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Bayesian Inference on the Generalized Gamma Distribution using Conjugate Priors

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Abstract:
This paper focuses on the three-parameter generalized gamma distribution and uses Bayesian techniques to estimate its parameters. Many authors con-sidered estimating the parameters of the generalized gamma distribution in a Bayesian framework using Jeffrey's priors. Others used different loss functions and the least squares approach. This study uses Bayesian techniques to estimate the three-parameter generalized gamma distribution by using conjugate priors. The random Metropolis algorithm is used to simulate the Bayesian estimates of the three parameters. Then these estimates are compared to the maximum like-lihood estimates using the mean error through simulation. It has been shown in this paper that the obtained estimates using this approach is more accurate than the traditional methods of estimation such as the Maximum likelihood method. The same approach is then used to estimate the parameters of mixtures of the generalized gamma parameters using conjugate priors.

Keywords:
Bayesian techniques, generalized gamma distribution, conjugate prior, max-imum likelihood estimator, method-of-moments.

1 Introduction

The generalized gamma distribution is a lifetime distribution that is often used to model real lifetime data. It is a flexible distribution that contains, as special cases, the exponential, Weibull, lognormal, and the gamma distribution.

We consider the three-parameter generalized gamma distribution first introduced by joseph in [11]. The main goal of this paper is to estimate the parameters of this distribution using the conjugate priors and Bayesian techniques. If $f(x \mid \theta)$ is an exponential family, with density $f(x \mid \theta) = C(\theta)h(x)\exp(\phi(\theta)S(x))$, then a conjugate prior distribution for θ exists and the prior distribution $P(\theta) \propto C(\theta)\exp(\phi(\theta)b)$ is conjugate to the likelihood of the exponential distribution (see [6]).

A random variable X is said to have a generalized gamma distribution if its probability density function (pdf) has the following form

$$f(x, a, \gamma, \delta) = \frac{\delta a^{\frac{\gamma+1}{\delta}}}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} x^{\gamma} e^{-ax^{\delta}}, \ x > 0, \delta > 0, a > 0, \gamma > -1. \tag{1.1}$$

In the above density a is the scale parameter, γ and δ are the shape parameters. The mean and the variance of this distribution, are respectively given by

$$E(X) = \frac{\Gamma\left(\frac{\gamma+2}{\delta}\right)}{a^{\frac{1}{\delta}}\Gamma\left(\frac{\gamma+1}{\delta}\right)} \tag{1.2}$$

$$V(X) = \frac{\Gamma\left(\frac{\gamma+3}{\delta}\right)}{a^{\frac{2}{\delta}}\Gamma\left(\frac{\gamma+1}{\delta}\right)} - \left(\frac{\Gamma\left(\frac{\gamma+2}{\delta}\right)}{a^{\frac{1}{\delta}}\Gamma\left(\frac{\gamma+1}{\delta}\right)}\right)^2. \tag{1.3}$$

Many authors considered estimating the parameters of the generalized gamma distribution in a Bayesian framework. For example, Naqash et. al. [16] obtained Bayesian estimators of the unknown parameters of the three parameter generalized gamma distribution, based on several priors using different loss functions. Clifford Cohen and Betty Jones Whitten [7] were concerned with the modifications of both maximum likelihood and moment estimators for parameters of the three-parameter gamma distribution. Stacy and Mihram [23] derived parameter estimation techniques for the generalized gamma distribution. Vani et. al. [26] estimated the three-parameter gamma distribution by using likelihood, spacings and least squares approach. Upadhyay et. al. [25] proposed Bayesian inference in life testing and reliability by using Markov Chain Monte Carlo (MCMC). Pang et. al. [18] used MCMC techniques to carry out a Bayesian estimation procedure using Hirose's simulated data. Pandey and Rao [17] derived Bayesian estimation of scale parameter of generalized gamma distribution using precautionary loss function. Balakrishnan et. al. [4] proposed some simple efficient estimators for the three-parameter gamma distribution. Shukla and Kumar [22] obtained Bayes estimators of the scale parameter of a generalized gamma type model by using several priors. Hirose [9,10] used maximum likelihood parameter estimation and continuation method in the three-parameter gamma distribution. Reshi et. al. [21] derived Bayesian analysis of size-biased generalized gamma distribution. Bai et. al. [3] used methods of moments and maximum likelihood in the three-parameter gamma and lognormal distributions. Ahmad et. al. [2] employed Bayesian method of estimation to estimate the parameters of generalized gamma distribution using Jeffrey's and extension of Jeffrey's priors.

Ramos et. al. [19] proposed an objective Bayesian estimation approach for the parameters of the generalized gamma distribution. Gaurav et. al. [8] compared the Bayesian estimates with the maximum likelihood estimates of the scale parameter in generalized gamma type distribution with known shape parameters under different loss functions. Koutrouvelis and Canavos [12] used the empirical moment generating function for the estimation of the shape, scale, and location parameters of a three-parameter gamma distribution. Tzavelas [24] obtained the maximum likelihood parameter estimation in the three-parameter gamma distribution with the use of Mathematica. Munilla [15] obtained Bayesian conjugate analysis using a generalized inverted Wishart distribution accounts for differential uncertainty among the genetic parameters–an application to the maternal animal model. Ramos [20] obtained Bayesian reference analysis for the Generalized Gamma distribution.

