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Exact distribution of time between failures under gamma lifetime model

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Abstract:

This paper is concerned with the distribution of the time between failures (TBF) under the gamma lifetime model. The exact distribution of TBF is computed in closed form. This allows interested researchers to compute the reliability of certain system models. To better visualize the distributions of TBF, its probability distribution function (pdf) is represented using matrices. Moreover, the moments of TBF is computed and given in closed form. Wolfram Mathematica cods are given in the appendix for fast and easy implementation. Finally, simulation studies for the TBF are performed together with relative tests results.

Keywords:

Gamma distribution; order statistics; time between failures; reliability.

1 introduction

Suppose that a system has n identical and independent components that are subject to failure according to a Poisson process with a fixed rate λ per unit time. Assume that the system fails if and only if k of its components fail, where $k \leq n$. Then, the time to failure (TTF), noted as T, of this system has a gamma distribution with shape parameter k and scale parameter λt. To see this, notice that the reliability function $R_T(t)$ of T is the probability that the system functions beyond time t. In other words, $R_T(t) = P(T > t)$; i.e., the probability that less than k components fail by time t.

Hence,

$$R_T(t) = \sum_{j=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} = 1 - \sum_{j=k}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}.$$

Therefore, the cumulative distribution function (cdf) of T is

$$F_T(t) = 1 - R_T(t) = \sum_{j=k}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}.$$

This implies that T has a gamma distribution with a positive integer shape parameter k and scale parameter λt (see Page 100 of [2]). The probability density function (pdf) of T is

$$f_T(t) = \frac{\lambda^k t^{k-1}}{\Gamma(k)} e^{-\lambda t}, \quad t > 0.$$

An important measure in reliability engineering is the mean time between failures (MTBF) of the system, which is the average time from repair to failure ([3]). It enables us study the reliability of $k-\mathrm{out}-\mathrm{of}-n$ systems ([5]). In general, past records of TBF are used to study the behaviour of the system in the future. It is known that the distributional forms of the failure data include the exponential, gamma, Weibull, and lognormal models ([6]). In reliability engineering, researchers are interested in estimating the distribution of TBF. Most of them use the maximum likelihood (ml) method to estimate the TBF distributions, others use the method-of-moments (mm) technique (see, for example, [8]). In this paper, an exact distribution of of the TBF distribution under the gamma model is derived.

In the following, the exact distribution of the TBF is derived when the failure time of the components of the system follows the gamma distribution. In fact, the distribution of the TBF turns out to be a mixture of gamma distributions. The TBF is the gap (or spacing) between successive order statistics associated with the time of failures of the system.

Suppose that m such systems were put on test and let $T_1, T_2, \ldots, T_m$ be the corresponding lifetimes (the time of failure) of the systems (or simply, let $T_1, T_2, \ldots, T_m$ be some recorded m failure times of the system). Arrange the lifetimes in ascending order as $T_{1:m}, T_{2:m}, \ldots, T_{m:m}$. That is, $T_{1:m} < T_{2:m} < \cdots < T_{m:m}$. For a non-negative integer i satisfying $0 \leq i < m$, define the sequence of random variables $\mathrm{T}_i, \ldots, \mathrm{T}_{m-1}$ by $\mathrm{T}_i := T_{i+1:m} - T_{i:m}$. That is, T_i, the time between two consecutive failures, is the i^{th} spacing (or gap) between the $(i+1)^{\text{th}}$ and the i^{th} order statistics. By convention, we set $T_{0:m} = 0$ and, consequently, $\mathrm{T}_{0,m} = T_{1:m}$.

Some authors used a different settings of the system, where they used the system failure times to derived a closed-formula for the time between failures under the Weibull failure distribution ([9]). Since the authors used a set of naturally sorted failure times of the system in increasing order, they didn't have to use the notion of order statistics. They also used a Weibull failure rate instead of the fixed exponential failure rate that has been used in this paper.

A related work can also be found in [1], where the authors determined the distribution fitting to time between failure of a group of power stations.

In [4], the author used Bayesian techniques to estimate the reliability for certain Markov models under the Exponential, Gamma, and Weibull failure time distributions.

