$$r(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$$

Proof. Let $\eta = r(x)$ and $\epsilon > 0$, then $\eta + \epsilon$ is not a lower bound. So there exists $k \in \mathbb{N}$ such that $\|x^k\|^{\frac{1}{k}} \leq \eta + \epsilon$. Now for any $n \in \mathbb{N}$, there exists $p(n), q(n)$ such that $n = p(n)k + q(n)$ where $p(n) \geq 0$ and $0 \leq q(n) \leq k-1$ and hence, $\lim_{n\to\infty} \frac{p(n)k}{n} = 1$ and $\lim_{n\to\infty} \frac{q(n)}{n} = 0$. Therefore,

$$\|x^n\| = \|x^{p(n)k+q(n)}\| = \|x^{p(n)k}\|\|x^{q(n)}\| \leq \|x^k\|^{p(n)}\|x\|^{q(n)}$$

So,

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \|x^k\|^{\frac{p(n)}{n}} \lim_{n\to\infty} \|x\|^{\frac{q(n)}{n}} \leq \|x^k\|^{\frac{1}{k}}.1 \leq \eta + \epsilon.$$

But $\epsilon > 0$ is arbitrary which implies, $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \eta$.
Conversely, $\eta = \inf\{\|x^n\|^{\frac{1}{n}} : n = 1, 2, ...\} \leq \|x^n\|^{\frac{1}{n}}$ and so $\eta \leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$. Thus, $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \eta = r(x)$. ■

Theorem 3.17. *Let $(X, \|.\|)$ be a Banach hyperalgebra with a unit element and $x \in X$ such that $r(x) < 1$. Then, $(1-x)$ is invertible and $(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n$*

Proof. Since $r(x) < 1$, there exists $\gamma > 0$ such that $r(x) < \gamma < 1$ and so $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < \gamma$ which implies, for sufficiently large n, $\|x^n\|^{\frac{1}{n}} < \gamma$ and $\|x^n\| < \gamma^n < 1$. Hence, $\sum_{n=0}^{\infty} \|x^n\|$ is convergent and so $\sum_{n=0}^{\infty} x^n$ is absolutely convergent in a Banach hyperalgebra X which implies, $\sum_{n=0}^{\infty} x^n$ converges with sum $s \in X$. Let $s_n = 1 + x + x^2 + ... + x^{n-1}$, then $s_n \to s$ and $\|x^n\| \to 0$ as $n \to \infty$ and so we have, $\lim_{n\to\infty}(1-x)s_n = \lim_{n\to\infty} s_n(1-x) = \lim_{n\to\infty}(1-x^n) = 1$ and hence, $(1-x)s = s(1-x) = 1$. Thus, $(1-x)$ is invertible and $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n = 1 + \sum_{n=1}^{\infty} x^n$. ■

Corollary 3.18. *Let $(X, \|.\|)$ be a Banach hyperalgebra with a unit element and $x \in X$ such that $\|1-x\| < 1$. Then, x is an invertible element in X.*

Proof. Since $r(1-x) \leq \|1-x\| < 1$, so by Theorem 3.17, $1-(1-x) = x$ is an invertible element. ■

4 Quasi-Inverse in Banach hyperalgebra

We study the inverse of elements in Banach hyperalgebras without a unit element by two methods. The first way is by the adjunction of a unit element. The second method is by using the concept of quasi-inverse. Here we present the two methods as follow:

Definition 4.1. *Let $(X, +, \circ, F)$ be a hypervector space over a field F. Then X is called a multiplicative hyperalgebra over the field F if there exist a hypermultiplication operation $\star : X \times X \to P^*(X)$ of (x, y) into $x \star y \in P^*(X)$ such that:*

1. $(x \star y) \star z = x \star (y \star z)$
2. $(x + y) \star z = x \star z + y \star z,$
3. $x \star (y + z) = x \star y + x \star z$
4. $(c \circ x) \star y = c \circ (x \star y) = x \star (c \circ y)$ $\forall x, y, z \in X$, $\forall c \in F$.

Let X be a multiplicative hyperalgebra and Y be a nonempty subset of X. Then Y is said to be a multiplicative subhyperalgebra of X if Y is subhypervector space such that $x \star y \subseteq Y$ whenever $x, y \in Y$.