2 Maximum Likelihood Estimation

The likelihood for a random sample $x_1, x_2, \ldots, x_n$ of size n from the generalized gamma distribution is

$$\begin{aligned} L(\gamma, \delta, a) &= \prod_{i=1}^{n} f(x_i \mid \gamma, \delta, a) \\ &= \frac{\delta^n a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^n\left(\frac{\gamma+1}{\delta}\right)} P^{\gamma} e^{-a\sum_{i=1}^{n} x_i^{\delta}}, \end{aligned} \tag{2.1}$$

where $P = \prod_{i=1}^{n} x_i, \delta > 0, a > 0, \gamma > -1, x_i > 0$ for all $i = 1, 2, \ldots, n$.

The corresponding log-likelihood function is

$$\begin{aligned} \log L(\gamma, \delta, a) =& \frac{n(\gamma+1)\log a}{\delta} + n\log\delta + \gamma\sum_{i=1}^{n}\log x_i - n\log\left(\Gamma\left(\frac{\gamma+1}{\delta}\right)\right) \\ &- a\sum_{i=1}^{n} x_i^{\delta}. \end{aligned} \tag{2.2}$$

Let $g_1(\gamma, \delta) = \int_0^\infty t^{\frac{\gamma+1}{\delta}-1} e^{-t} \log t \; dt$. We let $\frac{\partial \log L}{\partial a} = 0$, $\frac{\partial \log L}{\partial \gamma} = 0$ and $\frac{\partial \log L}{\partial \delta} = 0$ to get

$$\frac{n(\gamma+1)}{a\delta} - \sum_{i=1}^{n} x_i^{\delta} = 0 \tag{2.3}$$

$$\log a - \frac{g_1(\gamma,\delta)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} + \frac{n}{\delta}\sum_{i=1}^{n} \log x_i = 0 \tag{2.4}$$

$$-\log a + \frac{g_1(\gamma,\delta)}{\Gamma\left(\frac{\gamma+1}{\delta}\right)} + \frac{\delta}{\gamma+1} - \frac{a\delta^2}{n\,(\gamma+1)}\sum_{i=1}^{n} x_i^{\delta}\log x_i = 0. \tag{2.5}$$

Then, we simultaneously solve these equations for γ, δ, and a.

3 Bayesian Inference

In order to use the Bayesian techniques to estimate the parameters of the generalized gamma distribution, we consider the following cases.

3.1 Case 1: Unknown scale parameter a

When the scale parameter a is unknown and both the shape parameters γ and δ are known. By ignoring terms that contain γ, δ in (2.1), the likelihood function is given by:

$$L\,(x,\gamma,\delta \mid a) \propto a^{n\left(\frac{\gamma+1}{\delta}\right)} e^{-a\sum_{i=1}^{n} x_i^{\delta}}. \tag{3.1}$$

The conjugate prior $\pi\,(a)$ is the gamma distribution with hyperparameters $r > 0$ and $k > 0$

$$\pi\,(a) = \frac{1}{k^r\Gamma(r)} a^{r-1} e^{-\frac{a}{k}}. \tag{3.2}$$

By ignoring terms that contain γ, δ, the posterior distribution $\pi\,(a \mid \gamma,\delta,x)$ is the gamma distribution with hyperparameters $\acute{r} = \frac{n(\gamma+1)}{\delta} + r$ and $\acute{k} = \frac{k}{1+k\sum_{i=1}^{n} x_i^{\delta}}$

$$\pi\,(a \mid \gamma,\delta,x) \propto a^{\acute{r}-1} e^{\frac{-a}{\acute{k}}}. \tag{3.3}$$

3.2 Case 2: Unknown shape parameter γ

When the shape parameter γ is unknown and both the scale parameter a and the shape parameter δ are known. By ignoring terms that contain a, δ in (2.1), the likelihood function is given by:

$$L\left(x,a,\delta \mid \gamma\right) \propto \frac{a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{n}\left(\frac{\gamma+1}{\delta}\right)} P^{\gamma}, \tag{3.4}$$

where $P=\prod_{i=1}^{n} x_i$.

The conjugate prior $\pi\left(\gamma\right)$ with hyperparameters $s>0, b>0$ and $t>0$ is

$$\pi\left(\gamma\right) \propto \frac{a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{b}\left(\frac{\gamma+1}{\delta}\right)} t^{\gamma}. \tag{3.5}$$

By ignoring terms that contain a,δ, the posterior distribution $\pi\left(\gamma \mid a,\delta,x\right)$ with hyperparameters $\acute{s}=s+n, \acute{b}=b+n$ and $\acute{t}=tP$ is

$$\pi\left(\gamma \mid a,\delta,x\right) \propto \frac{a^{\acute{s}\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} (\acute{t})^{\gamma}, \tag{3.6}$$

where $P=\prod_{i=1}^{n} x_i$.