2 The pdf of T_i

The pdf of T_i is given in ([7]) as:

$$f_{\mathrm{T}_i}(r \mid k, \lambda) = \sum_{h=0}^{(k-1)(n-i)} \sum_{a_k=0}^{i-1} \sum_{a_{k-1}=0}^{i-1-a_k} \cdots \sum_{a_1=0}^{i-1-a_2-\cdots-a_k} A_{i,n}(\mathbf{a_k}) e^{-r(n-i)/\lambda} \cdot \sum_{b_{k-1}=0}^{n-i-1} \sum_{b_{k-2}=0}^{n-i-1-b_{k-1}} \cdots \sum_{b_1=0}^{n-i-1-b_2-\cdots-b_{k-1}} B_{i,n}(\mathbf{b_k}) r^{n_b+k-h-1}, \quad (2.1)$$

where

$$A_{i,n}(\mathbf{a_k}) = \frac{(-1)^{i-1-a_1} n!\Gamma(n_a+h+k)}{p_a \Gamma(k)^2 \Gamma(i-m_a)(n-a_1)^{n_a+h+k}} \quad \text{and} \quad (2.2)$$

$$B_{i,n}(\mathbf{b_k}) = \frac{(r/\lambda)^{n-i-2+k-b_1+n_b-h} \Gamma(n-i-1-b_1+n_b+k)\lambda^{-1}}{p_b h! \Gamma(n-i-m_b)\Gamma(n-i-1-b_1+n_b-h+k)}, \quad (2.3)$$

where $\mathbf{a_k} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{b_k} = (b_1, b_2, \ldots, b_{k-1})$, $k \geq 1$.

The symbols n_a, m_a, p_a, n_b, m_b, and p_b are given as

$$n_a = \sum_{k=1}^{\alpha}(k-1)a_k \; m_a = \sum_{k=1}^{\alpha} a_k \; p_a = \prod_{k=1}^{\alpha}(a_k)!((k-1)!)^{a_k}, \text{for } \alpha \geq 1; \quad (2.4)$$

$$n_b = \sum_{k=1}^{\alpha-1}(k-1)b_k \; m_b = \sum_{k=1}^{\alpha-1} b_k \; p_b = \prod_{k=1}^{\alpha-1}(b_k)!(k!)^{b_k}, \text{for } \alpha \geq 2 \quad (2.5)$$

and $n_b = m_b = 0$, $p_b = 1$, when $k = 1$.

Since the equation of the pdf of T_i is so tedious, I wrote a *Wolfram Mathematica* code (see appendix 6) for easily computing f_{T_t} for all values of k, i, and n. The time used by *Wolfram Mathematica* to compute (2.1) increases as both of k and $(n-i)$ increase. One way to overcome this problem is by using a suitable matrix representation. The representation of the pdf of T_i as a mixture of gamma distributions makes it easy to deal with as follows.

Instead of writing f_{T_t} as a mixture of distributions, we may use matrices to present it. That is, f_{T_t} is represented as the product of two matrices, a weight matrix and a component matrix. Let W denote the weights matrix. W can be obtained by integrating (2.1), term-by-term. In fact, W is the diagonal matrix with $\{w_0, w_1, \ldots, w_{(k-1)(n-i)}\}$ as its diagonal entries, where w_j is the wight of the j^{th} component of the mixture representing f_{T_t}. Then we sum the resulting integrals over the variables $a_1, a_2, \ldots, a_k$ satisfying the constraint $\sum_{j=1}^{k} a_j = i-1$; and over the variables $b_1, b_2, \ldots, b_{k-1}$ satisfying the constraint $\sum_{j=1}^{k-1} b_j = n-i-1$. Now, let M be diagonal matrix whose non-zero entries are the respective components of the mixture. That is,

1.5

$$M = \begin{pmatrix} \frac{(n-j)}{0!} e^{-(n-i)r\lambda}\lambda & 0 & \cdots & 0 \\ 0 & \frac{(n-j)^2}{1!} e^{-(n-i)r\lambda} r\lambda^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{(n-j)^{d+1}}{j!} e^{-(n-i)r\lambda} r^d \lambda^{d+1} \end{pmatrix},$$

where $d = (k-1)(n-i)$.