Definition 4.2. *A normed multiplicative hyperalgebra is a multiplicative hyperalgebra which is normed as a hypervector space and in which* $\sup \|x \star y\| \leq \|x\| \, \|y\|$, *where* $\sup \|x \star y\| = \sup\{\|z\| : z \in x \star y\}$.

Definition 4.3. *A Banach multiplicative hyperalgebra is a complete normed multiplicative hyperalgebra.*

A Banach multiplicative hyperalgebra is called real or complex when the underlying multiplicative hyperalgebra is real or complex. Also, any hyperalgebra is a multiplicative hyperalgebra.

Definition 4.4. *Let X be a multiplicative hyperalgebra. Then,*

1. *An element $e \in X$ is said to be a left (right) weak identity if $x \in e \star x (x \in x \star e)$ for $x \in X$. An element e is called a weak identity element if it is both left and right $weak$ identity element.*
2. *An element $e \in X$ is said to be a left (right) scalar identity if there is $x \in X$ such that $\{x\} = e \star x (\{x\} = x \star e)$ for all $x \in X$. An element e is called a scalar identity if it is both left and right scalar identity element.*
3. *An element x is called left (right) invertible with respect to a weak identity e, if there is $y \in X$ such that $e \in y \star x (e \in x \star y)$. An element x is called invertible if it is both left and right invertible.*

Definition 4.5. *Let X, Y be two multiplicative hyperalgebras over the same field F. A mapping $f : X \to Y$ is called:*

1. *Homomorphism if and only if*
   - $f(x + y) = f(x) + f(y)$
   - $f(\lambda \circ x) \subseteq \lambda \circ f(x)$
   - $f(x \star y) \subseteq f(x) \star f(y)$ *for all* $x, y \in X$, $\lambda \in F$
2. *Strongly homomorphism if and only if*
   - $f(x + y) = f(x) + f(y)$
   - $f(\lambda \circ x) = \lambda \circ f(x)$
   - $f(x \star y) = f(x) \star f(y)$ *for all* $x, y \in X$, $\lambda \in F$
3. *Strongly isomorphism if and only if f is a bijective Strongly homomorphism.*

Definition 4.6. *Let X, Y be two normed multiplicative hyperalgebras over the same field F. A mapping $f : X \to Y$ is called an isometric strongly isomorphism of X onto Y if f is a strongly isomorphism of the hyperalgebra X onto the hyperalgebra Y and is also an isometric map of the metric space X onto the metric space Y. i.e ($\|f(x) - f(y)\| = \|x - y\|$) which is equivalent to $\|f(x)\| = \|x\|$.*

Definition 4.7. *The unitization of a normed hyperalgebra X over a field F is denoted by $X + F$ is the normed multiplicative hyperalgebra consisting of the set $X \times F$ with operations of addition, hyperscalar multiplication and hypermultiplication defined as follows for all $x, y \in X$ and $\alpha, \beta \in F$:*

1. $(x,\alpha)+(y,\beta)=(x+y,\alpha+\beta)$

2. $\beta\circ(x,\alpha)=\{(u,\alpha\beta):u\in\beta\circ x\}$

3. $(x,\alpha)\star(y,\beta)=\{(xy+s+t,\alpha\beta):s\in\beta\circ x,t\in\alpha\circ y\}$

and with a norm defined by $\|(x,\alpha)\|=\|x\|+|\alpha|$.

Remark 4.8. *$X+F$ is a normed multiplicative hyperalgebra with a weak identity $(0,1)$ such that $\|(0,1)\|=1$.*