3.3 Case 3: Unknown shape parameter δ

When the shape parameter δ is unknown and both the scale parameter a and the shape parameter γ are known. By ignoring terms that contain a,γ in (2.1), the likelihood function is given by:

$$L\left(x,a,\gamma \mid \delta\right) = \frac{\delta^{n} a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{n}\left(\frac{\gamma+1}{\delta}\right)} e^{-a\sum_{i=1}^{n} x_i^{\delta}}. \tag{3.7}$$

The conjugate prior $\pi\left(\delta\right)$ with hyperparameters $m>1, s>0, b>0$ and $c>0$ is

$$\pi\left(\delta\right) \propto \frac{\delta^{c} a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{b}\left(\frac{\gamma+1}{\delta}\right)} e^{-am^{\delta}}. \tag{3.8}$$

By ignoring terms that contain a,γ, the posterior distribution $\pi\left(\delta \mid a,\gamma,x\right)$ with hyperparameters $\acute{m}=\sum_{i=1}^{n} x_i^{\delta}+m^{\delta}, \acute{s}=s+n, \acute{b}=b+n$ and $\acute{c}=c+n$ is

$$\pi\left(\delta \mid a,\gamma,x\right) \propto \frac{\delta^{\acute{c}} a^{\acute{s}\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} e^{-a\acute{m}}. \tag{3.9}$$

3.4 Case 4: Unknown both scale parameter a and shape parameter γ

When both the scale parameter a and the shape parameter γ are unknown and the shape parameter δ is known. By ignoring terms that contain δ in (2.1), the likelihood function is given by:

$$L\left(x,\delta \mid a,\gamma\right) \propto \frac{a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{n}\left(\frac{\gamma+1}{\delta}\right)} P^{\gamma} e^{-a\sum_{i=1}^{n} x_i^{\delta}}, \tag{3.10}$$

where $P = \prod_{i=1}^{n} x_i$.

The joint conjugate prior $\pi\left(a,\gamma\right)$ with hyperparameters $m > 1, s > 0, b > 0$ and $t > 0$ is

$$\pi\left(a,\gamma\right) \propto \frac{a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{b}\left(\frac{\gamma+1}{\delta}\right)} t^{\gamma} e^{-am^{\delta}}. \tag{3.11}$$

By ignoring terms that contain δ, the joint posterior distribution $\pi\left(a,\gamma \mid \delta, x\right)$ with hyperparameters $\acute{m} = \sum_{i=1}^{n} x_i^{\delta} + m^{\delta}, \acute{s} = s+n, \acute{b} = b+n$ and $\acute{t} = tP$ is

$$\pi\left(a,\gamma \mid \delta, x\right) \propto \frac{a^{\acute{s}\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} (\acute{t})^{\gamma} e^{-a\acute{m}}, \tag{3.12}$$

where $P = \prod_{i=1}^{n} x_i$.

3.5 Case 5: Unknown both scale parameter a and shape parameter δ

When both the scale parameter a and the shape parameter δ are unknown and the shape parameter γ is known. By ignoring terms that contain γ in (2.1), the likelihood function is given by:

$$L\left(x,\gamma \mid a,\delta\right) = \frac{\delta^{n} a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{n}\left(\frac{\gamma+1}{\delta}\right)} e^{-a\sum_{i=1}^{n} x_i^{\delta}}. \tag{3.13}$$

The joint conjugate prior $\pi\left(a,\delta\right)$ with hyperparameters $m > 1, s > 0, b > 0$ and $c > 0$ is

$$\pi\left(a,\delta\right) \propto \frac{\delta^{c} a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{b}\left(\frac{\gamma+1}{\delta}\right)} e^{-am^{\delta}}. \tag{3.14}$$

By ignoring terms that contain γ, the joint posterior distribution $\pi\left(a,\delta \mid \gamma, x\right)$ with hyperparameters $\acute{m} = \sum_{i=1}^{n} x_i^{\delta} + m^{\delta}, \acute{s} = s+n, \acute{b} = b+n$ and $\acute{c} = c+n$ is

$$\pi\left(a,\delta \mid \gamma, x\right) \propto \frac{\delta^{\acute{c}} a^{\acute{s}\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} e^{-a\acute{m}}. \tag{3.15}$$

3.6 Case 6: Unknown both shape parameter γ and δ

When both the shape parameters γ and δ are unknown and the scale parameter a is known. The likelihood function from (2.1) is given by:

$$L\left(x, a \mid \gamma, \delta\right) = \frac{\delta^n a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^n\left(\frac{\gamma+1}{\delta}\right)} P^\gamma e^{-a\sum_{i=1}^n x_i^\delta}, \tag{3.16}$$

where $P = \prod_{i=1}^n x_i$.

The joint conjugate prior $\pi\left(\gamma, \delta\right)$ with hyperparameters $m > 1, s > 0, b > 0, c > 0$ and $t > 0$ is

$$\pi\left(\gamma, \delta\right) \propto \frac{\delta^c a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^b\left(\frac{\gamma+1}{\delta}\right)} t^\gamma e^{-am^\delta}. \tag{3.17}$$

By ignoring terms that contain a, the joint posterior distribution $\pi\left(\gamma, \delta \mid a, x\right)$ with hyperparameters $\acute{m} = \sum_{i=1}^n x_i^\delta + m^\delta, \acute{s} = s + n, \acute{b} = b + n$ and $\acute{c} = c + n$ is

$$\pi\left(\gamma, \delta \mid a, x\right) \propto \frac{\delta^{\acute{c}} a^{(s+n)\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} (\acute{t})^\gamma e^{-a\acute{m}}, \tag{3.18}$$

where $P = \prod_{i=1}^n x_i$.

3.7 Case 7: Unknown three parameters a, γ and δ

When both the shape parameters γ and δ and the scale parameter a are unknown, the likelihood function is given by:

$$L\left(x \mid a, \gamma, \delta\right) = \frac{\delta^n a^{n\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^n\left(\frac{\gamma+1}{\delta}\right)} P^\gamma e^{-a\sum_{i=1}^n x_i^\delta}, \tag{3.19}$$

where $P = \prod_{i=1}^n x_i$.