Then the pdf of T_i have the matrix representation

$$f_{T_i}(r) = \sum W \cdot M.$$

In other words, the pdf of T_i have the matrix representation

$$f_{T_i}(r) = \sum_{i,j} C, \qquad (2.6)$$

where C is the diagonal matrix with $\{w_0 b_0, w_1 b_1, \ldots, w_{(k-1)(n-i)} b_{(k-1)(n-i)}\}$ as its diagonal entries, and $b_j = \frac{(n-i)^{j+1} r^j \lambda^{j+1}}{j!} e^{-(n-i)r\lambda}$, $j = 0, 1, \ldots, (k-1)(n-i)$. That is,

$$f_{T_i}(r) = \sum_{j=0}^{(k-1)(n-i)} w_j \frac{(n-i)^{j+1}}{j!} r^j \lambda^{j+1} e^{-(n-i)r\lambda}. \tag{2.7}$$

Example 2.1 *If* $k = 3, i = 2$, *and* $n = 4$, *then*

1.5

$$A = \begin{pmatrix} w_0 & 0 & 0 & 0 & 0 \\ 0 & w_1 & 0 & 0 & 0 \\ 0 & 0 & w_2 & 0 & 0 \\ 0 & 0 & 0 & w_3 & 0 \\ 0 & 0 & 0 & 0 & w_4 \end{pmatrix} \quad \text{and}$$

$$M = \begin{pmatrix} 2e^{-2r\lambda}\lambda & 0 & 0 & 0 & 0 \\ 0 & 4e^{-2r\lambda} r\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 4e^{-2r\lambda} r^2\lambda^3 & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{3}e^{-2r\lambda} r^3\lambda^4 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3}e^{-2r\lambda} r^4\lambda^5 \end{pmatrix}.$$

Then, the pdf of T_2 is given as

$$f_{T_2}(r) = 2\lambda w_0 e^{-2\lambda r} + 4\lambda^2 r w_1 e^{-2\lambda r} + \cdots + \frac{4}{3}\lambda^5 r^4 w_4 e^{-2\lambda r}.$$

Theorem 2.1 *The* m^{th} *moment of* T_i *is*

$$E[T_i^m] = \sum_{j=0}^{(k-1)(n-i)} \frac{w_j (j+m)!}{j!\lambda^m (n-i)^m}$$

$$= \sum_{j=0}^{(k-1)(n-i)} \frac{\binom{j+m}{j} m! w_j}{\lambda^m (n-i)^m}, \quad m = 1, 2, \ldots.$$

Proof. The proof follows by multiplying the right-hand side of (2.7) by r^m and then integrating term-by-term with respect to r over $(0,\infty)$ and noting that

$$\int_0^\infty \frac{\lambda^{j+1} w_j (n-i)^{j+1} r^{j+m} e^{-(n-i)r\lambda}}{j!} dr = \frac{w_j (j+m)!}{j! \lambda^m (n-i)^m}.$$

3 The weights matrix

The weights associated with the pdf of T_i can be obtained by integrating the different terms of (2.1). One way to do express the weights is use a suitable matrix representation, where the weights can be obtained by summing over the columns of some matrix that might be called "the weights matrix." In Appendix 7, I wrote a mathematica code for easily computing the weights matrix.

Now, let

$$S_a(k,i,n) = \sum_{a_k=0}^{i-1} \sum_{a_{k-1}=0}^{-a_k+i-1} \cdots \sum_{a_1=0}^{i-1-a_2-a_3-\ldots-a_k} s_a, \quad \text{where}$$

$$s_a = \frac{n!(-1)^{-a_1+i-1} \Gamma\left(a_2 + 2a_3 + \cdots + (k-1)a_k + h + k\right)}{(n-a_1)^{a_2+2a_3+\cdots+(k-1)a_k+h+k} ((k-3)!)^{a_{k-2}} ((k-2)!)^{a_{k-1}} ((k-1)!)^{a_k}}$$
$$\times \frac{1}{a_1! a_2! \cdots a_k! \left(i-1-a_1-a_2-\cdots-a_k\right)!},$$

Similarly, let

$$S_b(k,i,n) = \sum_{b_{k-1}=0}^{-i+n-1} \sum_{b_{k-2}=0}^{-b_{k-1}-i+n-1} \cdots \sum_{b_1=0}^{n-i-1-b_2-b_3-\cdots-b_{k-1}} s_b, \quad \text{where}$$

$$s_b = \frac{\mathrm{Boole}\left(b_1 + h + i + 1 < n + b_2 + 2b_3 + \cdots + (k-2)b_{k-1} + k\right)}{\Gamma(k)^2 \left(((k-3)!)^{b_2} ((k-2)!)^{b_3} \cdots\right) ((k-1)!)^{b_{k-1}}}$$
$$\times \Gamma\left(n-i-1-b_1+b_2+2b_3+\cdots+(k-2)b_{k-1}+k\right)$$
$$\frac{(n-i)^{-k+1+h+i-n+b_1-b_2-2b_3-\cdots-(k-2)b_{k-1}}}{h!\left(b_1!\left(n-i-1-b_1-b_2-\cdots-b_{k-1}\right)! b_2! \cdots b_{k-1}!\right)}$$
$$\times \mathrm{Boole}\left(b_1 + h + i + 1 < n + b_2 + 2b_3 + \cdots + (k-2)b_{k-1} + k\right),$$

