Hereditary Properties in Isotonic Spaces

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Abstract

An isotonic space (X, cl) is a set X with isotonic operator $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ which satisfies $cl(\emptyset) = \emptyset$ and $cl(A) \subseteq cl(B)$ whenever $A \subseteq B \subseteq X$. Many properties which hold in topological spaces hold in isotonic spaces as well. We explore the topological concepts of lower separation axioms, higher separation axioms and connectedness for isotonic spaces, and we establish that some of these concepts are hereditary properties in isotonic spaces and some are not.

Keywords: generalized closure spaces, isotonic spaces, neighborhood spaces, hereditary properties, lower separation axioms, higher separation axioms, connectedness.

AMS Subject classification: 54A05

*This research is supported by a grant from the Deanery of Scientific Research, Islamic University of Gaza.

1 Introduction

Closure spaces and (more generally) isotonic spaces have already been studied by Hausdorff [13], Day [2], Hammer [12, 11], Gnilka [4, 5, 6], Stadler [14, 15], and Habil and Elzenati [7, 8]. In [14, 15], Stadler studied lower separation axioms and higher separation axioms in isotonic spaces. In [7, 8], the notions of connectedness, Z-conectedness, strong connectedness, and topological properties in isotonic spaces have been studied. In this paper, we explore all these notions and axioms, and we further study hereditary properties in isotonic spaces. As in topological spaces, there are many hereditary properties that hold in isotonic spaces, and we note that not every property which holds in topological spaces must hold in isotonic spaces. However, since every topological space is an isotonic space, we note that if a property does not hold in a topological space, it must not hold in any isotonic space either.

Let X be a set, $\mathscr{P}(X)$ its power set and $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ be an arbitrary set-valued set-function, called a *closure function*. We call $cl(A)$, $A \subseteq X$, the *closure* of A, and we call the pair (X, cl) a *generalized closure space*. Consider the following axioms of the closure function for all $A, B, A_\lambda \in \mathscr{P}(X), \lambda \in \Lambda$:

K0) $cl(\emptyset) = \emptyset$.

Kl) $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (isotonic).

K2) $A \subseteq cl(A)$ (expanding).

K3) $cl(A \cup B) \subseteq cl(A) \cup cl(B)$ (sub-additive).

K4) $cl(cl(A)) = cl(A)$ (idempotent).

The dual of a closure function is the *interior function* $int : \mathscr{P}(X) \to \mathscr{P}(X)$ which is defined by

$$int(A) := X \setminus cl(X \setminus A). \tag{1}$$

Given the interior function $int : \mathscr{P}(X) \to \mathscr{P}(X)$, the closure function is recovered by

$$cl(A) := X \setminus int(X \setminus A) \tag{2}$$

for all $A \in \mathscr{P}(X)$. A set $A \in \mathscr{P}(X)$ is *closed* in the generalized closure space (X, cl) if $cl(A)$=A holds. It is *open* if its complement $X \setminus A$ is closed or equivalently $A = int(A)$.

It should be noted that the open and closed sets will not play a central role in our discussion. From now on, the word *space* will mean a generalized closure space.

Definition 1.1. [14] Let *cl* and *int* be a closure and its dual interior function on X. Then the *neighborhood function* $\mathscr{N} : X \rightarrow \mathscr{P}(\mathscr{P}(X))$ assigns to each $x \in X$ the collection

$$\mathscr{N}(x) := \{ N \in \mathscr{P}(X) \mid x \in int(N)\} \tag{3}$$

of its neighborhoods. A set V is a *neighborhood* of A, in symbols $V \in \mathscr{N}(A)$, if $V \in \mathscr{N}(x)\ \forall x \in A$.

The proof of the next lemma follows immediately from the definitions.

Lemma 1.1. [14, 15] *For any space* (X, cl)*,* $V \in \mathscr{N}(A)$ *if and only if* $A \subseteq int(V)$*.*

The next theorem illustrates the intimate relationship between closures of sets and neighborhoods of points.

Theorem 1.1. [14, 15] *Let* $\mathscr{N}$ *be the neighborhood function defined in equ.(3). Then* $x \in cl(A)$ *if and only if* $X \setminus A \notin \mathscr{N}(x)$*.*

It should be noted that there are equivalent properties for (Ki), $i = 0, 1, ..., 4$, which can be expressed in terms of interior or neighborhood functions (see[14, 15, 10]).