The joint conjugate prior $\pi\left(a, \gamma, \delta\right)$ with hyperparameters $m > 1, s > 0, b > 0, c > 0$ and $t > 0$ is

$$\pi\left(a, \gamma, \delta\right) \propto \frac{\delta^c a^{s\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^b\left(\frac{\gamma+1}{\delta}\right)} t^\gamma e^{-am^\delta}. \tag{3.20}$$

The joint posterior distribution $\pi\left(a, \gamma, \delta \mid x\right)$ with hyperparameters $\acute{m} = \sum_{i=1}^n x_i^\delta + m^\delta, \acute{s} = s + n, \acute{b} = b + n$ and $\acute{c} = c + n$ is

$$\pi\left(a, \gamma, \delta \mid x\right) \propto \frac{\delta^{\acute{c}} a^{\acute{s}\left(\frac{\gamma+1}{\delta}\right)}}{\Gamma^{\acute{b}}\left(\frac{\gamma+1}{\delta}\right)} (\acute{t})^\gamma e^{-a\acute{m}}, \tag{3.21}$$

where $P = \prod_{i=1}^n x_i$.

4 Finite Mixtures of Generalized Gamma Distribution

The general form of the k-finite generalized gamma mixture is given by

$$f(x \mid a, \gamma, \delta, \lambda) = \sum_{j=1}^{k} \lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x^{\gamma_j} e^{-a_j x^{\delta_j}}, \tag{4.1}$$

where $x > 0, a = (a_j)_{j=1}^k$ with $a_j > 0, \gamma = (\gamma_j)_{j=1}^k$ with $\gamma_j > -1, \delta = (\delta_j)_{j=1}^k$ with $\delta_j > 0, \lambda = (\lambda_j)_{j=1}^k$ with $0 < \lambda_j < 1, \sum_{j=1}^k \lambda_j = 1$ and $k > 1$.

Let $x_1, x_2, \ldots, x_n$ are independent and identically distributed observations according to the k-finite generalized gamma mixture. For each observation $x_i, i = 1, 2, \ldots, n$, the indicator parameter z_i is introduced as follows:

$$z_{ij} = \begin{cases} 1, & \text{if the observation } \ x_i \text{ belongs to the } j^{th} \text{ component of the mixture} \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

For fixed i, $\sum_{j=1}^k z_{ij} = 1$. It is clear that $f(z_{ij} = 1 \mid \lambda) = \lambda_j$ and $f(z_{ij} = 0 \mid \lambda) = 1 - \lambda_j$. Therefore, $z_{ij} \sim$ Bernoulli(λ_j) and $z_i \mid \lambda \sim$ multinomial $(1, \lambda_1, \lambda_2, \ldots, \lambda_k)$. Also, $f(x_i \mid z_{ij} = 1)$ is a generalized gamma distribution with the parameters $a_j > 0, \gamma_j > -1$ and $\delta_j > 0$.

$$f(z_i \mid \lambda) = \prod_{j=1}^{k} \lambda_j^{z_{ij}}. \tag{4.3}$$

Since $z_1, z_2, \ldots, z_n$ are independent, we write the joint indicator density as the following form:

$$f(z \mid \lambda) = \prod_{i=1}^{n} \prod_{j=1}^{k} \lambda_j^{z_{ij}}. \tag{4.4}$$

Let $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$ denote the complete data. Since for fixed i, only one of z_{ij} equal to 1, the joint pdf of the observed observation x_i and the unobserved indicator parameter z_i can be written as

$$f(x_i, z_i \mid a, \gamma, \delta, \lambda) = \prod_{j=1}^{k} \left(\lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x_i^{\gamma_j} e^{-a_j x_i^{\delta_j}} \right)^{z_{ij}}. \tag{4.5}$$

Since $(x_1, z_1), (x_2, z_2), \ldots, (x_n, z_n)$ are independent, the complete data likelihood of this mixture is given by:

$$L(x, z \mid a, \gamma, \delta, \lambda) = \prod_{j=1}^{k} \left[\frac{\lambda_j \delta_j a_j^{\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} \right]^{\sum_{i=1}^n z_{ij}} P_j^{\gamma_j} e^{-a_j \sum_{i=1}^n z_{ij} x_i^{\delta_j}}, \tag{4.6}$$

where $P_j = \prod_{i=1}^{n} x_i^{z_{ij}}$, for all $j = 1, 2, \ldots, k$.

4.1 z_i Posterior

When the indicator parameter z_i is unknown, for all observation $x_i, i = 1, 2, \ldots, n$ and the scale parameter a, the shape parameters γ, δ and the weight parameter λ are known.

For all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, the conjugate prior $\pi(z_{ij})$ of z_{ij} is Bernoulli(λ_j) with $0 < \lambda_j < 1$ and $\sum_{j=1}^{k} \lambda_j = 1$. Thus for all $i = 1, 2, \ldots, n$, the conjugate prior $\pi(z_i)$ of z_i is multinomial with hyperparameters $(1, \lambda_1, \lambda_2, \ldots, \lambda_k)$.

