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Higher Rank Transmuted Families of Distributions

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Abstract

In this paper, a generalized method for generating new families of distributions is proposed. The new proposed method generalizes some known methods by introducing higher rank transmutation maps to generate more flexible and tractable lifetime families of probability models. A number of the known methods turn out to be special cases of the proposed method in this paper. Some examples are given to demonstrate how the method works on the Gompertz, exponential, and Weibull probability distributions.

Keywords:
Exponentiated, transmuted, exponential, Gompertz, Weibull.

Introduction:

In statistics, it is always desired to extend classical distributions and generate new flexible distributions in order to adequately fit real lifetime date. The literature is rich of studies that aim at proposing and introducing methods to generate and extend new families of univariate continuous probability models (see, for example, [Lee et al., 2013]). Among those methods that have recently attracted statisticians and gained their attention are exponentiation and transmutation. The idea of exponentiated distributions was introduced in 1998 by [Gupta et al., 1998]. A random variable Z is said to have an exponentiated distribution if its cumulative distribution function (cdf) $G(x)$ and probability density function (pdf) $g(x)$ are given by

$$G(x) = [F(x)]^{\alpha},$$

$$g(x) = \alpha f(x)[F(x)]^{\alpha-1}, \quad \alpha > 0.$$

where $F(x)$ and $f(x)$ are the cdf and pdf of some baseline distribution X, respectively.

Transmuted distributions were introduced by [Shaw and Buckley, 2009] in 2009 as a method to extend known non-Gaussian distributions by adding extra parameters to their distribution functions. Transmuted distributions provide statisticians with tools to control the skewness and kurtosis of the distribution in order to fit their real data.

Given a baseline probability distribution with cdf $G(x)$ and pdf $g(x)$, a random variable Z is said to have a transmutation map of quadratic map if its cdf $F(x)$ and pdf $f(x)$ have the following simple forms

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x),$$

$$f(x) = g(x)[1+\lambda-2\lambda G(x)], \quad |\lambda| \leq 1.$$

Given a cdf G(x) of a continuous random variable, the exponentiated generalized class of distributions is defined in [Cordeiro et al., 2013] by

$$F(x) = \{1-[1-G(x)]^{\alpha}\}^{\beta},$$

where $\alpha > 0$ and $\beta > 0$ are two additional shape parameters.

In the last decade, many authors introduced ideas to generalize the transmutation methods. For example, [Nofal et al., 2017] introduced a new class of continuous distributions called the generalized transmuted-G family which extends the quadratic rank transmutation map. [Rahman et al., 2018] proposed a general transmuted family of distributions with the emphasis on the cubic transmuted family of distributions. [Granzotto et al., 2017] introduced a new family of transmuted distributions and called it the cubic rank transmutation map distribution. Some other authors also proposed methods by using cubic rank transmutation maps like [Cordeiro et al., 2013] and [Nofal et al., 2017].

Many authors applied some of the above methods to certain distributions including the Rayleigh distribution ([Merovci, 2013]), the Gompertz distribution ([Abdul-Moniem and Seham, 2015]), the Gompertz-Makeham distribution ([El-Bar, 2017]), the Weibull distribution ([Granzotto1 et al., 2018, khan et al., 2017]), etc.

In [Granzotto etal., 2017], the authors proposed a cubic ranking transmutation map as follows.

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)G^2(x) + (1-\lambda_2)G^3(x),$$

where $0 \leq \lambda_1 \leq 1$, $-1 \leq \lambda_2 \leq 1$, and $G(x)$ and $g(x)$ are, respectively, the cdf and pdf of the baseline distribution.

The corresponding pdf is

$$f(x) = g(x)\{\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)G^2(x)\},$$

where $0 \leq \lambda_1 \leq 1, -1 \leq \lambda_2 \leq 1$.

In this paper, we propose a new method of extending classical lifetime probability models by generalizing some methods used to produce transmuted families of distributions through higher rank transmutation maps. By using the proposed method, we can generate more flexible and tractable lifetime probability distributions that meet the complexity of the real data. We will refer to the new distributions generated by this method as a higher rank transmuted (HRT-G) distributions, where $G(x)$ is the cdf of a baseline distribution. The proposed method in this paper is specially good for transmuting distributions having cdf of the form $G(x) = 1 - h(x)$ for some function $h(x)$, a property that is true for many lifetime distributions.

In some situations, higher rank transmuted distributions are needed to capture the complexity of the real data to be analyzed and studied. They are also needed when the real data to be fitted comes from a mixture of distributions.

In general, the HRT-G family has cdf and pdf given, respectively, as

$$F_k(x) = 1 - (1 - G(x))^{\alpha_1}\{1 - \lambda_1 + \sum_{j=2}^{k-1} (\lambda_{j-1} - \lambda_j)(1 - G(x))^{\alpha_j} + \lambda_{k-1}(1 - G(x))^{\alpha_k}\}, \quad (1.1)$$

where $k \geq 2$ and the α_js and the λ_js are all real parameters obeying certain conditions that we will state below.

Another way to produce higher rank transmutation maps is as in the following equation

$$\tilde{F}_k(x) = G(x)^{\alpha_1}\{1 - \lambda_1 + \sum_{j=2}^{k-1} (\lambda_{j-1} - \lambda_j)G(x)^{\alpha_j} + \lambda_{k-1}G(x)^{\alpha_k}\}, \quad (1.2)$$

where $k \geq 2$ and the α_js and the λ_js are real parameters obeying certain conditions that we will state below.

The method presented by (1.2) generalizes a method for producing cubic rank transmutation maps introduced by [Aslam et al., 2018].