Proof. Firstly, $\sup\|(x,\alpha)\star(y,\beta)\|=\sup\{\|(xy+s+t,\alpha\beta)\|:s\in\beta\circ x,t\in\alpha\circ y\}=\sup\{\|xy+s+t\|+|\alpha\beta|:s\in\beta\circ x,t\in\alpha\circ y\}\leq\|xy+|\beta|\|x\|+|\alpha|\|y\|\|+|\alpha\beta|\leq\|xy\|+|\beta|\|x\|+|\alpha|\|y\|+|\alpha\beta|\leq\|x\|\|y\|+|\beta|\|x\|+|\alpha|\|y\|+|\alpha\beta|=(\|x\|+|\alpha|)(\|y\|+|\beta|)=\|(x,\alpha)\|\|(y,\beta)\|$. Thus, $X+F$ is a normed multiplicative hyperalgebra. Secondly, since $(x,\alpha)\in(0,1)\star(x,\alpha)=\{(0.x+s+t,1.\alpha):s\in 0\circ\alpha,t\in 1\circ x\}=\{(t,\alpha):t\in 1\circ x\}$ and $x\in 1\circ x$ then $(x,\alpha)\in(0,1)\star(x,\alpha)$. Similarly,$(x,\alpha)\in(x,\alpha)\star(0,1)$. Thus, $(0,1)$ is a left and right weak identity and hence, $(0,1)$ is a weak identity. Finally, $\|(0,1)\|=\|0\|+|1|=1$ ■

Remark 4.9. *The mapping $T:X\to X+F$ given by $T(x)=(x,0)$ is an isometric strongly isomorphism of X onto hypersubalgebra of $X+F$.*

Proof. Let X be a hyperalgebra and $S=\{(x,0):x\in X\}$, then S is a hypersubalgebra of $X+F$ because for any $(x,0),(y,0)\in S$ and $\alpha\in F$ we have, $(x,0)+(y,0)=(x+y,0)\in S$, $\alpha\circ(x,0)=\{(z,0):z\in\alpha\circ x\}\subseteq S$ and $(x,0)(y,0)=\{(xy+s+t,0):s\in 0\circ x,t\in 0\circ y\}=\{(xy,0)\}\subseteq S$ .Now define $T:X\to S$ be a mapping defined by $T(x)=(x,0)$ Then

1. Let $(x,0)\in S$, then there is $x\in X$ such that $T(x)=(x,0)$. Hence, T is an onto map.
2. Let $\alpha,\beta\in F$ and $x,y\in X$. Then
   - $T(x+y)=(x+y,0)=(x,0)+(y,0)=T(x)+T(y)$
   - $T(\alpha\circ x)=\{T(v):v\in\alpha\circ x\}=\{(v,0):v\in\alpha\circ x\}=\{(v,\alpha.0):v\in\alpha\circ x\}=\alpha\circ T(x)$.
   - $T(x.y)=(xy,0)$ and $(x,0)\star(y,o)=\{(v,0):v=xy+s+t:s\in 0\circ x,t\in 0\circ y\}=(xy,0)$. Thus, $T(x.y)=(xy,0)=(x,0)\star(y,0)=T(x)\star T(y)$.

   Therefore, T is a strongly homomorphism.
3. Consider $\|T(x)\|=\|(x,0)\|=\|x\|+|0|=\|x\|$. Thus, T is an isometric map and so T is a 1-1 map.

Therefore, X is an isometric strongly isomorphism from X onto a hypersubalgebra S of $X+F$. ■

Definition 4.10. *Let X be a hyperalgebra without a unit element and x,y be elements in X. The quasi-product of x,y is the element $x\bullet y$ of X defined by*

$$x\bullet y=x+y-xy.$$

It's clear that for any $x,y,z\in X$, we have $(x\bullet y)\bullet z=x\bullet(y\bullet z)$ and $x\bullet 0=0\bullet x=x$. Thus, a hyperalgebra without a unit element together with its quasi product is a semi group with identity 0.
In the following paragraph,we shall define the quasi-inverses which are the inverses of elements with respect to this semigroup.

Definition 4.11. *Let X be a hyperalgebra without a unit element and $x \in X$. An element $y \in X$, is called a left quasi-inverse of x if $y \bullet x = 0$. Also, an element $z \in X$ is called a right quasi-inverse of x if $x \bullet z = 0$. A quasi-inverse of x is an element in X which is both a left quasi-inverse and a right quasi-inverse of x.*

A quasi-invertible element is an element which has a quasi-inverse and the other elements which don't have a quasi-inverse are called quasi-singular elements. Similarly, x is called a left or (right) quasi-singular if it is has no left or (right) quasi-inverse respectively. Moreover, if x has a left quasi-inverse y and a right quasi-inverse z. Then $y = y \bullet 0 = y \bullet (x \bullet z) = (y \bullet x) \bullet z = 0 \bullet z = z$.