By using Bayeś theorem, the conjugate posterior distribution $\pi(z_{ij} \mid x, a, \gamma, \delta, \lambda)$ of z_{ij} is Bernoulli(w_{ij}), where $w_{ij} = \dfrac{\lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x_i^{\gamma_j} e^{-a_j x_i^{\delta_j}}}{\sum_{j=1}^{k} \lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x_i^{\gamma_j} e^{-a_j x_i^{\delta_j}}}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ and it is given by:

$$\pi(z_{ij} = 1 \mid x_i, a, \gamma, \delta, \lambda) = \frac{\lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x_i^{\gamma_j} e^{-a_j x_i^{\delta_j}}}{\sum_{j=1}^{k} \lambda_j \frac{\delta_j a_j^{\frac{\gamma_j+1}{\delta_j}}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)} x_i^{\gamma_j} e^{-a_j x_i^{\delta_j}}} = w_{ij}. \tag{4.7}$$

Since each z_{ij} takes two values only 1 or 0, then

$$\pi(z_{ij} = 0 \mid x, a, \gamma, \delta, \lambda) = 1 - w_{ij}. \tag{4.8}$$

Therefore, $\pi(z_i \mid x, a, \gamma, \delta, \lambda)$ of z_i has a multinomial distribution, namely multinomial $(1, w_{i1}, w_{i2}, \ldots, w_{ik})$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.

4.2 λ Posterior

When the weight parameter λ is unknown and the scale parameter a and the shape parameters γ, δ are known. By ignoring terms that contain a, γ, δ in (4.6), the complete data likelihood function is given by:

$$L(x, z, a, \gamma, \delta \mid \lambda) \propto \prod_{j=1}^{k} \lambda_j^{\sum_{i=1}^{n} z_{ij}}$$
$$\propto \prod_{j=1}^{k} \lambda_j^{N_j}, \tag{4.9}$$

where N_j is the number of the observations, which belong to the j^{th} component, for $j = 1, 2, \ldots, k$ and we compute N_j as the following

$$N_j = \sum_{i=1}^{n} w_{ij}. \tag{4.10}$$

The conjugate prior $\pi(\lambda)$ is a Dirichlet distribution with hyperparameters $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$

$$\pi(\lambda) = \frac{\Gamma\left(\sum_{j=1}^{k} \mu_j\right)}{\prod_{j=1}^{k} \Gamma(\mu_j)} \prod_{j=1}^{k} \lambda_j^{\mu_j - 1}, \tag{4.11}$$

where $0 < \lambda_j < 1, \mu_j > 0$, for all $j = 1, 2, \ldots, k$ and $\sum_{j=1}^{k} \lambda_j = 1$.

By ignoring terms that contain a, γ, δ, the posterior distribution $\pi(\lambda \mid a, \gamma, \delta, x, z)$ is a Dirichlet with hyperparameters $(\mu_1 + \sum_{i=1}^{n} w_{i1}, \mu_2 + \sum_{i=1}^{n} w_{i2}, \ldots, \mu_k + \sum_{i=1}^{n} w_{ik})$ and it is given by:

$$\pi(\lambda \mid a, \gamma, \delta, x, z) = \prod_{j=1}^{k} \lambda_j^{\sum_{i=1}^{n} w_{ij} + \mu_j - 1}. \tag{4.12}$$

4.3 a_j Posterior

When the scale parameter a_j is unknown, for some $j = 1, 2, \ldots, k$ and both the weight parameter λ and the shape parameters γ, δ are known. By ignoring terms that contain $\gamma, \delta, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k$ in (4.6), the complete data likelihood function is given by:

$$L(x, z, \gamma, \delta, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \mid a_j) \propto a_j^{\left(\frac{\gamma_j + 1}{\delta_j}\right) \sum_{i=1}^{n} z_{ij}} e^{-a_j \sum_{i=1}^{n} z_{ij} x_i^{\delta_j}}. \tag{4.13}$$

The conjugate prior $\pi(a_j)$ is the gamma distribution with hyperparameters $r_j > 0$ and $k_j > 0$

$$\pi(a_j) = \frac{1}{k_j^{r_j} \Gamma(r_j)} a_j^{r_j - 1} e^{-\frac{a_j}{k_j}}. \tag{4.14}$$

By ignoring terms that contain $\gamma, \delta, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k$, the posterior distribution $\pi\left(a_j \mid \gamma, \delta, x, z, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k\right)$ is the gamma distribution with hyperparameters $\acute{r}_j = \left(\frac{\gamma_j+1}{\delta_j}\right) \sum_{i=1}^{n} z_{ij} + r_j$ and $\acute{k}_j = \frac{k_j}{1+k_j \sum_{i=1}^{n} z_{ij} x_i^{\delta_j}}$

$$\pi\left(a_j \mid \gamma, \delta, x, z, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k\right) \propto a_j^{\acute{r}_j - 1} e^{\frac{-a_j}{\acute{k}_j}}. \tag{4.15}$$

4.4 γ_j Posterior

When the shape parameter γ_j is unknown, for some $j = 1, 2, \ldots, k$ and the scale parameter a, the shape parameter δ and the weight parameter λ are known. By ignoring terms that contain $a, \delta, \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k$ in (4.6), the complete data likelihood function is given by:

$$L\left(x, z, a, \delta, \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k \mid \gamma_j\right) \propto \frac{a_j^{\left(\frac{\gamma_j+1}{\delta_j}\right) \sum_{i=1}^{n} z_{ij}}}{\Gamma^{\sum_{i=1}^{n} z_{ij}} \left(\frac{\gamma_j+1}{\delta_j}\right)} P_j^{\gamma_j}, \tag{4.16}$$

where $P_j = \prod_{i=1}^{n} x_i^{z_{ij}}$.