2 Isotonic Spaces

Almost all approaches to extend the framework of topology at least assume that the closure functions are isotonic [4, 12, 11, 2], and many properties which hold in general topological spaces also hold in spaces with the isotonic property. The hierarchy of separation axioms that is familiar for topological spaces generalizes to spaces with isotonic, expansive closure functions [14, 15]. Neither additivity nor idempotency of the closure function must be assumed.

Definition 2.1. [14, 15] An *isotonic space* is a pair (X, cl), where X is a set and $cl : \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ satisfies the axioms $(K0)$ and $(K1)$. An isotonic space (X, cl) that satisfies $(K2)$ is called a *neighborhood space*. A *closure space* is a neighborhood space that satisfies $(K4)$. A *topological space* is a closure space that satisfies $(K3)$.

Most topology books establish the last definition of topological space as a theorem, where the operator $cl : \mathscr{P}(X) \rightarrow \mathscr{P}(X)$ is called *Kuratowski closure operation* (see for example [3, 9]).

Lemma 2.1. [11, Lemma10] *The following conditions are equivalent for an arbitrary closure function* $cl : \mathscr{P}(X) \to \mathscr{P}(X)$*:*

(K1) $A \subseteq B \subseteq X$ *implies* $cl(A) \subseteq cl(B)$.

(K1^I) $cl(A) \cup cl(B) \subseteq cl(A \cup B)$ *for all* $A, B \in \mathscr{P}(X)$.

(K1^{II}) $cl(A \cap B) \subseteq cl(A) \cap cl(B)$ *for all* $A, B \in \mathscr{P}(X)$.

It is easy to derive equivalent conditions for the associated interior function by repeated application of $int(A) = X \setminus cl(X \setminus A)$, as the following lemma shows.

Lemma 2.2. [14] The following conditions are equivalent for an arbitrary interior function: $int : \mathscr{P}(X) \to \mathscr{P}(X)$:

(K1^{III}) $A \subseteq B \subseteq X$ implies $int(A) \subseteq int(B)$.

(K1^{IV}) $int(A) \cup int(B) \subseteq int(A \cup B)$ for all $A, B \in \mathscr{P}(X)$.

(K1^V) $int(A \cap B) \subseteq int(A) \cap int(B)$ for all $A, B \in \mathscr{P}(X)$.

An isotonic space can be described by means of interior functions, as the following lemma shows. Its proof follows immediately from the definitions and, therefore, is omitted.

Lemma 2.3. *A space* (X, cl) *is isotonic if and only if the interior function* $int : \mathscr{P}(X) \to \mathscr{P}(X)$ *satisfies:*

I0) $int(X) = X$*;*

(K1^{III}) $A \subseteq B \subseteq X$ *implies* $int(A) \subseteq int(B)$.

Definition 2.2. [14, 15] Let (X, cl_X) and (Y, cl_Y) be two spaces. A function $f : X \to Y$ is *continuous* if $cl_X(f^{-1}(B)) \subseteq f^{-1}(cl_Y(B))$ $\forall B \in \mathscr{P}(Y)$.

Definition 2.3. [14] Let (X, cl_X) and (Y, cl_Y) be two spaces. Then $f : X \to Y$ is *closure-preserving* if for all $A \in \mathscr{P}(X)$, $f(cl_X(A)) \subseteq cl_Y(f(A))$.

Theorem 2.1. [14] *Let* (X, cl_X) *and* (Y, cl_Y) *be isotonic spaces. Then the following properties are equivalent:*

(i) $f : X \to Y$ *is continuous.*

(ii) $f : X \to Y$ *is closure-preserving.*

(iii) $f(A) \subseteq B$ *implies* $f(cl_X(A)) \subseteq cl_Y(B)$ *for all* $A \in \mathscr{P}(X)$ *and* $B \in \mathscr{P}(Y)$.

3 Hereditary Properties

In this section we present the notions of lower separation axioms, higher separation axioms, connectedness, Z-connectedness, and strong connectedness for isotonic spaces, and we show that many of these properties are hereditary.