Remark 4.12. *The set of all quasi-invertible elements of a hyperalgebra X is denoted by q-Inv(X). Also, q-Sing(X) denotes to the set of all quasi-singular elements in X.*

Proposition 4.13. *Let X be a hyperalgebra and $x, y \in X$. Then:*

1. *If X without a unit element and x has the quasi-inverse y in X then, $(0,1) - (x,0)$ has the inverse $(0,1) - (y,0)$ with respect to a weak identity $(0,1)$ in $X + F$.*

2. *If X has a unit element then x has a quasi-inverse y in X if and only if $(1 - x)$ has an inverse $(1 - y)$ in X.*

Proof.

1. Let x has a quasi inverse y in X, then $x \bullet y = y \bullet x = 0$ that is $x + y - xy = y + x - yx = 0$. Consider $((0,1) - (y,0)) \star ((0,1) - (x,0)) = (-y,1) \star (-x,1) \in X + F$, then $(-y,1) \star (-x,1) = \{(yx + s + t, 1) : s \in 1 \circ -y, t \in 1 \circ -x\}$ but $-x \in 1 \circ -x$ and $-y \in 1 \circ -y$. So $(xy - y - x, 1) \in (-y,1) \star (-x,1)$ and so $(0,1) \in (-y,1) \star (-x,1)$. Similarly, $(0,1) \in (-x,1) \star (-y,1)$ and hence, $(0,1) - (x,0)$ has the inverse $(0,1) - (y,0)$ with respect to a weak identity $(0,1)$ in $X + F$.

2. Similar to the above prove.

■

Proposition 4.14. *Let X be a hyperalgebra and $x, y \in X$. Then, if xy is left or right quasi-invertible, then yx is also left or right quasi-invertible respectively.*

Proof. Let xy be a left quasi-invertible, then there exists $z \in X$ such that $xy \bullet z = 0$. Now consider $yx \bullet (yzx - yx) = yx + yzx - yx - yx(yzx - yx) = yzx - yx(yzx - yx) = y(z - x(yz - y))x = y(z - xyz + xy)x = y(xy \bullet z)x = y.0.x = 0$. Thus, yx is a left quasi-invertible and it's left quasi-inverse is $yzx - yx$. Similarly the right quasi-invertible case. ■

Theorem 4.15. *Let x be an element in a Banach hyperalgebra X with a unit element such that $r(x) < 1$. Then, x is a quasi-invertible and $x^{\bullet} = -\sum_{n=1}^{\infty} x^n$. Where $x^{\bullet}$ denotes to the quasi-inverse of x.*

Proof. Let $r(x) < 1$, then $(1-x)$ is invertible in X and $r((x,0)) = \lim_{n\to\infty} \|(x,0)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(x^n,0)\|^{\frac{1}{n}} = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = r(x) < 1$. So $r((x,0)) < 1$ in $X + F$ which implies $(0,1) - (x,0)$ is invertible in $X + F$ and this implies, by Proposition 4.13, x is a quasi-invertible in X. Now, $((0,1) - (x,0))^{-1} = (-x,1)^{-1} = (0,1) + \sum_{n=1}^{\infty} (x,0)^n = (0,1) - \sum_{n=1}^{\infty} -(x,0)^n = (0,1) - (-\sum_{n=1}^{\infty} (x^n,0))$. Hence, by Proposition 4.13, $x^{\bullet} = \sum_{n=1}^{\infty} -(x,0)^n$ ■

5 The Spectrum of Elements of a Complex Normed Normal Hyperalgebra

The purpose of this section is to study the spectrum of any element of a complex normed normal hyperalgebra whether this hyperalgebra has or does not have a unit element. Let us begin with the following definition of the spectrum.

Definition 5.1. *Let X be a complex normed normal hyperalgebra and $x \in X$. The spectrum of x is the set $\sigma_X(x)$ of a complex numbers defined as follows:*

1. *If X has a unit element, then $\sigma_X(x) = \{\lambda \in C : z_{\lambda \circ e} - x \in Sing(X)\}$*
2. *If X doesn't have a unit element, then*

$$\sigma_X(x) = \{0\} \cup \{\lambda \in C/\{0\} : z_{\lambda^{-1} \circ x} \in q - Sing(X)\}$$

Where $z_{\lambda \circ e}$ and $z_{\lambda^{-1} \circ x}$ are the essential points of $\lambda \circ e$ and $\lambda^{-1} \circ x$ respectively.