The prior $\pi\left(\gamma_j\right)$ with hyperparameters $s_j > 0, b_j > 0$ and $t_j > 0$ is

$$\pi\left(\gamma_j\right) \propto \frac{a_j^{s_j \left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma^{b_j} \left(\frac{\gamma_j+1}{\delta_j}\right)} t_j^{\gamma_j}. \tag{4.17}$$

By ignoring terms that contain the parameters $a, \delta, \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k$, the posterior distribution of $\pi\left(\gamma_j \mid a, \delta, x, z, \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k\right)$ with hyperparameters $\acute{s}_j = s_j + \sum_{i=1}^{n} z_{ij}$, $\acute{b}_j = b_j + \sum_{i=1}^{n} z_{ij}$ and $\acute{t}_j = t_j P_j$ is

$$\pi\left(\gamma_j \mid a, \delta, x, z, \gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_k\right) \propto \frac{a_j^{\acute{s}_j \left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma^{\acute{b}_j} \left(\frac{\gamma_j+1}{\delta_j}\right)} (\acute{t}_j)^{\gamma_j}, \tag{4.18}$$

where $P_j = \prod_{i=1}^{n} x_i^{z_{ij}}$.

4.5 δ_j Posterior

When the shape parameter δ_j is unknown, for some $j = 1, 2, \ldots, k$ and the scale parameter a, the shape parameter γ and the weight parameter λ are known. By ignoring terms that contain $a, \gamma, \delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_k$ in (4.6), the complete data likelihood function is given by:

$$L\left(x, za, \gamma, \delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_k \mid \delta_j\right) = \frac{\delta_j^{\sum_{i=1}^n z_{ij}} a_j^{\left(\frac{\gamma_j+1}{\delta_j}\right)\sum_{i=1}^n z_{ij}}}{\Gamma^{\sum_{i=1}^n z_{ij}}\left(\frac{\gamma_j+1}{\delta_j}\right)} e^{-a_j \sum_{i=1}^n z_{ij} x_i^{\delta_j}}. \tag{4.19}$$

The conjugate prior $\pi\left(\delta_j\right)$ with hyperparameters $m_j > 1, s_j > 0, b_j > 0$ and $c_j > 0$ is

$$\pi\left(\delta_j\right) \propto \frac{\delta_j^{c_j} a_j^{s_j\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma^{b_j}\left(\frac{\gamma_j+1}{\delta_j}\right)} e^{-a_j m_j^{\delta_j}}. \tag{4.20}$$

By ignoring terms that contain $a, \gamma, \delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_k$, the posterior distribution $\pi\left(\delta_j \mid a, \gamma, x, z, \delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_k\right)$ with hyperparameters $\acute{m}_j = \sum_{i=1}^n z_{ij} x_i^{\delta_j} + m_j^{\delta_j}, \acute{s}_j = s_j + \sum_{i=1}^n z_{ij}, \acute{b}_j = b_j + \sum_{i=1}^n z_{ij}$ and $\acute{c}_j = c_j + \sum_{i=1}^n z_{ij}$ is

$$\pi\left(\delta_j \mid a, \gamma, x, z, \delta_1, \ldots, \delta_{j-1}, \delta_{j+1}, \ldots, \delta_k\right) \propto \frac{\delta_j^{\acute{c}_j} a_j^{\acute{s}_j\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma^{\acute{b}_j}\left(\frac{\gamma_j+1}{\delta_j}\right)} e^{-a_j \acute{m}}. \tag{4.21}$$

4.6 Joint Posterior of a, γ, δ

When the weight parameter λ is known and the shape parameters γ, δ and the scale parameter a are unknown. By ignoring terms that contain λ in (4.6), the complete data likelihood function is given by:

$$L\left(x, z, \lambda \mid a, \gamma, \delta\right) = \prod_{j=1}^k \left[\frac{\delta_j a_j^{\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)}\right]^{\sum_{i=1}^n z_{ij}} P_j^{\gamma_j} e^{-a_j \sum_{i=1}^n z_{ij} x_i^{\delta_j}}, \tag{4.22}$$

where $P_j = \prod_{i=1}^n x_i^{z_{ij}}$, for $j = 1, 2, \ldots, k$.

The joint conjugate prior $\pi\left(a, \gamma, \delta, \lambda\right)$ with hyperparameters $m_j > 1, s_j > 0, b_j > 0, c_j > 0$ and $t_j > 0$ is

$$\pi\left(a, \gamma, \delta\right) \propto \prod_{j=1}^k \frac{\delta_j^{c_j} a_j^{s_j\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\left[\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)\right]^{b_j}} t_j^{\gamma_j} e^{-a_j m_j^{\delta_j}}. \tag{4.23}$$

By ignoring terms that contain λ, the joint posterior distribution $\pi\left(a, \gamma, \delta, \lambda \mid x, z\right)$ with hyperparameters $\acute{m}_j = \sum_{i=1}^n z_{ij} x_i^{\delta_j} + m_j^{\delta_j}, \acute{s}_j = s_j + \sum_{i=1}^n z_{ij}, \acute{b}_j = b_j + \sum_{i=1}^n z_{ij}, \acute{c}_j =$

$c_j + \sum_{i=1}^{n} z_{ij}$ and $\acute{t_j} = t_j P_j$ is given by:

$$\pi\left(a, \gamma, \delta \mid x, z, \lambda\right) \propto \prod_{j=1}^{k} \frac{\delta_j^{\acute{c_j}} a_j^{\acute{s_j}\left(\frac{\gamma_j+1}{\delta_j}\right)}}{\left[\Gamma\left(\frac{\gamma_j+1}{\delta_j}\right)\right]^{\acute{b_j}}} (\acute{t_j})^{\gamma_j} e^{-a_j \acute{m_j}}, \tag{4.24}$$

where $P_j = \prod_{i=1}^{n} x_i^{z_{ij}}$, for $j = 1, 2, \ldots, k$.