Definition 3.1. [15] Let (X, cl) be a space and $Y \subseteq X$. Then $c_Y : \mathscr{P}(Y) \to \mathscr{P}(Y)$, $A \longmapsto Y \cap cl(A)$ is the *relativization* of cl to Y. The pair (Y, c_Y) is called a *subspace* of (X, cl). If $A \subseteq Y$, then the *relative interior* of A is given by

$$int_Y(A) := Y \setminus c_Y(Y \setminus A) = Y \cap int(A \cup (X \setminus Y))$$

and the *relative neighborhoods* of A are

$$\mathscr{N}_Y(A) := \{N \cap Y | N \in \mathscr{N}(A)\}.$$

Definition 3.2. [15] A property $\mathbb{B}$ of a space (X, cl) is *hereditary* if every subspace (Y, c_Y) of (X, cl) also has the property $\mathbb{B}$.

The proof of the following lemma is obvious.

Lemma 3.1. *The properties (K0), (K1), (K2), and (K3) are hereditary in any space (X, cl).*

Definition 3.3. Let (X, cl) and (Y, cl) be spaces, $f : X \to Y$ and $A \subseteq X$. We will use $f|A$ to denote the *restriction of f to A* which is defined by $(f|A)(x) := f(x)$ for each $x \in A$.

Theorem 3.1. [7] *Let (X, cl) and (Y, cl) be isotonic spaces and $A \subseteq X$. Then $(f|A) : A \to Y$ is continuous, whenever $f : X \to Y$ is continuous.*

3.1 Lower Separation Axioms

Definition 3.4. [14] A space (X, cl) is a $T_0 -$ *space* if and only if $\forall\, x, y \in X, x \neq y$, $\exists\, N_x \in \mathscr{N}(x)$ such that $y \notin N_x$ or $\exists\, N_y \in \mathscr{N}(y)$ such that $x \notin N_y$.

An equivalent definition of a T_0-space can be given by using closure functions, as the following result shows.

Proposition 3.1. [14] *An isotonic space (X, cl) is a T_0-space if and only if whenever x and y are distinct points in X, we have $x \notin cl(\{y\})$ or $y \notin cl(\{x\})$.*

Definition 3.5. A space (X, cl) is a T_1-space if $\forall\, x, y \in X,\ x \neq y,\ \exists\, N^{'} \in \mathscr{N}(x)$ and $N^{''} \in \mathscr{N}(y)$ such that $x \notin N^{''},\ \ y \notin N^{'}$.

Proposition 3.2. [14] *An isotonic space (X, cl) is a T_1-space if and only if $cl(\{x\}) \subseteq \{x\}\ \forall\, x \in X$.*

Definition 3.6. A space (X, cl) is a $T_2 -$ *space* if and only if $\ \forall\, x, y \in X\ \ , x \neq y,\ \exists\, N^{'} \in \mathscr{N}(x)$ and $N^{''} \in \mathscr{N}(y)$ such that $N^{'} \cap N^{''} = \emptyset$.

The next lemma is a generalization of ([3, Lemma VII-1.2])

Lemma 3.2. *An isotonic space (X, cl) is a $T_2 -$ space if and only if for any two distinct points $x, y \in X$, there is $U \in \mathscr{N}(x)$ such that $y \notin cl(U)$.*

Proof. Let (X, cl) be an isotonic space and $x \neq y$ in X. Then (X, cl) is T_2 if and only if there are $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $U \cap V = \emptyset$. Since $V \subseteq X \setminus U$, Lemma 2.3 implies that $int(V) \subseteq int(X \setminus U)$; so the above statement is equivalent to the existence of $U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(y)$ such that $y \in int(V) \subseteq int(X \setminus U)$. This, in turn, is equivalent to the existence of $U \in \mathscr{N}(x)$ such that $X \setminus U \in \mathscr{N}(y)$. Thus, it follows from Theorem 1.1 that (X, cl) is T_2 if and only if there exists $U \in \mathscr{N}(x)$ such that $y \notin cl(U)$. $\square$

Definition 3.7. A space (X, cl) is a $T_{2\frac{1}{2}} -$ *space* if and only if $\ \forall\, x, y \in X\ , x \neq y,\ \exists\, N^{'} \in \mathscr{N}(x)$ and $N^{''} \in \mathscr{N}(y)$ such that $cl(N^{'}) \cap cl(N^{''}) = \emptyset$.