Proposition 5.2. *Let X be a complex normed normal hyperalgebra and $x \in X$. If Y is the unitization $X + F$ of X. Then, $x \in q - Sing(X)$ if and only if $(x, 0) \in q - Sing(Y)$.*

Proof. ($\Rightarrow$) Let $(x, 0) \notin q - Sing(Y)$, then there exists $(y, \eta) \in Y$ such that $(x, 0) \bullet (y, \eta) = \{(0, 0)\}$ and so,

$$(x, 0) + (y, \eta) - (x, 0) \star (y, \eta) = \{(0, 0)\}$$

which implies

$$\{(x + y, \eta)\} - \{(xy + s + t, 0) : s \in \eta \circ x, t \in o \circ y\} = \{(0, 0)\}$$

and hence,

$$\{(x + y - (xy + s + t), \eta) : s \in \eta \circ x, t \in o \circ y\} = \{(0, 0)\}$$

which implies, $\eta = 0$, $t \in o \circ y = \{0\}$ and so $t = 0$ and $s \in \eta \circ x = 0 \circ x = \{0\}$ implies $s = 0$. Thus we have $x + y - xy = 0$ which means, $x \bullet y = 0$ and so, $x \in q - Inv(X)$. Thus, by contrapositive we have, if $x \in q - Sing(X)$, then$(x, 0) \in q - Sing(Y)$.
($\Leftarrow$) Let $x \notin q - Sing(X)$, then there exists $y \in X$ such that $x \bullet y = 0$. Now consider,

$$(x, 0) \bullet (y, 0) = (x, 0) + (y, 0) - (x, 0) \star (y, 0)$$

but $(x, 0)(y, 0) = \{(xy + s + t, 0) : s \in 0 \circ x, t \in 0 \circ y\} = \{(xy, 0)\}$. So,

$$(x, 0) \bullet (y, 0) = (x, 0) + (y, 0) - (x, 0)(y, 0) = \{(x + y, 0)\} - \{(xy, 0)\} = \{(x + y - xy, 0)\} = \{(x \bullet y, 0)\}$$

Since $x \bullet y = 0$, we have $(x, 0) \bullet (y, 0) = \{(0, 0)\}$ and thus, $(x, 0) \in q - Inv(Y)$. By contrapositive, we have, if $(x, 0) \in q - Sing(Y)$, then $x \in q - Sing(X)$. Hence,

$$x \in q - Sing(X) \text{ if and only if} (x, 0) \in q - Sing(Y).$$

∎

Proposition 5.3. *Let X be a complex normed normal hyperalgebra such that X doesn't have a unit element and let Y be the unitization $X + \mathbb{C}$ of X. Then $\sigma_Y((x, 0)) \subseteq \sigma_X(x)$.*

Proof. Case (1): If $\lambda = 0 \in \sigma_Y((x, 0))$, then $0 \in \sigma_X(x)$ holds by the Definition 5.1.

Case (2): If $0 \neq \lambda \notin \sigma_X(x)$, then $z_{\lambda^{-1}\circ x} \notin q - Sing(X)$. That is $z_{\lambda^{-1}\circ x} \in q - Inv(X)$. By Proposition 4.13, $(0, 1) - (z_{\lambda^{-1}\circ x}, 0) \in Inv(Y)$. Since $\lambda \neq 0$, $\lambda(0, 1) - \lambda(z_{\lambda^{-1}\circ x}, 0) \in Inv(Y)$ which implies, $\lambda(0, 1) - (x, 0) \in Inv(Y)$ and so, $\lambda \notin \sigma_Y((x, 0))$. Thus, $\sigma_Y((x, 0)) \subseteq \sigma_X(x)$. ■

Proposition 5.4. *Let $x, y \in X$. Then, $\sigma_X(xy)/\{0\} = \sigma_X(yx)/\{0\}$.*