5 Simulation Results

In this section, a simulation study using Monte Carlo methods is presented to compare the efficiency of MLE method with Bayesian method of estimation using by computing the mean of the sum of the modulus of the bias (MBias), and the root-mean square error (RMSE), where the smaller RMSE and MBias indicates a better overall quality of the estimates. To take care of small, medium and large data sets, we investigate the performance of the proposed prior distributions through a simulation study with samples of size $100, 150, 200, 250$ and 300 generated from the generalized gamma distribution with parameters $(a = 6, \gamma + 1 = 3, \delta = 5)$. To find the MLE estimators, the Newton–Raphson method was adopted. The parameters $(a, \gamma+1, \delta)$ are estimated with random walk Metropolis method (RWM) of estimation using the joint prior in (3.20) with hyperparameters $(c = 2, m = 4.5, t = 6, s = 4$ and $b = 3$, where the simulation study was carried out $10,000$ times for $(a, \gamma + 1, \delta)$. The working of the random walk Metropolis method is greatly affected by the standard deviations of the distribution that gives the proposals. A large standard deviation allows for large values for the proposal, and then the ratio is often small, which results in small acceptance probabilities and frequent rejection of the proposal. Table 1 present the estimates (Est.) and the RMSE and MBias values by MLE and RWM method. The smaller RMSE and MBias for each sample size is highlighted in bold. Looking at these tables we observe that: In general when sample size increases, the relative MBias and the RMSE inecrease for random walk Metropolis method, and decrease for MLE method. Moreover, we obtained that the random walk Metropolis method is uniformly better than MLE.

On the other hand, We consider the GG distribution to analyze a number of real lifetime data sets. The parameters $a, \gamma + 1$ and δ are estimated with random walk Metropolis method (RWM) of estimation for all fifteen real lifetime data sets $1 - 4$. For each case, $10,000$ samples were simulated using the joint prior in (3.20) with hyperparameters $(c = 3, m = 5, t = 4, s = 5$ and $b = 2)$ to be used to get the posterior summaries. Table 3 display the means, medians, standard deviations and $CI_{95\%}$ of the RWM estimators.

Data set 1: This data set related to the cycles to failure for a batch of 60 electrical appliances in a life test introduced by Lawless [13]. The data is as follows:
0.014, 0.034, 0.059, 0.061, 0.069, 0.080, 0.123, 0.142, 0.165, 0.210, 0.381, 0.464, 0.479,

sample size	method	$\hat{a}$	$\hat{\gamma}+1$	$\hat{\delta}$	RMSE	MBias
100	MLE	7.66573	3.21369	5.48645	6.99739	4.25769
	RWM	5.90041	2.90081	4.89663	**0.174503**	**0.302181**
150	MLE	6.76174	3.06352	5.398	3.87066	2.85274
	RWM	5.89964	2.89991	4.89754	**0.174909**	**0.302934**
200	MLE	6.53246	3.07509	5.26657	2.03767	2.22731
	RWM	5.89918	2.90106	4.8965	**0.175139**	**0.303292**
250	MLE	6.53246	3.07509	5.26657	2.03767	2.22731
	RWM	5.89918	2.90106	4.8965	**0.175139**	**0.303292**
300	MLE	6.27541	3.0097	5.22432	1.53954	1.67023
	RWM	5.89853	2.90014	4.89633	**0.176133**	**0.30503**

Table 1: MBias and RMSE of the MLE estimates and the RWM estimators for Generalized Gamma distribution

sample size	$CI_{95\%}$ of $\hat{a}$	$CI_{95\%}$ of $\hat{\gamma}+1$	$CI_{95\%}$ of $\hat{\delta}$
100	$(5.9004, 5.90043)$	$(2.9008, 2.90081)$	$(4.89658, 4.89668)$
150	$(5.89963, 5.89965)$	$(2.89991, 2.89992)$	$(4.89751, 4.89758)$
200	$(5.89917, 5.8992)$	$(2.90105, 2.90107)$	$(4.89646, 4.89653)$
250	$(5.90049, 5.9005)$	$(2.89981, 2.89982)$	$(4.89561, 4.89569)$
300	$(5.89851, 5.89854)$	$(2.90014, 2.90015)$	$(5.89851, 5.89854)$

Table 2: 95% credibility intervals of the RWM estimators for sample of sizes $n = 100, 150, 200, 250$ and 300

0.556, 0.574, 0.839, 0.917, 0.969, 0.991, 1.064, 1.088, 1.091, 1.174, 1.270, 1.275, 1.355, 1.397, 1.477, 1.578, 1.649, 1.702, 1.893, 1.932, 2.001, 2.161, 2.292, 2.326, 2.337, 2.628, 2.785, 2.811, 2.886, 2.993, 3.122, 3.248, 3.715, 3.790, 3.857, 3.912, 4.100, 4.106, 4.116, 4.315, 4.510, 4.580, 4.580, 4.580, 4.580, 4.580, 4.580.