Theorem 3.2. *The lower separation axioms T_0, T_1, T_2 and $T_{2\frac{1}{2}}$ are hereditary properties in any isotonic space.*

Proof. Let (X, cl) be an isotonic space and let (Y, c_Y) be a subspace of X.

(T_0) If X is a T_0-space, then for distinct points x and y in Y, we have $x, y \in X$; hence, by Proposition 3.1, $x \notin cl\{y\}$ or $y \notin cl\{x\}$. Since $c_Y\{x\} = Y \cap cl\{x\}$ and $c_Y\{y\} = Y \cap cl\{y\}$, it follows that $x \notin c_Y\{y\}$ or $y \notin c_Y\{x\}$. Therefore, Y is a T_0-space.

(T_1) If X is T_1 and $x \in Y$, then $x \in X$, so that, by Proposition 3.2, $cl\{x\} \subseteq \{x\}$. Hence, $cl_Y\{x\} = Y \cap cl\{x\} \subseteq \{x\}$. Therefore, Y is a T_1-space.

(T_2) If X is T_2 and $x, y \in Y,\ x \neq y$, then there are $N^{'} \in \mathscr{N}(x)$ and $N^{''} \in \mathscr{N}(y)$ such that $N^{'} \cap N^{''} = \emptyset$. Since $Y \cap N^{'} \in \mathscr{N}_Y(x)$ and $Y \cap N^{''} \in \mathscr{N}_Y(y)$, and $(Y \cap N^{'}) \cap (Y \cap N^{''}) \subseteq N^{'} \cap N^{''} = \emptyset$, it follows that Y is a T_2-space.

$(T_{2\frac{1}{2}})$ Suppose that X is $T_{2\frac{1}{2}}$, $x, y \in Y, x \neq y$. Then there are $N^{'} \in \mathscr{N}(x)$ and $N^{''} \in \mathscr{N}(y)$ such that $cl(N^{'}) \cap cl(N^{''}) = \emptyset$. Setting $N_x^{'} = Y \cap N^{'}$ and $N_y^{''} = Y \cap N^{''}$, we have $N_x^{'} \in \mathscr{N}_Y(x)$, $N_y^{''} \in \mathscr{N}_Y(y)$, and $c_Y(N_x^{'}) = Y \cap cl(N_x^{'}) \subseteq cl(N_x^{'}) \subseteq cl(N^{'})$. Similarly, $c_Y(N_y^{''}) \subseteq cl(N^{''})$. Hence, $c_Y(N_x^{'}) \cap c_Y(N_y^{''}) = \emptyset$. Therefore, Y is a $T_{2\frac{1}{2}}$-space. □

3.2 Regular and Completely Regular Spaces

Definition 3.8. A function $v : X \to [0,1]$ -where X is any space with a closure function and $[0,1]$ is the closed unit interval in $\mathbb{R}$ with the usual topology- is called a *Urysohn function separating* A and B in X if v is continuous, $v(A) \subseteq \{0\}$ and $v(B) \subseteq \{1\}$. Two subsets A and B of X are *Urysohn separated* if there is a Urysohn function $v : X \to [0,1]$ separating A and B. In this case, we write $A \parallel_v B$ or simply $A \parallel B$.

Definition 3.9. Let X be a space and $A, B \subseteq X$. We say that A is *completely within* B, and we write $A \Subset B$, if there is a continuous function $v : X \to [0,1]$ such that $v(A) \subseteq \{0\}$ and $v(X \setminus B) \subseteq \{1\}$.

By definition, we have $A \Subset B$ iff $A \parallel X \setminus B$, and by Lemma 5.1 of [8], $A \Subset B$ implies $A \subseteq B$.

A separation condition stronger than T_2 is obtained by replacing one of the points by a closed set.

Definition 3.10. [14, 15] A space (X, cl) is *regular* if for all $x \in X$ and all nonempty $A \in \mathscr{P}(X)$ such that $x \notin cl(A)$, $\exists\, U \in \mathscr{N}(x)$ and $V \in \mathscr{N}(A)$ such that $U \cap V = \emptyset$. A space (X, cl) is a T_3-*space* if it is T_0 and regular.