2 *Construction of the HRT-G Family*

From now on, we assume that the integer $k \geq 2$.

2.1 First Construction

Let $G(x)$ and $g(x)$ be the cdf and pdf of some baseline distribution, respectively. The construction of the HRT-G family of distributions is described in the following theorem when the parameters $\alpha_1, \dots, \alpha_k$ of (1.1) are positive integers.

Theorem 2.1 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then the cumulative probability function of the HRT-G family is given as*

$$F_k(x) = 1 - (1-G(x))^{n_1}\{1-\lambda_1 + \sum_{j=2}^{k-1}(\lambda_{j-1}-\lambda_j)(1-G(x))^{n_j} + \lambda_{k-1}(1-G(x))^{n_k}\}, \tag{2.1}$$

where $n_1 \geq 1$, $n_{j+1}, n_k \geq 0$, $0 \leq \lambda_1 \leq 1$, and $\lambda_j - 1 \leq \lambda_{j+1} \leq \lambda_j$ for $j = 1, \ldots, k-2$.

Proof. Let N be a random variable with possible values $m_1, \ldots, m_k$ and respective probabilities $p_1, \ldots, p_k$, where $m_j \geq 1$ and $0 \leq p_j \leq 1$ with $\sum_{j=1}^{k} p_j = 1$. Let $X_1, \ldots, X_N$ be a random sample from the distribution of X and let $Y = \min\{X_1, \ldots, X_N\}$. Then, the cdf of Y, $F_k(x)$, is given by

$$F_k(x) = P(Y \leq x) = 1 - P(Y > x)$$
$$= 1 - \sum_{j=1}^{k} P(N = m_j)P(Y > x | N = m_j)$$
$$= 1 - \sum_{j=1}^{k-1} p_j[1-G(x)]^{m_j} - \left(1 - \sum_{j=1}^{k-1} p_j\right)[1-G(x)]^{m_k}. \tag{2.2}$$

Now, let $p_1 = 1 - \lambda_1$ and $p_j = \lambda_{j-1} - \lambda_j$ for $j = 2, \ldots, k-1$. Note that

$$p_k = \left(1 - \sum_{j=1}^{k-1}\right)p_j = \lambda_{k-1}.$$

Then, the cdf of Y can be written as

$$F_k(x) = 1 - [1-G(x)]^{n_1}\{1-\lambda_1 + \sum_{j=2}^{k-1}(\lambda_{j-1}-\lambda_j)[1-G(x)]^{n_j} + \lambda_{k-1}[1-G(x)]^{n_k}\}, \tag{2.3}$$

where $n_1 = m_1$, $n_{j+1} = m_{j+1} - m_1$, $n_k = m_k - m_1$, $0 \leq \lambda_1 \leq 1$, and $\lambda_j - 1 \leq \lambda_{j+1} \leq \lambda_j$, for $j = 1, \ldots, k-2$.

Now, we state a general theorem for producing the HRT-G family of distributions.

Theorem 2.2 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then a HRT-G random variable has a cdf given by*

$$F_k(x) = 1 - (1-G(x))^{\alpha_1}\{1 - \lambda_1 + \sum_{j=2}^{k-1} (\lambda_{j-1} - \lambda_j)(1-G(x))^{\alpha_j} + \lambda_{k-1}(1-G(x))^{\alpha_k}\}, \quad (2.4)$$

where $\alpha_1 \geq 1$, $\alpha_{j+1} \geq 0$, and $0 \leq \lambda_1 \leq 1$ and $\lambda_j - 1 \leq \lambda_{j+1} \leq \lambda_j$ for $j = 1, \ldots, k-1$.

Proof. Consider the function $p_k(t)$ defined by

$$p_k(t) = \alpha_1(1-\lambda_1)t^{\alpha_1-1} - \sum_{j=2}^{k-1} (\alpha_1+\alpha_j)(\lambda_j - \lambda_{j-1})t^{\alpha_1+\alpha_j-1} + (\alpha_1+\alpha_k)\lambda_{k-1}t^{\alpha_1+\alpha_k-1}, \quad (2.5)$$

where $0 \leq t \leq 1$.

Under the conditions imposed on the λ_js and the α_js in the statement of this theorem, it is easy to see that $p_k(t)$ is a pdf with support $[0,1]$. First, note that $p_k(t)$ can be expressed as

$$p_k(t) = \alpha_1(1-\lambda_1)t^{\alpha_1-1} + \sum_{j=2}^{k-1} (\alpha_1+\alpha_j)(\lambda_{j-1} - \lambda_j)t^{\alpha_1+\alpha_j-1} + (\alpha_1+\alpha_k)\lambda_{k-1}t^{\alpha_1+\alpha_k-1}. \quad (2.6)$$

Since $0 \leq 1-\lambda_1 \leq 1$, $\alpha_j \geq 0$, $0 \leq \lambda_{j-1} - \lambda_j \leq 1$, for $2 \leq j \leq k$, and $\alpha_1 \geq 1$, each term of (2.6) is positive. Second, note that

$$\begin{aligned} &\int_0^1 (\alpha_1+\alpha_j)(\lambda_{j-1}-\lambda_j)t^{\alpha_1+\alpha_j-1}\,dt = \lambda_{j-1}-\lambda_j \\ &\int_0^1 (\alpha_1+\alpha_k)\lambda_{k-1}t^{\alpha_1+\alpha_k-1}\,dt = \lambda_{k-1} \quad (2.7) \\ &\int_0^1 \alpha_1(1-\lambda_1)t^{\alpha_1-1}\,dt = 1-\lambda_1. \end{aligned}$$

It follows from (2.7) that

$$\int_0^1 = p_k(t) = 1 - \lambda_1 + \sum_{j=2}^{k-1} (\lambda_{j-1}-\lambda_j) + \lambda_{k-1} = 1 - \lambda_1 + \lambda_1 - \lambda_{k-1} + \lambda_{k-1} = 1. \quad (2.8)$$

Therefore, $p_k(t)$, $0 \le t \le 1$, is a legitimate probability density function.