Data set 2: This data set represents the life of fatigue fracture of Kevlar 373 epoxy subjected to constant pressure at 90% stress level until all had failed. The data was extracted from [1] and it has previously been used by Barlow et al. [5]. The data is as follows:
0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Data set 3: This data set represents remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [14]. The data is as follows:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 6.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26,5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Data set 4: This data set that represents the remission times (in months) of a random sample of 128 bladder cancer patients (see [14]). The data is as follows:
0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.60, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Data set	Parameter	Mean	Median	SD	$CI_{95\%}$
data 1	a	0.000120369	0.000118997	0.0000416538	$(0.000119553, 0.000121186)$
	$\gamma+1$	0.720643	0.712901	0.0958507	$(0.718764, 0.722522)$
	δ	5.69288	5.69293	0.00095577	$(5.69286, 5.6929)$
data 2	a	0.836035	0.822737	0.143874	$(0.833215, 0.838855)$
	$\gamma+1$	1.71862	1.71649	0.239448	$(1.71392, 1.72331)$
	δ	1.05568	1.05706	0.0197382	$(1.05529, 1.05607)$
data 3	a	1.3785	1.37685	0.00570541	$(1.37838, 1.37861)$
	$\gamma+1$	1.99841	1.99909	0.131522	$(1.99583, 2.00099)$
	δ	0.514015	0.51405	0.0172657	$(0.513676, 0.514353)$
data 4	a	1.36728	1.36297	0.0354017	$(1.36658, 1.36797)$
	$\gamma+1$	1.97137	1.98427	0.169011	$(1.96806, 1.97468)$
	δ	0.510272	0.510736	0.017294	$(0.509933, 0.510611)$

Table 3: Posterior means, medians, standard deviations and 95% credibility intervals for data sets $1-4$

Also, we choose samples of size $50, 100, 150, 200$ and 250 to generate the data set of two-component mixture generalized gamma distribution with parameters $(a_1 = 3, a_2 = 4, \gamma_1+1 = 2, \gamma_2+1 = 6, \delta_1 = 5, \delta_2 = 2, K = 2, \lambda = 0.5)$. The parameters $\lambda, a_1, a_2, \gamma_1+1, \gamma_2+1, \delta_1$ and δ_2 are estimated with MLE and random walk Metropolis method (RWM). Then we compare the performance of MLE and Bayes estimators. For each simulated sample, 1000 iterations were performed using random walk Metropolis method. Bayes estimator is obtained by using the joint prior (4.23) and (4.11) with hyperparameters $(c_1 = 3, m_1 = 5, t_1 = 4, s_1 = 5, b_1 = 2, \mu_1 = 1, c_2 = 2, m_2 = 4.5, t_2 =$

n	method	$\hat{a_1}$	$\hat{\gamma_1}+1$	$\hat{\delta_1}$	$\hat{a_2}$	$\hat{\gamma_2}+1$	$\hat{\delta_2}$	$\hat{\lambda}$	RMSE	MBias
50	MLE	4.8890	2.1031	6.0730	3.9056	6.0041	2.0970	0.49350	6.7669	4.723
	RWM	3.0096	2.0099	5.0095	4.0100	6.0098	2.0099	0.46245	**0.1425**	**0.2192**
100	MLE	3.5145	2.0113	6.2940	4.047	6.1709	1.9753	0.52780	1.7387	3.7980
	RWM	3.009	2.0100	5.0095	4.010	6.011	2.0099	0.6589	**0.2445**	**0.3793**
150	MLE	3.1758	1.9553	6.2862	4.388	6.3253	1.9514	0.51411	3.6565	2.9248
	RWM	3.0098	2.0099	5.009	4.009	6.050	2.0098	0.43227	**0.1383**	**0.2485**
200	MLE	3.2313	1.9910	6.2156	3.6733	5.7243	2.1217	0.49158	3.5031	3.168
	RWM	3.0098	2.0099	5.0095	4.0097	6.0105	2.0100	0.51472	**0.0821**	**0.1490**
250	MLE	3.2204	2.0545	5.5675	3.9007	5.8255	1.9888	0.48829	2.450	2.4372
	RWM	3.0101	2.0099	5.0097	4.0097	6.0096	2.0099	0.52958	**0.07884**	**0.1451**

Table 4: MBias and RMSE of the MLE estimates and the RWM estimators for two-component mixture Generalized Gamma distribution

$6, s_2 = 4, b_2 = 3, \mu_2 = 1)$ and the proposal is generated from six-variate normal distribution that has independent components with means of zero and known standard deviations. To avoid the local maximum, we restart the EM algorithm 20 times and choose the result with the highest log-likelihood. Table 4 present the estimates (Est.) and the RMSE and MBias values by MLE and RWM method. The smaller RMSE and MBias for each sample size is highlighted in bold. It is observed that Bayes estimator fairs better than MLE in all cases.

6 Conclusions

In this paper, we focus on family of conjugate priors for the Generalized Gamma Distribution. The conjugate priors are presented for different cases of known parameters and also, for mixtures of distributions. It takes more efforts to choose a suitable conjugate priors for these parameters. From the simulation results, it is observed that Bayesian method of estimation is better than classical method estimation.

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