Definition 3.11. A space (X, cl) is *completely regular* if for all $x \in X$ and all $N \in \mathscr{N}(x)$ there is $N^{'} \in \mathscr{N}(x)$ such that $N^{'} \Subset N$. A space (X, cl) is a $T_{3\frac{1}{2}}$-*space* if it is T_1 and completely regular.

Lemma 3.3. [8] *Every completely regular, isotonic space is regular, and hence* $(T_{3\frac{1}{2}})$ *implies* (T_3).

Theorem 3.3. *Regularity,* T_3*, complete regularity and* $T_{3\frac{1}{2}}$ *are hereditary properties in any isotonic space.*

Proof. Let (X, cl) be an isotonic space and (Y, c_Y) be a subspace of (X, cl).

(Regularity) Let X be a regular space, and let $y \in Y$, $A \subseteq Y$ be such that $y \notin c_Y(A)$. Then $y \notin Y \cap cl(A)$ implies $y \notin cl(A)$, and hence, by regularity of X, there are $U \in \mathscr{N}(y)$ and $V \in \mathscr{N}(A)$ such that $U \cap V = \emptyset$. Since $U \cap Y \in \mathscr{N}_Y(y)$, $V \cap Y \in \mathscr{N}_Y(A)$ and $(U \cap Y) \cap (V \cap Y) = U \cap V \cap Y = \emptyset$, it follows that Y is a regular space.

(T_3) Immediate from (regularity) and Theorem 3.2.

(Complete regularity) Let X be a completely regular space, $y \in Y$ and $N \in \mathscr{N}_Y(y)$. Then $N = U \cap Y$ for some $U \in \mathscr{N}(y)$. Since X is completely regular, there is $V \in \mathscr{N}(y)$ such that $V \Subset U$. Hence, there is a continuous function $v : X \to [0, 1]$ such that $v(V) \subseteq \{0\}$ and $v(X \setminus U) \subseteq \{1\}$. Since $V \cap Y \subseteq V$ and $(X \setminus U) \cap Y = Y \setminus (U \cap Y)) \subseteq X \setminus U$, it follows that $v(V \cap Y) \subseteq \{0\}$ and $v(Y \setminus (U \cap Y)) \subseteq \{1\}$. Now setting $N_1 := V \cap Y$ and noting that, by Theorem 3.1, the restriction of v to Y is continuous, we have shown that there is $N_1 \in \mathscr{N}_Y(y)$ such that $N_1 \Subset N$. Therefore, Y is completely regular.

($T_{3\frac{1}{2}}$) Immediate from (complete regularity) and Theorem 3.2. □

3.3 Normal and Completely Normal Spaces

In the last two subsections, we have discussed the separation of points, lower separation axioms, the separation of points from closed sets, regular and completely regular spaces. Carrying out this theme further, we may consider spaces in which disjoint closed sets can be separated by means of neighborhoods. Such spaces are called normal spaces.

Four kinds of normal spaces will be given in this subsection. Every one has a special name with a special definition; t-normal, quasi-normal, normal and Urysohn-normal spaces.

Definition 3.12. [15] An isotonic space (X, cl) is

(tN) *t-normal* if any two non-empty disjoint closed sets $A, B \subseteq X$ are separated; that is, there are neighborhoods $U \in \mathscr{N}(A)$ and $V \in \mathscr{N}(B)$ such that $U \cap V = \emptyset$;

(QN) *quasi-normal* if, for all non-empty sets $A, B \subseteq X$ satisfying $cl(A) \cap cl(B) = \emptyset$, there are neighborhoods $U \in \mathscr{N}(A)$ and $V \in \mathscr{N}(B)$ such that $U \cap V = \emptyset$;

(T_4) if it is (T_1) and quasi-normal;

(N) *normal* if, for all non-empty sets $A, B \subseteq X$ satisfying $cl(A) \cap cl(B) = \emptyset$, there are neighborhoods $U \in \mathscr{N}(cl(A))$ and $V \in \mathscr{N}(cl(B))$ such that $U \cap V = \emptyset$;

(UN) *Urysohn-normal* if, for all non-empty sets $A, B \subseteq X$ satisfying $cl(A) \cap cl(B) = \emptyset$, there is a Urysohn function separating A and B, $A \parallel B$;

(T_4U) if it is (T_1) and Urysohn-normal.