Now, let $H(x)$ be defined by

$$H(x) = 1 - \int_0^{1-G(x)} p_k(t)\ dt, \quad -\infty < x < \infty. \tag{2.9}$$

Note that $H(x)$ is a continuous non-decreasing positive function with the following properties.

$$\lim_{x \to -\infty} H(x) = 0, \quad \text{and} \quad \lim_{x \to \infty} H(x) = 1.$$

That is, $H(x)$ is a cumulative distribution function of a certain random variable. In fact, $H(x) = F_k(x)$.

As a special case, when $k = 3$, the cdf of the HRT-G family has the following form

$$F_3(x) = 1 - [1 - G(x)]^{\alpha_1}\{1 - \lambda_1 + (\lambda_1 - \lambda_2)[1 - G(x)]^{\alpha_2} + \lambda_2[1 - G(x)]^{\alpha_3}\}, \tag{2.10}$$

where $\alpha_1 \ge 1$, $\alpha_2, \alpha_3 \ge 0$, $0 \le \lambda_1 \le 1$, and $-1 \le \lambda_2 \le 1$.

The corresponding pdf has the following form

$$f_3(x) = g(x)[1 - G(x)]^{\alpha_1 - 1}\{\alpha_1(1 - \lambda_1) + (\alpha_1 + \alpha_2)(\lambda_1 - \lambda_2)[1 - G(x)]^{\alpha_2}$$
$$+(\alpha_1 + \alpha_3)\lambda_2[1 - G(x)]^{\alpha_3}\}, \quad x \ge 0. \tag{2.11}$$

In the next theorem, we introduce another method of constructing a cubic rank transmuted distributions.

Theorem 2.3 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then, the cdf of the HRT-G family is given by*

$$\tilde{F}_3(x) = G(x)^{n_1}\{1 - \lambda_1 + (\lambda_1 - \lambda_2)G(x)^{n_2} + \lambda_2 G(x)^{n_3}\}, \tag{2.12}$$

with $0 \leq \lambda_1 \leq 1, -1 \leq \lambda_2 \leq 1, n_1 \geq 1$, and $n_2, n_3 \geq 0$.

Proof. The proof is similar to the proof of Theorem 2.1. Let $Y = \max\{X_1, \dots, X_N\}$, where the random variable N takes the positive integer values k, m, and n with respective probabilities: p, q, and $1 - p - q$.

We start with the cdf of Y,

$$\tilde{F}_3(x) = P(N = k)P(Y > x|N = k) + P(N = m)P(Y > x|N = m)$$

$$+P(N = n)P(Y > x|N = n)$$

$$= pG(x)^k + qG(x)^m + (1 - p - q)G(x)^n. \quad (2.13)$$

Now, let $p = 1 - \lambda_1$ and $q = \lambda_1 - \lambda_2$. Then, the cdf of Y can be written as

$$\tilde{F}_3(x) = G(x)^{n_1}\{1 - \lambda_1 + (\lambda_1 - \lambda_2)G(x)^{n_2} + \lambda_2 G(x)^{n_3}\},$$

where $n_1 = k, n_2 = m - k$, and $n_3 = n - k$.

The next theorem generalizes Theorem 2.3.

Theorem 2.4 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then, the cdf of the HRT-G family is given by*

$$\tilde{F}_k(x) = G(x)^{\alpha_1}\{1 - \lambda_1 + \sum_{j=2}^{k-1} (\lambda_{j-1} - \lambda_j)G(x)^{\alpha_j} + \lambda_{k-1}G(x)^{\alpha_k}\}, \quad (2.14)$$

with $0 \leq \lambda_1 \leq 1, \lambda_{j-1} \leq \lambda_{j+1} \leq \lambda_{j-1}, \alpha_1 \geq 1$, and $\alpha_{j+1}, \alpha_k \geq 0$ for $j = 1, \dots, k-2$.

Proof. The proof is similar to the proof of Theorem 2.2. We use the same function $p_k(t)$ given in (2.5), but with a function $\widetilde{H}(x)$ given as

$$\widetilde{H}(x) = \int_0^{G(x)} p_k(t)\, dt, \quad -\infty < x < \infty. \tag{2.15}$$

2.2 Second Construction

Here, we introduce a second construction for a special case, namely when $k = 2$.

Theorem 2.5 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then, the cumulative distribution function of the HRT-G distribution is given by*

$$F_2(x) = 1 - [1 - G(x)]\{1 - \lambda + \lambda[1 - G(x)]^{n-1}\}, \tag{2.16}$$

where either $1 - n \leq \lambda \leq 1$ when $1 \leq n < 2$, or $-1/(n-1) \leq \lambda \leq 1$ when $n \geq 2$.

Proof. Let $X_1, \dots, X_n$ be independent and identically distributed random variables with common cdf $G(x)$ and corresponding order statistics $X_{1:n}, \dots, X_{n:n}$. Consider the random variable Y defined by $Y \stackrel{d}{=} X_{i:n}$ with probability p_i, $i = 1, \dots n$, where $p_n = 1 - \sum_{i=1}^{n-1} p_i$. Then the cdf $F_2(x)$ of the random variable Y is given by

$$F_2(x) = p_1 P(X_{1:n} \leq x) + \cdots + p_n P(X_{n:n} \leq x)$$
$$= \left(1 - \sum_{i=2}^{n} p_i\right) G_{1:n}(x) + \sum_{i=2}^{n} p_i G_{i:n}(x), \tag{2.17}$$

where

$$G_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} G^k(x)[1 - G(x)]^{n-k} \tag{2.18}$$

is the cdf of the ith order statistic $X_{i:n}$ ([Arnold et al., 1992]).