where

$$\mathrm{Boole}\left(b_1 + h + i + 1 < n + b_2 + 2b_3 + \cdots + (k-2)b_{k-1} + k\right) =$$
$$\begin{cases} 1, & if \;\; b_1 + h + i + 1 < n + b_2 + 2b_3 + \cdots + (k-2)b_{k-1} + k \\ 0, & otherwise. \end{cases}$$

and, similarly,

$$\text{Boole}\left(n-i-2+k-b_1+b_2+2b_3+\cdots+(k-2)b_{k-1}-h=p\right)=$$
$$\begin{cases}1, & if \;\; n-i-2+k-b_1+b_2+2b_3+\cdots+(k-2)b_{k-1}-h=p\\ 0, & otherwise.\end{cases}$$

The weights matrix of the pdf of T_i is obtain by computing the product of S_a and S_b when h and p run from 0 to $(k-1)(n-i)$.

$$W(k,i,n)=\left[S_a S_b \mid \{h,0,(k-1)(n-i)\},\{p,0,(k-1)(n-i)\}\right].$$

As a result, the pdf of T_i can be expressed as the product of the weights matrix and the components matrix.

Example 3.1 *Suppose that* $k=2$, $i=2$, *and* $n=5$. *Then the weights matrix corresponding to* T_2, *the time between the first failure and the second failure is 1.5*

$$W=\begin{pmatrix} 0 & \frac{13}{300} & \frac{13}{225} & \frac{13}{450} \\ \frac{113}{1000} & \frac{113}{750} & \frac{113}{1500} & 0 \\ \frac{2463}{10000} & \frac{2463}{20000} & 0 & 0 \\ \frac{3231}{20000} & 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

The weights are the sums of the columns of (3.1):

$$\{10417/20000, 6343/20000, 599/4500, 13/450\}.$$

The components matrix is

1.5

$$M=\begin{pmatrix} 3e^{-3r\lambda}\lambda & 0 & 0 & 0 \\ 0 & 9e^{-3r\lambda}r\lambda^2 & 0 & 0 \\ 0 & 0 & \frac{27}{2}e^{-3r\lambda}r^2\lambda^3 & 0 \\ 0 & 0 & 0 & \frac{27}{2}e^{-3r\lambda}r^3\lambda^4 \end{pmatrix} \tag{3.2}$$

Hence, the matrix representing $f_{T_2}(r)$ is the product of W and M, namely 1.5

$$WM = \begin{pmatrix} 0 & \frac{39}{100}e^{-3r\lambda}r\lambda^2 & \frac{39}{50}e^{-3r\lambda}r^2\lambda^3 & \frac{39}{100}e^{-3r\lambda}r^3\lambda^4 \\ \frac{339e^{-3r\lambda}\lambda}{1000} & \frac{339}{250}e^{-3r\lambda}r\lambda^2 & \frac{1017e^{-3r\lambda}r^2\lambda^3}{1000} & 0 \\ \frac{7389e^{-3r\lambda}\lambda}{10000} & \frac{22167e^{-3r\lambda}r\lambda^2}{20000} & 0 & 0 \\ \frac{9693e^{-3r\lambda}\lambda}{20000} & 0 & 0 & 0 \end{pmatrix} \quad (3.3)$$

The pdf f_{T_2} is obtained by summing the entries of the matrix in (3.3):

$$f_{T_2}(r) = \frac{39}{100}\lambda^4 r^3 e^{-3\lambda r} + \frac{1797\lambda^3 r^2 e^{-3\lambda r}}{1000} + \frac{57087\lambda^2 r e^{-3\lambda r}}{20000} + \frac{31251\lambda e^{-3\lambda r}}{20000}.$$

Finally, the mean and variance of T_2, respectively are

$$E(T_2) = 0.55668/\lambda$$

$$V(T_2) = 0.258965/\lambda^2$$

So, when $\lambda = 2$, we get

$$E(T_2) = 0.27834 \quad (3.4)$$

$$V(T_2) = 0.0647411 \quad (3.5)$$

4 Simulation

We simulate the experiment of computing the a data of size $m = 10000$ from the distribution of T_2, which is the time between the second and third failures. That is, we have in our simulation example $n = 5, k = 2, i = 2$.