Theorem 3.4. [15] *If (X, cl) is an isotonic space, then $(UN) \Rightarrow (N) \Rightarrow (tN)$ and $(QN) \Rightarrow (tN)$.*

Remark 3.1. It should be noted that, in isotonic spaces, (tN), (QN) (and hence (T_4)), (N), and (UN) (and hence (T_4U)) are not hereditary properties. Indeed, by Theorem 3.4, it suffices to show that t-normality (tN) is not hereditary. To this end, note that t-normality is not hereditary in topological spaces (see Example 4, page 145 of [3]); hence t-normality is not hereditary in isotonic spaces, since every topological space is an isotonic space.

Definition 3.13. In an isotonic space (X, cl), two subsets $A, B \subseteq X$ are called *semi-separated* if $cl(A) \cap B = A \cap cl(B) = \emptyset$.

Definition 3.14. [15] A space (X, cl) is

(CN) *completely normal* if any two semi-separated sets are separated;

(T_5) if it is (T_1) and completely normal;

(CUN) *completely Urysohn-normal* if each pair of semi-separated sets is Urysohn-separated;

(T_5U) if it is (T_1) and completely Urysohn-normal.

Lemma 3.4. [8] *In isotonic spaces, (CUN) implies (CN).*

Theorem 3.5. *(CN), (T_5), (CUN) and (T_5U) are hereditary properties in any isotonic space.*

Proof. Let (X, cl) be an isotonic space and let (Y, c_Y) be a subspace of (X, cl).

(CN) Let X be a completely normal space, and let $A, B \subseteq Y$ be semi-separated in Y; that is, $c_Y(A) \cap B = c_Y(B) \cap A = \emptyset$. Then $Y \cap cl(A) \cap B = Y \cap cl(B) \cap A = \emptyset$, and hence $B \cap cl(A) = A \cap cl(B) = \emptyset$. Now A, B are semi-separated in X, and X is completely normal, hence A, B are separated in X; that is, there are $U \in \mathscr{N}(A)$ and $V \in \mathscr{N}(B)$ such that $U \cap V = \emptyset$. Let $U_Y = Y \cap U$ and $V_Y = Y \cap V$. Then we have $U_Y \in \mathscr{N}_Y(A)$, $V_Y \in \mathscr{N}_Y(B)$, and $U_Y \cap V_Y = \emptyset$. Thus A, B are separated in Y, and therefore Y is (CN).

(T_5) Immediate from (CN) and Theorem 3.2.

(CUN) Let X be an $CUN-$space, and let $A, B \subseteq Y$ be semi-separated in Y; that is, $c_Y(A) \cap B = c_Y(B) \cap A = \emptyset$. Then, as above, we have $B \cap cl(A) = A \cap cl(B) = \emptyset$, and hence A, B are semi-separated in X. So, as X is (CUN), there is a continuous function $\nu : X \to [0,1]$ such that $\nu(A) \subseteq \{0\}$ and $\nu(B) \subseteq \{1\}$. Let $\nu_Y : Y \to [0,1]$ be the restriction of ν to Y. Then, by Theorem 3.1, ν_Y is continuous, $\nu_Y(A) = \nu(A) \subseteq \{0\}$, and $\nu_Y(B) = \nu(B) \subseteq \{1\}$. Thus, ν_Y is a continuous Urysohn function separating A and B in Y, and therefore Y is (CUN).

(T_5U) Immediate from (CUN) and Theorem 3.2. $\square$

3.4 Connectedness in Isotonic Spaces

Definition 3.15. [14] A set $Y \in \mathscr{P}(X)$ is *connected* in a space (X, cl) if it is not a disjoint union of a nontrivial semi-separated pair of sets $A, Y \setminus A$, where $A \neq \emptyset$, and $A \neq Y$. We say that a space (X, cl) is *connected* if X is connected in (X, cl).