By induction on n, it can be easily shown that

$$\sum_{i=2}^{n} G_{i:n}(x) = \sum_{k=1}^{n} (-1)^k \binom{n}{k} G(x)^k + nG(x). \tag{2.19}$$

Note also that

$$G_{1:n}(x) = 1 - [1 - G(x)]^n. \tag{2.20}$$

Using (2.19), we see that

$$\sum_{i=2}^{n} G_{i:n}(x) = nG(x) + \sum_{k=1}^{n} (-1)^k \binom{n}{k} G(x)^k$$

$$= nG(x) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} G(x)^k - 1$$

$$= nG(x) + (1 - G(x))^n - 1. \tag{2.21}$$

Let us, for simplicity, let $p_i = p$ for $i = 2, \dots, n$. Then by using (2.21) and (2.20), we see that

$$F_2(x) = p_1 G_{1:n}(x) + p \sum_{i=2}^{n} G_{i:n}(x)$$

$$= p_1\{1 - [1 - G(x)]^n\} + p\{nG(x) + [1 - G(x)]^n - 1\}. \tag{2.22}$$

We now let $p = (1 - \lambda)/n$. Then $p_1 = (1 + (n-1)\lambda)/n$ and

$$F_2(x) = \lambda - \lambda[1 - G(x)]^n + (1 - \lambda)G(x)$$

$$= 1 - (1 - G(x))\{1 - \lambda + \lambda(1 - G(x))^{n-1}\}.$$

The condition: either $1 - n \le \lambda \le 1$ when $1 \le n < 2$, or $-1/(n-1) \le \lambda \le 1$ when $n \ge 2$, guarantees that $0 \le p_1, \dots, p_n \le 1$ and $\sum_{i=1}^{n} p_i = 1$.

Corollary 2.1 *Let X be a random variable with cumulative distribution function $G(x)$ and probability density function $g(x)$. Then, the cdf of the HRT-G random variable appears in Theorem 2.5 has a cdf given by*

$$F_{2,n}(x) = 1 - [1 - G(x)]\{1 - \lambda + \lambda[1 - G(x)]^{n-1}\}$$
$$= G(x) + \lambda[1 - G(x)] - \lambda[1 - G(x)]^n$$
$$= G(x)\left\{1 - \lambda + \lambda \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} [G(x)]^{k-1}\right\}. \quad (2.23)$$

The pdf corresponding to $F_{2,n}(x)$ is

$$f_{2,n}(x) = g(x)\left\{(1 - \lambda) + \lambda \sum_{k=1}^{n} (-1)^{k+1} k \binom{n}{k} [G(x)]^{k-1}\right\}. \quad (2.24)$$

In many cases, $f_{2,n}(x)$ is a mixture of distributions.

When $n = 2$, $f_{2,2}(x) = g(x)\{1 + \lambda - 2\lambda G(x)\}$, the pdf of the quadratic transmuted distribution. When $n = 3$, $f_{2,3} = g(x)\{1 + 2\lambda - 6\lambda G(x) + 3\lambda G(x)^2\}$, the pdf of the cubic transmuted distribution.

3 Remarks and Sub-results

Remark 3.1 *Theorem 2.1 generalizes many known results about transmuted families of distributions.*

1. By letting $\alpha_1 = 1, \alpha_2 = 1$, and $\lambda_2 = 0$, we get the quadratic rank transmutation map

$$F_2(x) = G(x)(1 + \lambda_1 - \lambda_1 G(x)).$$

2. By letting $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = 2$, replacing λ_1 by $\lambda_1 + \lambda_2 - 2$, and replacing λ_2 by $1 - \lambda_2$, we get the cubic rank transmutation as stated in [Granzotto et al., 2017].

$$F_3(x) = G(x)(\lambda_1 + (\lambda_2 - \lambda_1)G(x) + (1 - \lambda_2)G(x)^2).$$

3. By letting $k = 2, \alpha_1 = \alpha_2 = 1, \lambda_1 = \lambda$, and using the cdf of the exponentiated baseline distribution $[G(x)]^\alpha$ instead of $G(x)$ in (2.4), we get a result proved in [Alizadeh et al., 2017], namely, the pdf of the transmuted distribution is given by

$$f(x) = \alpha g(x)[G(x)]^{\alpha-1}\{1 + \lambda - 2\lambda[G(x)]^\alpha\}.$$

Remark 3.2 *[Rahman et al., 2018] generated a general transmuted family of distributions using the following higher rank transmutation map.*

$$\tilde{F}(x) = (1 - G(x)) \sum_{i=1}^{k} \tilde{\lambda}_i G(x)^i + G(x). \tag{3.1}$$

It turns out that this family of distributions is a special case of the HRT-G family of distributions produced in this paper. To see this, by letting $\alpha_1 = 1$ and $\alpha_j = j - 1$ for $j = 2, \dots, k + 1$ in (2.4), we get

$$F_{k+1}(x) = 1 - (1 - G(x))\left\{1 - \lambda_1 + \sum_{j=2}^{k} \left(\lambda_{j-1} - \lambda_j\right)(1 - G(x))^{j-1} + \lambda_k(1 - G(x))^k\right\}. \tag{3.2}$$

Now, by letting

$$\lambda_j = \sum_{i=j}^{k} (-1)^{j-1} \binom{i-1}{j-1} \tilde{\lambda}_i,$$

$F(x)$ and $\tilde{F}(x)$ coincide.