Proof. Since $\lambda \in \sigma_X(xy)/\{0\}$ if and only if $z_{\lambda^{-1}\circ xy} \in q-Sing(X)$ and this holds if and only if $\lambda \circ z_{\lambda^{-1}\circ xy} \subseteq q - Sing(X)$, but $\lambda \circ z_{\lambda^{-1}\circ xy} = \lambda \circ \lambda^{-1} \circ xy = 1 \circ xy$ and since $xy \in 1 \circ xy$, we have. $xy \in q - Sing(X)$. By Proposition 4.14, $xy \in q - Sing(X)$ if and only if $yx \in q - Sing(X)$ and so for any $\lambda \neq 0$, $\lambda^{-1} \circ yx \subseteq q - Sing(X)$ and this holds if and only if $z_{\lambda^{-1}\circ yx} \in q - Sing(X)$ and thus, $\lambda \in \sigma_X(yx)/\{0\}$. ■

Proposition 5.5. *Let T be a strongly homomorphism from a hyperalgebra X into a hyperalgebra Y such that $Z_{\lambda\circ x}$, $E_{\eta\circ y}$ are the sets of all essential points of $\lambda \circ x$ in X and $\eta \circ y$ in Y respectively. Then, for $z_{\lambda\circ x} \in Z_{\lambda\circ x}$, we have $T(z_{\lambda\circ x}) \in E_{\lambda\circ T(x)}$.*

Proof. Let T be a strongly homomorphism from the hyperalgebra X into the hyperalgebra Y and consider an essential point $z_{\lambda\circ x}$ in X. In order to prove $T(z_{\lambda\circ x}) \in E_{\lambda\circ T(x)}$ we need to prove that $T(z_{\lambda\circ x})$ is an essential point of $\lambda \circ T(x)$. That is, $T(z_{\lambda\circ x}) \in \lambda \circ T(x)$ and $T(x) \in \lambda^{-1} \circ T(z_{\lambda\circ x})$.
Firstly, since $z_{\lambda\circ x} \in \lambda \circ x$, then $T(z_{\lambda\circ x}) \in T(\lambda \circ x)$ and T is a strongly homomorphism, so $T(\lambda \circ x) = \lambda \circ T(x)$. Thus, $T(z_{\lambda\circ x}) \in \lambda \circ T(x)$.
Secondly, since $\lambda \circ x = 1 \circ \lambda \circ x = 1 \circ z_{\lambda\circ x}$ and T is a strongly homomorphism, $\lambda \circ T(x) = T(\lambda \circ x) = T(1 \circ z_{\lambda\circ x}) = 1 \circ T(z_{\lambda\circ x})$ and so $\lambda^{-1} \circ \lambda \circ T(x) = \lambda^{-1} \circ 1 \circ T(z_{\lambda\circ x})$ which implies, $1 \circ T(x) = \lambda^{-1} \circ T(z_{\lambda\circ x})$. But, $T(x) \in 1 \circ T(x)$. Thus, $T(x) \in \lambda^{-1} \circ T(z_{\lambda\circ x})$ and therefore, $T(z_{\lambda\circ x}) \in E_{\lambda\circ T(x)}$. ■

Corollary 5.6. *Let T be a strongly homomorphism from normed normal hyperalgebra X into normed normal hyperalgebra Y. Then, if $z_{\lambda\circ x}$ is the essential point of $\lambda \circ x$ in X then, $T(z_{\lambda\circ x}) = z_{\lambda\circ T(x)}$. That is $T(z_{\lambda\circ x})$ is the essential point of $\lambda \circ T(x)$.*

Proof. By the Proposition 5.5, $T(z_{\lambda\circ x})$ is an essential point of $\lambda \circ T(x)$ in Y where Y is normed normal hyperalgebra which implies the essential points are unique. Thus, $z_{\lambda\circ T(x)} = T(z_{\lambda\circ x})$. ■

Proposition 5.7. *Let X be a complex normal hyperalgebra with a unit element. Let T be a strongly homomorphism of X into a normal hyperalgebra Y with a unit such that $T(1)$ is the unit element of Y. Then, for any $x \in X$, $\sigma_Y(T(x)) \subseteq \sigma_X(x)$.*