Theorem 3.6. [7] *A set $Y \in \mathscr{P}(X)$ is connected in a space (X, cl) if and only if for each proper subset $A \subsetneqq Y$,*

$$[cl(A) \cap (Y \setminus A)] \cup [cl(Y \setminus A) \cap A] \neq \emptyset.$$

Theorem 3.7. [7] *An isotonic space (X, cl) is connected if and only if for all T_1-isotonic doubleton spaces $Y = \{0, 1\}$, any continuous function $f : X \to Y$ is constant.*

It should be noted that connectedness is not a hereditary property in isotonic spaces, as the following example shows.

Example 3.1. Let $X = \{a, b, c\}$ and define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows: $cl(\emptyset) = \emptyset$, $cl(X) = X$, $cl(\{a\}) = \{a\}$, $cl(\{b\}) = X$, $cl(\{c\}) = \{c\}$, $cl(\{a, b\}) = cl(\{b, c\}) = cl(\{a, c\}) = X$. Clearly, (X, cl) is an isotonic space, and the only semi-separated sets in X are $\{a\}$ and $\{c\}$. Since $\{a\} \cup \{c\} \neq X$, (X, cl) is connected.

Now, if $Y = \{a, c\}$, then (Y, cl_Y) is a subspace of (X, cl), where $cl_Y(\emptyset) = \emptyset$, $cl_Y(Y) = Y$, $cl_Y(\{a\}) = \{a\}$, $cl_Y(\{c\}) = \{c\}$. Moreover, $\{a\}$ and $\{c\}$ are semi-separated sets in Y with $\{a\} \cup \{c\} = Y$. Therefore, (Y, cl_Y) is not a connected subspace of (X, cl).

Definition 3.16. [1] Let (Z, cl) be an isotonic space with more than one element. An isotonic space (X, cl) is called Z*-connected* if and only if any continuous function $f : X \to Z$ is constant.

It should be noted that Z-connectedness is not a hereditary property in isotonic spaces, as the following example shows.

Example 3.2. Let $X = \{a, b, c\}$ and define $cl : \mathscr{P}(X) \to \mathscr{P}(X)$ as follows: $cl(\emptyset) = \emptyset$, $cl(X) = X$, $cl(\{a\}) = \{a\}$, $cl(\{b\}) = \{b\}$, $cl(\{c\}) = \{a\}$, $cl(\{a, b\}) = cl(\{a, c\}) = cl(\{b, c\}) = \{a, b\}$. Clearly, (X, cl) is an isotonic space. It is easy to check that for all proper subsets $A \subsetneqq X$,

$$[cl(A) \cap (X \setminus A)] \cup [cl(X \setminus A) \cap A] \neq \emptyset.$$

Thus, by Theorem 3.6, (X, cl) is connected.

Now, let $Z = \{0, 1\}$ and define $cl_Z : \mathscr{P}(Z) \to \mathscr{P}(Z)$ as follows: $cl_Z(\emptyset) = \emptyset$, $cl_Z(Z) = Z$, $cl_Z(\{0\}) = \{0\}$, $cl_Z(\{1\}) = \{1\}$. By Proposition 3.2, (Z, cl_Z) is a T_1-isotonic space. Let $f : X \to Z$ be any continuous function. Then, by Theorem 3.7, f must be constant. Therefore, (X, cl) is Z-connected.

Let $Y = \{a, b\}$, so that (Y, cl_Y) is a subspace of (X, cl), where $cl_Y(\emptyset) = \emptyset$, $cl_Y(Y) = Y$, $cl_Y(\{a\}) = \{a\}$, $cl_Y(\{b\}) = \{b\}$. Let $g : Y \to Z$ be defined by $g(a) = 0$, $g(b) = 1$. Clearly, $g(cl_Y(A)) \subseteq cl_Z(g(A))$ for all $A \in \mathscr{P}(Y)$, so that g is closure-preserving and hence, by Theorem 2.1, g is continuous. Since g is not constant, it follows that Y is not Z-connected.

Definition 3.17. [7] A space (X, cl) is *strongly connected* if there is no countable collection of pairwise semi-separated sets $\{A_i\}$ such that $X = \bigcup A_i$.

It should be noted that strong connectedness is not a hereditary property in isotonic spaces, as Example 3.1 shows. Indeed, the isotonic space (X, cl) of Example 3.1 is strongly connected but the subspace $Y = \{a, c\}$ is not strongly connected.

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