Example 3.1 *When $k = 2$,*

$$F_3(x) = G(x)(1 + \lambda_1 + \lambda_2 - (\lambda_1 + 2\lambda_2)G(x) + \lambda_2 G(x)^2).$$

We let $\lambda_1 = \tilde{\lambda}_1 + \tilde{\lambda}_2$ and $\lambda_2 = -\tilde{\lambda}_2$ to get

$$\tilde{F}(x) = G(x)(1 + \lambda_1 + (\lambda_2 - \lambda_1)G(x) - \lambda_2 G(x)^2)$$

Remark 3.3 *By combining the two above methods (described by $F_k(x)$ and $\tilde{F}_k(x)$) for generating transmuted distributions, we can produce what is known ẗhe exponentiated generalized class of distributionsïntroduced in [Cordeiro et al., 2013].*

Let $G(x)$ be the cdf of some baseline distribution. Then, for $k = 2, \lambda_1 = 0$, and $\alpha_1 = \alpha$, we see that

$$F_2(x) = 1 - [1 - G(x)]^{\alpha}. \tag{3.3}$$

Now, by using $F_2(x)$ as a new baseline distribution along with $\tilde{F}_2(x)$ and letting $\lambda_1 = 0$ and $\alpha_1 = \beta$, where $\beta \geq 1$, we get

$$\tilde{F}_2(x) = \{1 - [1 - G(x)]^{\alpha}\}^{\beta}. \tag{3.4}$$

4 Examples

In all of the following examples, we use the pdf of the cubic rank transmutation map expressed in (2.11) with $\alpha_1 = 1, \alpha_2 = \beta > 0$, and $\alpha_3 = \gamma > 0$.

4.1 The exponential distribution

Let the random variable X_e be exponentially distributed with parameter θ. Then, the pdf and cdf of X_e are respectively,

$$G_{X_e}(x) = 1 - e^{-\theta x},$$

$$g_{X_e}(x) = \theta e^{-\theta x}, \quad x \geq 0.$$

The pdf of the HRT-Exponential random variable is given by

$$f_e(x) = \theta e^{-\theta x}\{1 - \lambda_1 + \beta(\lambda_1 - \lambda_2)e^{-(\beta-1)\theta x} + \gamma\lambda_2 e^{-(\gamma-1)\theta x}\}.$$

The hazard rate function of the HRT-Exponential random variable is

$$h_e(x) = \frac{\theta\left((\lambda_1-1)e^{\theta x(\beta+\gamma)}-\beta\lambda_1 e^{(\gamma+1)\theta x}+\beta\lambda_2 e^{(\gamma+1)\theta x}-\gamma\lambda_2 e^{(\beta+1)\theta x}\right)}{(\lambda_1-1)e^{\theta x(\beta+\gamma)}-\lambda_2 e^{(\beta+1)\theta x}-\lambda_1 e^{(\gamma+1)\theta x}+\lambda_2 e^{(\gamma+1)\theta x}}.$$

The moment-generating function of the HRT-Exponential random variable is

$$M_e(t) = \theta\left\{\frac{\lambda_2 t(\beta-\gamma)}{(t-\beta\theta)(t-\gamma\theta)} - \frac{(\beta-1)\lambda_1 t}{(t-\theta)(t-\beta\theta)} + \frac{1}{\theta-t}\right\}.$$

The rth moment of HRT-Exponential random variable is

$$E[X_e^r] = r!\,(\beta\gamma\theta)^{-r}\{(\beta\gamma)^r + \lambda_1(\gamma^r-(\beta\gamma)^r) + \lambda_2(\beta^r-\gamma^r)\},\quad r \geq 1.$$