Proof. Let $\lambda \notin \sigma_X(x)$, then $z_\lambda - x \in Inv(X)$ and so there exists $y \in X$ such that $(z_\lambda - x)y = y(z_\lambda - x) = 1$. Since T is a strongly homomorphism, $T(z_\lambda - x)T(y) = T(y)T(z_\lambda - x) = T(1)$ and so $(T(z_\lambda) - T(x))T(y) = T(y)(T(z_\lambda) - T(x)) = T(1)$. By Proposition 5.5, $T(z_\lambda)$ is an essential point for $\lambda \circ T(1)$ and so we can write $T(z_\lambda) = z_{\lambda\circ T(1)}$. Hence, $(z_{\lambda\circ T(1)} - T(x))T(y) = T(y)(z_{\lambda\circ T(1)} - T(x)) = T(1)$ and $T(y) \in Y$. Thus, $(z_{\lambda\circ T(1)} - T(x)) \in Inv(Y)$ and so $\lambda \notin \sigma_Y(T(x))$. ■

6 Conclusion

In this article we prove the set of all bounded strongly homomorphisms operators on a complete normed hypervector space into itself is a Banach hyperalgebra as a new example of Banach hyperalgebra. Also, we prove the formula of an invertable elements in a unital normed hyperalgebra. Then we define the unitization of a hyperalgebra without a unit element and we show that the unitization is a multiplicative hyperalgebra with a weak identity $(0, 1)$. Also, we study the quasi inverse of elements in a nonunital hyperalgebra. Finally, we define the spectrum of elements in a normal hyperalgebra without a unit element and we prove that, if X is a complex normed normal hyperalgebra without a unit element and Y is the unitization $X + \mathbb{C}$ of X. Then $\sigma_Y((x, 0)) \subseteq \sigma_X(x)$.

References

[1] O. R. Dehghan and R. Ameri , Some Results on Hypervector Spaces, *Italian Journal of Pure and Applied Mathematics*, (41), 23-41, (2019).

[2] F. Marty, Sur nue generalization de la notion de group, In: *8-th Congress of the Scandinavic Mathematics*, Stockholm(1934)28, 45-49.

[3] R. Parvinianzadeh and A. Taghavi, Gelfand Theorem for Banach Hyperalgebras, *Italian Journal of Pure and Applied Mathematics*, 36 (2016), 913-922.

[4] M. Scafati Tallini, Weak Hypervector Spaces and Norms in such Spaces, Atti Convegno, *Algebraic Hyperstructures and Applications* , Iasi (Romania), (Luglio 1993), Hadronic Press, Plam Harbor, Fl. (U.S.A), (1994), 199-206.

[5] A. Taghavi and R. Hosseinzadeh , A Note on Dimension of Weak Hypervector Spaces , *Italian Journal of Pure and Applied Mathematics*, 33, 7-14, (2014).

[6] A. Taghavi and R. Hosseinzadeh, Operators on normed hypervector spaces, *Southeast Asian Bulletin of Mathimatics*, 35, (2011), 367-372.

[7] A. Taghav and R. Parvinianzadeh, The Spectrum in Banach Hyperalgebras, *Communications in Applied MATHEMATICS*, 20 (2016), 291-299.

[8] M. S. Tallini, Hypervector Spaces, *Proceedings of the Fourth International Congress on Algebraic Hyperstructures and Applications*, Xanthi, Greece, (1990), 167-174.

[9] M. S. Tallini, Matorial Hypervector Spaces, *Journal of Geometry*, 42(1999), 132-140.

[10] M. S. Tallini, Characterization of Remarkable Hypervector Spaces, *Proc. 8-th Int. Congress on Algebraic Hyperstructures and Applications*, Samotraki, Greece, (2002), Spanidis Press, Xanthi, (2003), 231-237.

[11] P. Raja and S. M. Vaezpour, Normed Hypervector Spaces, *Iranian Journal of Mathematical Sciences and Informatics*, 2(2), 35-44, (2007).

[12] P. Raja and S. M. Vaezpour, On The HyperBanach Spaces, *Italian Journal of Pure and Applied Mathematics*, 28, 261-272, (2011).

[13] P. Raja and S. M. Vaezpour, Convexity in Normed Hypervector Spaces, *Italian Journal of Pure and Applied Mathematics*, 28, 7-16, (2011).