The data is plotted in the following figure.

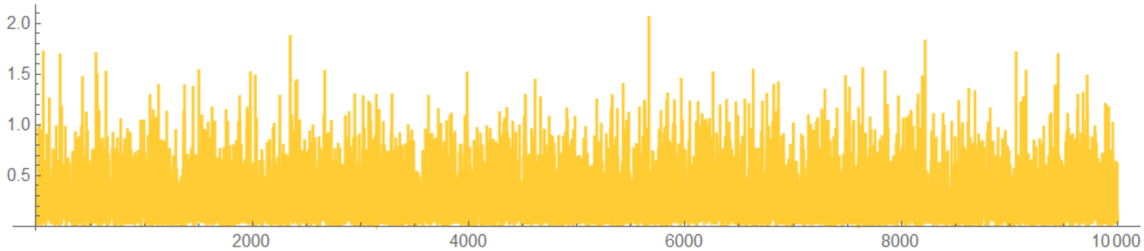

Figure 1: The data representing a random sample of size 1000 of T_2

The mean and the variance of the data are 0.278969 and 0.0652316, respectively. Compare with (3.4).

The estimated distribution has a pdf and cdf as in figure 2 below

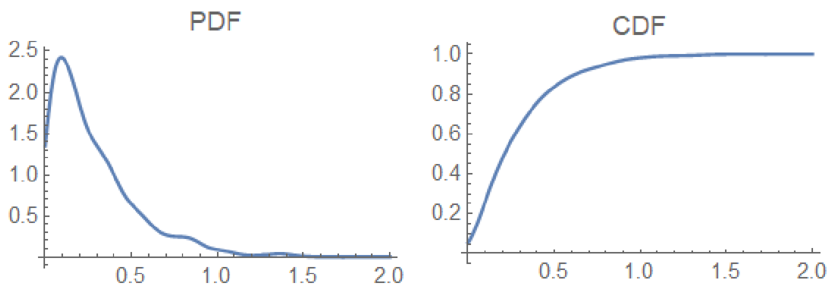

Figure 2: The pdf and cdf of T_2

In the following figure, the blue curve represents the pdf of T_2 and the brown curve represents the pdf of the estimated distribution of the data.

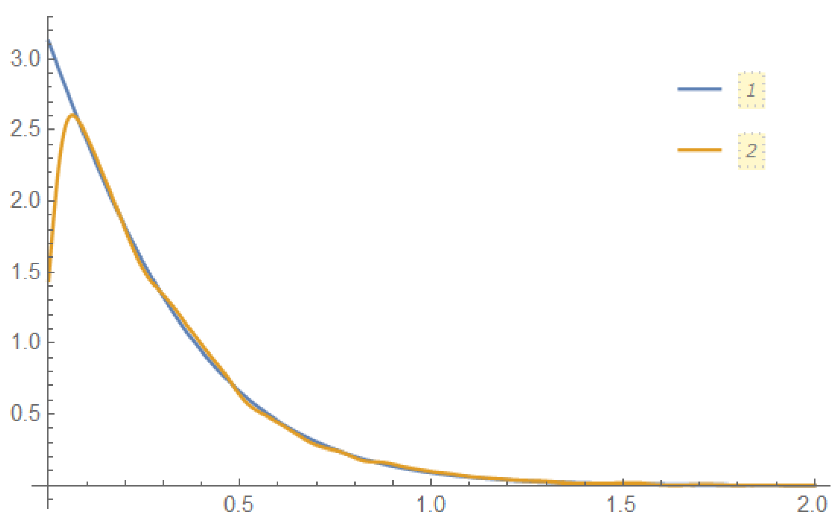

Figure 3: The pdf of T_2 (blue) and the pdf of the estimated distribution of the data (brown)

The Kolmogorov-Smirnov test performs the goodness-of-fit test with null hypothesis H_0 that data was drawn from a population with the distribution of T_2 and alternative hypothesis H_a that it was not.

The p-value of the Kolmogorov-Smirnov test is 0.882457. This implies that the null hypothesis can't be rejected.

5 Conclusion

This paper is concluded with the results obtained about the exact distribution of time between failures under the gamma failure distribution. The distribution of the TBF between successive failures of certain $k-\text{out}-\text{of}-n$ systems has a gamma mixture distribution, where both the pdf and weights of the mixture can be computed using the codes 7 and 6 in the appendix. Matrix representations are provided for both the pdf and the weights of the gamma mixture distribution corresponding to the distribution of TBF.