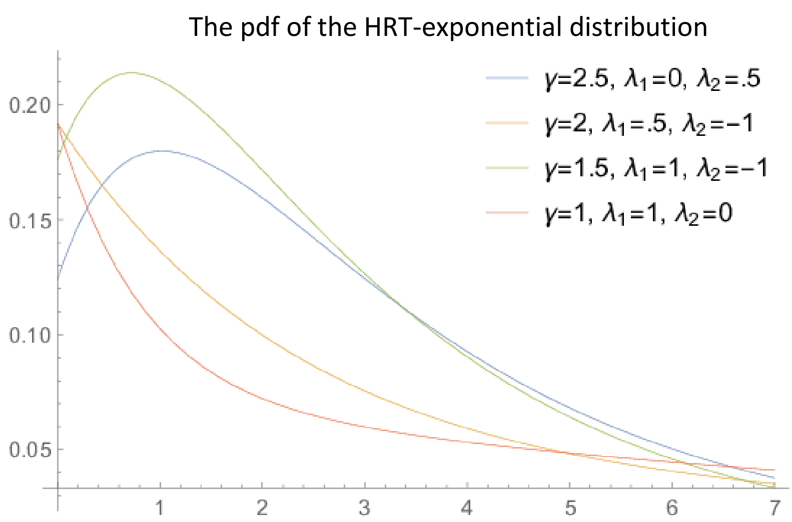

Figure 1: $\theta = .4, \beta = .2$

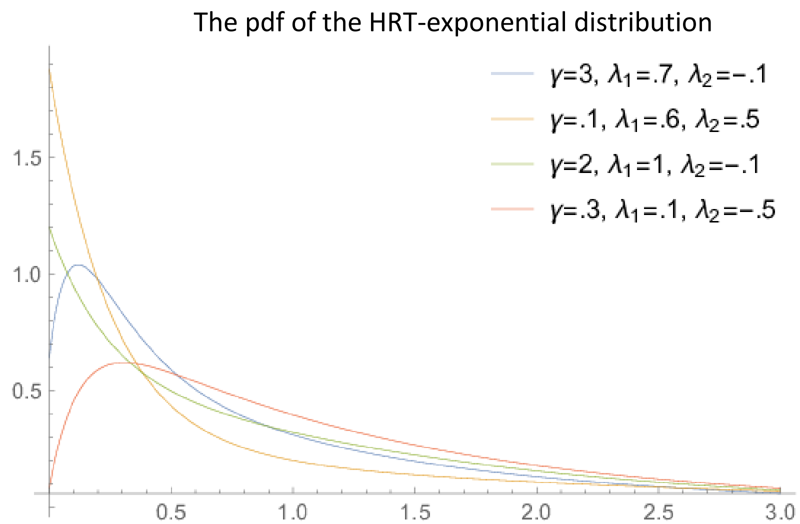

Figure 2: $\theta = 4, \beta = .2$

4.2 The Gompertz distribution

The cdf of the Gompertz random variable is

$$G(x) = 1 - e^{\xi - \xi e^{\delta x}}, \quad x \geq 0.$$

The pdf of the Gompertz random variable is

$$g(x) = \delta \xi e^{\xi - \xi e^{\delta x} + \delta x}, \quad x \geq 0.$$

The cdf of the HRT-Gompertz random variable is

$$F_g(x) = 1 - e^{\xi - \xi e^{\delta x}} \left\{ 1 - \lambda_1 + (\lambda_1 - \lambda_2) \left(e^{\xi - \xi e^{\delta x}} \right)^{\beta - 1} + \lambda_2 \left(e^{\xi - \xi e^{\delta x}} \right)^{\gamma - 1} \right\}.$$

The pdf of the HRT-Gompertz random variable

$$f(x) = \delta\xi e^{\xi(1-e^{\delta x})+\delta x}\left\{-\lambda_1 + \beta(\lambda_1-\lambda_2)e^{(\beta-1)\xi(1-e^{\delta x})} + \gamma\lambda_2 e^{(\gamma-1)\xi(1-e^{\delta x})} + 1\right\}.$$

The hazard rate function of the HRT-Gompertz random variable is

$$h(x) = \delta\xi e^{\xi(\beta+\gamma)(e^{\delta x}-1)+\delta x}$$

$$\times \frac{(\lambda_1-1)e^{\xi}+\beta(\lambda_2-\lambda_1)e^{\xi e^{\delta x}}e^{\beta\xi\left(1-e^{\delta x}\right)}-\gamma\lambda_2 e^{\xi e^{\delta x}}e^{\gamma\xi\left(1-e^{\delta x}\right)}}{(\lambda_1-1)e^{\xi+\xi(\beta+\gamma)\left(e^{\delta x}-1\right)}-\lambda_2 e^{\xi\left((\beta+1)e^{\delta x}-\beta\right)}+(\lambda_2-\lambda_1)e^{\xi\left((\gamma+1)e^{\delta x}-\gamma\right)}}. \quad (4.1)$$

The moment-generating function of the HRT-Gompertz random variable is

$$M_g(t) = \xi\left\{\beta(\lambda_1-\lambda_2)e^{\beta\xi}\mathrm{EI}\left(-\frac{t}{\delta},\beta\xi\right) + \gamma\lambda_2 e^{\gamma\xi}\mathrm{EI}\left(-\frac{t}{\delta},\gamma\xi\right) + (1-\lambda_1)e^{\xi}\mathrm{EI}\left(-\frac{t}{\delta},\xi\right)\right\},$$

where $EI(t,z) = \int_1^\infty y^{-t}e^{-yz}\,dy$ is the exponential integral.

To compute the HRT-Gompertz Moments, we use the transformation $Y = e^{\delta X}$. The pdf of Y is given by

$$f_Y(y) = \xi e^{\xi-\xi y}\left\{1-\lambda_1+\beta(\lambda_1-\lambda_2)e^{(\beta-1)\xi(-(y-1))}+\gamma\lambda_2 e^{(\gamma-1)\xi(-(y-1))}\right\}.$$

$$E[X^r] = \delta^{-r}E[(\log Y)^r]$$

$$= \int_1^\infty \delta\xi\delta^{-r-1}e^{\xi-\xi y}\log(y)\big(1-\lambda_1+\beta(\lambda_1-\lambda_2)e^{(\beta-1)\xi(-(y-1))}$$

$$+\gamma\lambda_2 e^{(\gamma-1)\xi(-(y-1))}\big)\,dy.$$

After algebraic manipulation, the moments of the HRT-Gompertz random variable are given by

$$E[X^r] = \xi r!\,\delta^{-r}\left\{(1-\lambda_1)e^{\xi}E_0^{(r)}(\xi) + \beta(\lambda_1-\lambda_2)e^{\beta\xi}E_0^{(r)}(\beta\xi) + \gamma\lambda_2 e^{\gamma\xi}E_0^{(r)}(\gamma\xi)\right\},$$

where $r \geq 1$ and $E_s^{(r)}(z)$ is the general integro-exponential function (see [Milgram, 1985]) given as

$$E_{(s)}^{r}(z) = \frac{1}{\Gamma(r+1)}\int_1^\infty \log^r(u)u^{-s}e^{-zu}\,du, z > 0. \quad (4.2)$$

In other words,

$$\int_1^{\infty} \log^r(u)u^{-s}e^{-zu}\, du = \Gamma(r+1)E_{(s)}^r(z), \quad r \geq 1. \qquad (4.3)$$

The pdf of the HRT-Gompertz distribution

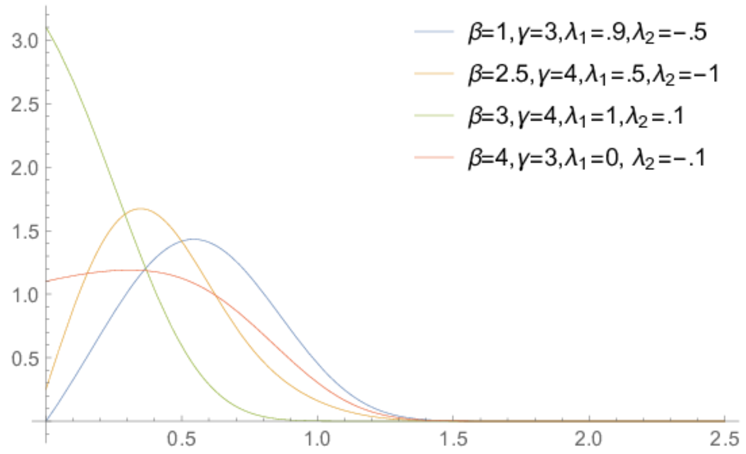

Figure 3: $\delta = 2, \xi = .5$

The pdf of the HRT-Gompertz distribution

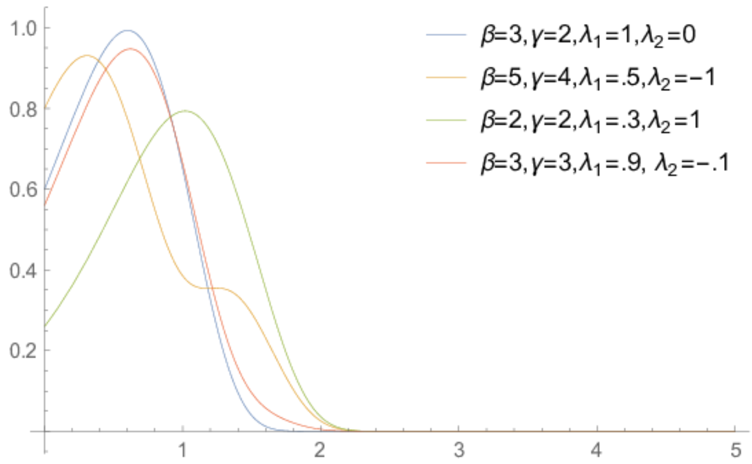

Figure 4: $\delta = 2, \xi = .1$

4.3 The Weibull distribution

The pdf of the Weibull random variable is

$$g(x) = \mu x^{-1} e^{-\left(\frac{x}{\beta}\right)^{\mu}} \left(\frac{x}{\beta}\right)^{\mu} \quad x > 0.$$

The cdf of the Weibull random variable is

$$G(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\mu}} \quad x > 0.$$

The pdf of the HRT-Weibull random variable is

$$f_w(x) = \delta^{-1} \mu e^{-\left(\frac{x}{\delta}\right)^{\mu}} \left(\frac{x}{\delta}\right)^{\mu-1} \left\{1 - \lambda_1 + \beta(\lambda_1 - \lambda_2) e^{-(\beta-1)\left(\frac{x}{\delta}\right)^{\mu}} + \gamma \lambda_2 e^{-(\gamma-1)\left(\frac{x}{\delta}\right)^{\mu}}\right\}.$$

The Cdf of the HRT-Weibull random variable is

$$F_w(x) = 1 - e^{-\left(\frac{x}{\delta}\right)^{\mu}} \left\{1 - \lambda_1 + (\lambda_1 - \lambda_2)\left(e^{-\left(\frac{x}{\delta}\right)^{\mu}}\right)^{\beta-1} + \lambda_2 \left(e^{-\left(\frac{x}{\delta}\right)^{\mu}}\right)^{\gamma-1}\right\}.$$