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Appendix

6 PDF code

```
pdf[k_,i_,n_,lambda_]:=Module[{vara,varb,na,nb,ma,mb,pa,pb, alist,blist,h,x,table,A,B,pDF}, vara[l_,m_]:=Table["a" <> ToString[L],{L,l,m}]//ToExpression; varb[l_,m_]:=Table["b" <> ToString[L],{L,l,m-1}]//ToExpression; x=Union[vara[1,k],varb[1,k]]; alist=Table[i-1-Total[vara[j,k]],{j,2,k+1}]; blist=Table[n-i-1-Total[varb[j,k]],{j,2,k}]; atable=Reverse[Transpose[{vara[1,k],ConstantArray[0,k],alist}]]; btable=Reverse[Transpose[{varb[1,k], ConstantArray[0,k-1],blist}]]; table=Join[{{h,0,(k-1)(n-i)}},Reverse[Transpose[{varb[1,k], ConstantArray[0,k-1],blist}]], Reverse[Transpose[{vara[1,k],ConstantArray[0,k],alist}]]]; na=Sum[(j-1)vara[1,k][[j]],{j,2,k}]; nb=Sum[(j-1)varb[1,k][[j]],{j,2,k-1}]; ma=Sum[vara[1,k][[j]],{j,1,k}];mb=Sum[varb[1,k][[j]],{j,1,k-1}]; pa=Product[(vara[1,k][[j]])!,{j,1,k}]* Product[((j-1)!)^vara[1,k][[j]],{j,3,k}]; pb=Product[(varb[1,k][[j]])!,{j,1,k-1}]* Product[((j-1)!)^varb[1,k][[j-1]],{j,3,k}]; A=(n! (-1)^(i-1-x[[1]]) Gamma[na+h+k])/((i-1-ma)!* (Gamma[k])^2 (n-x[[1]])^(na+h+k) (pa)); B=(lambda((lambda r)^(n-i-2+k-x[[k+1]]+nb-h))* Gamma[n-i-1-x[[k+1]]+nb+k])/(((n-i-1-mb)!)h!* Gamma[n-i-1-x[[k+1]]+nb-h+k](pb)); pDF=Sum[E^(-r(n-i)lambda)*A*B,##]&@@table; Return[pDF]//Expand ]
```

7 Weights code

```
w[k_,i_,n_]:=Module[{vara,varb,na,nb,ma,mb,pa,pb,x,alist,blist,
atable,h,btable,int,A,B,kernel,weights}, vara[l_,m_]:=Table["a" <> ToString[L],{L,l,m}]//ToExpression; varb[l_,m_]:=Table["b" <> ToString[L],{L,l,m-1}]//ToExpression; x=Union[vara[1,k],varb[1,k]]; alist=Table[i-1-Total[vara[j,k]],{j,2,k+1}]; blist=Table[n-i-1-Total[varb[j,k]],{j,2,k}]; atable=Reverse[Transpose[{vara[1,k],ConstantArray[0,k],alist}]]; btable=Reverse[ Transpose[{varb[1,k],ConstantArray[0,k-1],blist}]]; na=Sum[(j-1)vara[1,k][[j]],{j,2,k}]; nb=Sum[(j-1)varb[1,k][[j]],{j,2,k-1}]; ma=Sum[vara[1,k][[j]],{j,1,k}]; mb=Sum[varb[1,k][[j]],{j,1,k-1}]; pa=Product[(vara[1,k][[j]])!,{j,1,k}]* Product[((j-1)!)^vara[1,k][[j]],{j,3,k}]; pb=Product[(varb[1,k][[j]])!,{j,1,k-1}]* Product[((j-1)!)^varb[1,k][[j-1]],{j,3,k}]; int=(n-i)^(1+h+i-n-nb-k+x[[1+k]]) Boole[1+h+i+x[[k+1]]<n+nb+k]; A=(n! (-1)^(i-1-x[[1]]) Gamma[na+h+k])/ ((i-1-ma)! (Gamma[k])^2 (n-x[[1]])^(na+h+k) (pa)); B=Gamma[n-i-1-x[[k+1]]+nb+k]/(((n-i-1-mb)!)h!(pb)); kernel=Sum[int*A*B,##]&@@atable; weights=Table[ Sum[kernel*Boole[n-i-2+k-x[[k+1]]+nb-h==p],##]&@@btable, {h,0,(k-1)(n-i)}]; Return[Transpose[Table[weights,{p,0,(k-1)(n-i)}]]] ]
```