The pdf of the HRT-Weibull distribution

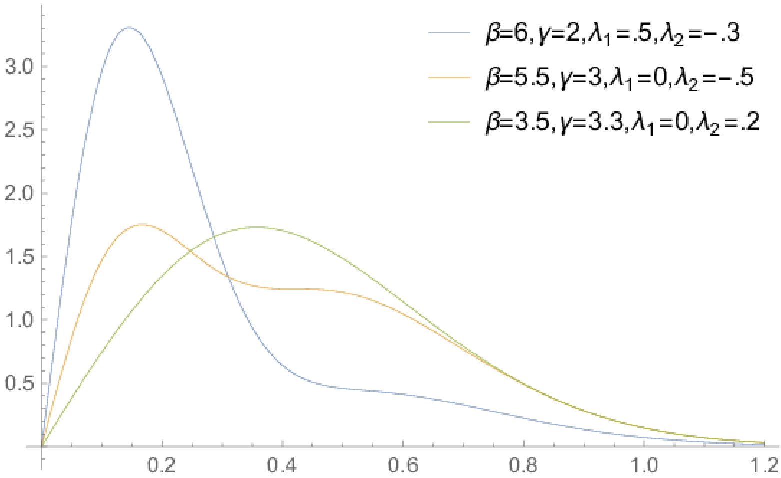

Figure 5: $\mu = 2, \delta = .5$

The pdf of the HRT-Weibull distribution

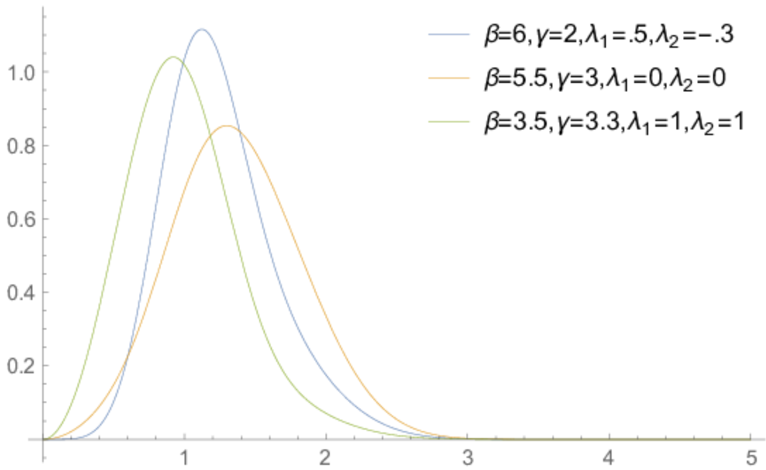

Figure 6: $\mu = 3, \delta = 1.5$

The hazard rate function of the HRT-Weibull random variable is

$$h(x) = \frac{\delta^{-1}\mu\left(\frac{x}{\delta}\right)^{\mu-1}\left(a-1+\beta(b-a)e^{-(\beta-1)\left(\frac{x}{\delta}\right)^{\mu}}-b\gamma e^{-(\gamma-1)\left(\frac{x}{\delta}\right)^{\mu}}\right)}{\left(a-1+(b-a)e^{-(\beta-1)\left(\frac{x}{\delta}\right)^{\mu}}-be^{-(\gamma-1)\left(\frac{x}{\delta}\right)^{\mu}}\right)}.$$

The moments of the HRT-Weibull random variable are described in

$$E(X_w^r) = \delta^r \left\{1 - \lambda_1 + (\lambda_1 - \lambda_2)\beta^{-\frac{r}{\mu}} + \lambda_2 \gamma^{-\frac{r}{\mu}}\right\}, \quad r \geq 1.$$

Conclusion

The proposed method in this paper for extending classical lifetime probability models generalizes some methods used to produce transmuted families of distributions through higher rank transmutation maps. By using the proposed method, researchers and statisticians can generate more flexible and tractable lifetime distributions that capture the complexity of the real lifetime data. The proposed method in this paper is specially good for transmuting distributions having cdf of the form $G(x) = 1 - h(x)$ for some function $h(x)$, a property that is true for many lifetime distributions. Meanwhile, an alternative, but equivalent method, is proposed in this paper to generalize lifetime distributions that have cdfs of other forms.

References

[Abdul-Moniem and Seham, 2015] Abdul-Moniem, I. B. and Seham, M. (2015). Transmuted gompertz distribution. *Computational and Applied Mathematics Journal*, 1(3):88–96.

[Alizadeh et al., 2017] Alizadeh, M., Merovci, F., and Hamedani, G. (2017). Generalized transmuted family of distributions: Properties and applications. *Hacettepe Journal of Mathematics and Statistics*, 46:645–667.

[Arnold et al., 1992] Arnold, B., Balakrishnan, N., and Nagaraja, H. (1992). *A first course in order statistics*. Wiley, New York.

[Aslam et al., 2018] Aslam, M., Hussain, Z., and Asghar, Z. (2018). Cubic transmuted-g family of distributions and its properties. *Stochastics and Quality Control*.

[Cordeiro et al., 2013] Cordeiro, G. M., Ortega, E. M. M., and da Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11:1–27.

[El-Bar, 2017] El-Bar, A. M. T. A. (2017). An extended gompertz-makeham distribution with application to lifetime data. *Communications in Statistics-Simulation and Computation*, pages 1–22.

[Granzotto1 et al., 2018] Granzotto1, D., Santos, C., and Louzada, F. (2018). The transmuted weibull regression model: an application to type 2 diabetes mellitus data. *International Journal of Statistics and Probability*, 7(2).

[Granzotto et al., 2017] Granzotto, D. C. T., Louzada, F., and Balakrishnan, N. (2017). Cubic rank transmuted distributions: inferential issues and applications. *Journal of Statistical Computation and Simulation*, 87(14):2760–2778.

[Gupta et al., 1998] Gupta, R., Gupta, R., and P.L.Gupta (1998). Modelling failure time data by lehman alternatives. *Communications in Statistics-Theory and Methods*, 27:887–904.

[khan et al., 2017] khan, M. S., King, R., and Hudson, I. (2017). Transmuted weibull distribution: Properties and estimation. *Communications in Statistics-Theory and Methods*, 46(11):5394–5418.

[Lee et al., 2013] Lee, C., Famoye, F., and Alzaatreh, A. Y. (2013). Methods for generating families of univariate continuous distributions in the recent decades. *WIREs Comput Stat*, 5(3):219–238.

[Merovci, 2013] Merovci, F. (2013). Transmuted rayleigh distribution. *Austrian Journal of Statistics*, 42(1):21–31.

[Milgram, 1985] Milgram, M. (1985). The generalized integro-exponential function. *Math. Comp.*, 44(170):443–458.

[Nofal et al., 2017] Nofal, Z., Afify, A., Yousof, H., and Cordeiro, G. (2017). The generalized transmuted-g family of distributions. *Communications in Statistics-Theory and Methods*, 46(8):4119–4136.

[Rahman et al., 2018] Rahman, M., Al-Zahrani, B., and Shahbaz, M. (2018). A general transmuted family of distributions. *Pak.j.stat.oper.res.*, 14(2):451–469.

[Shaw and Buckley, 2009] Shaw, W. T. and Buckley, I. R. (2009). The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. *arXiv:0901.0434*.