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Moments of order statistics from the Gompertz-Makeham Distribution

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Abstract

:Abstract

In this paper we study the moments of order statistics corresponding to a random sample of finite size from the three-parameter Gompertz-Makeham distribution. We will derive explicit closed-form formulas for the moment-generating function of order statistics from this distribution in terms of the integro-exponential function. Then, we derive explicit close-form formulas for the moments of all orders random variables follow this distribution. Then we prove an identity that relates the moments of order statistics from the Gompertz-Makeham distribution to the moments of random variables from this distribution.

Keywords:

Gompertz-Makeham distribution, Order statistics, Moments, Moment-generating function.

1. Introduction:

The Gompertz-Makeham probability distribution, developed in 1860 by [Makeham, 1860], is an extension of the Gompertz probability distribution that was introduced by [Gompertz, 1825] in 1825. The Gompertz-Makeham distribution is a lifetime probability distribution. It has been recently used together with its extensions in many fields of sciences including actuaries, biology, demography, gerontology, and computer science. Computer scientists use the Gompertz-Makeham distribution to model the failure rates of computer codes. In Marketing Science, the Gompertz-Makeham model is used as an individual-level simulation for customer lifetime value modeling. In network theory, particularly the Erdös-Rányi model, the walk length of a random self-avoiding walk (SAW) is distributed according to the Gompertz-Makeham distribution.

For the applications of the Gompertz-Makeham distribution in life insurance, the reader may refer to [Terzioand Sucu, 2015]. For information about the order statistics of an extended Gompertz-Makeham distribution, one can read [El-Bar, 2017].

In [Marshall and Olkin, 2007], the authors provide a comprehensive review of the history and theory of the Gompertz-Makeham probability distribution. [Golubev, 2004] emphasizes the practical importance of this probability distribution. Detailed information about the Gompertz-Makeham distribution, its mathematical and statistical properties, and its applications can be found in [Johnson et al., 1995] and [Dey et al., 2017].

The Gompertz-Makeham probability distribution has a somewhat mathematically complicated probability density function and distribution function, because these two functions involve the double exponential [Pollard and Valkovics, 1992]. This makes the task of computing the moments of random variables from this distributions complicated.

A random variable X is said to have a Gompertz-Makeham distribution with strictly positive real parameters α, β and λ, abbreviated as $X \sim GM(\alpha, \beta, \lambda)$, if its cumulative distribution function is given by

$$F(x; \alpha, \beta, \lambda) = 1 - \exp\{(\alpha/\beta)(1 - e^{\beta x}) - \lambda x\}, \quad x > 0.$$

The pdf of the X is given as

$$f(x; \alpha, \beta, \lambda) = (\lambda + \alpha e^{\beta x})\exp\{(\alpha/\beta)(1 - e^{\beta x}) - \lambda x\}, \quad x > 0.$$

In this paper, closed-form formulas for the moment-generating functions of order statistics from the Gompertz-Makeham distribution are derived. Then the moments of those order statics are also derived.

This work is motivated by a similar work in [Jodr 2013], where the author used Lambert W function to derive the moments of the first order statistics $X_{i:n}$, where $X \sim GM(\alpha, \beta, \lambda)$. Then he used a relation from [Balakrishnan and Rao, 1998] to derive the moments of the other order statistics. Our approach is to use different techniques and straightforward proofs to compute the moment-generating functions of the order statistics. Then we use the moment-generating function to derive the moments of order statistics associated with a random sample of size n from the Gompertz-Makeham.

For simplicity, we will denote $F(x; \alpha, \beta, \lambda)$ and $f(x; \alpha, \beta, \lambda)$ by $F(x)$ and $f(x)$, respectively.

2 *Gomperts-Makeham Order Statistics*

let $X_1, \dots, X_n$ be a random sample of size n from the Gompertz-Makeham probability distribution with parameters $\alpha > 0, \beta > 0$, and $\lambda > 0$, and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the corresponding order statistics obtained by arranging $X_i, i = 1, \dots, n$, in non-decreasing order of magnitude. The i^{th} element of this sequence, $X_{i:n}$, is called the i^{th} order statistic.

From [DasGupta, 2011], the pdf of the i^{th} order statistics is obtain from the equation

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1 - F(x)]^{n-i} f(x). \qquad (2.1)$$

Therefore, the pdf of the i^{th} order statistics is given as

$$f_{X_{i:n}}(x) = \alpha i \binom{n}{i}\left(\frac{\lambda}{\alpha} + e^{\beta x}\right)\left(1 - e^{\frac{\alpha}{\beta}(1-e^{\beta x}) - \lambda x}\right)^{i-1}$$

$$\times \exp\left\{(n-i)\left(\frac{\alpha}{\beta}\left(1-e^{\beta x}\right)-\lambda x\right)-\frac{\alpha}{\beta}\left(\frac{\beta}{\alpha}\lambda x+e^{\beta x}-1\right)\right\}. \tag{2.2}$$

By the Binomial Theorem,

$$\left(1-e^{\frac{\alpha}{\beta}(1-e^{\beta x})-\lambda x}\right)^{i-1}=\sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{i-1}{j}e^{(i-1-j)\left(\frac{\alpha}{\beta}(1-e^{\beta x})-\lambda x\right)}$$

$$=\sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{i-1}{j}e^{\frac{\alpha}{\beta}(i-1-j)(1-e^{\beta x})}e^{-(i-1-j)\lambda x}. \tag{2.3}$$

Substituting in (2.2), we get

$$f(x)_{X_{i:n}}=\sum_{j=0}^{i-1}(-1)^{i-1-j}i\alpha\binom{i-1}{j}\binom{n}{i}e^{(\alpha/\beta)(n-j)(1-e^{\beta x})}e^{-(n-j)\lambda x}\left(e^{\beta x}+\frac{\lambda}{\alpha}\right), \tag{2.4}$$

where $0<\alpha,\beta,\lambda<\infty$.

3 The Moment-Generating Function

In this section, we will derive and prove a closed-form formula for the moment-generating function of $X\sim GM(\alpha,\beta,\lambda)$.

Before proceeding to the main theorem of this section, let us introduce the general integro-exponential function (see [Milgram, 1985]), denoted as $E_s^{(k)}(z)$. It is given as

$$E_{(s)}^{k}(z)=\frac{1}{\Gamma(k+1)}\int_1^\infty \log^k(u)u^{-s}e^{-zu}\,du, \tag{3.1}$$

where $z>0, s$ are reals and $k\geq 0$ is an integer. In other words,

$$\int_1^\infty \log^k(u)u^{-s}e^{-zu}\,du=\Gamma(k+1)E_{(s)}^{k}(z). \tag{3.2}$$

Note that $E_{(s)}^{0}(z)$ is the exponential integral given as

$$E_{(s)}^{0}(z)=\int_1^\infty t^{-s}e^{-zt}\,dt. \tag{3.3}$$

Below, we are going to use the following identity from [Milgram, 1985].

$$E_{(s)}^{k-1}(z)=(s-1)E_{(s)}^{k}(z)+zE_{(s-1)}^{k}(z), z>0. \tag{3.4}$$

Theorem 3.1 *Let $X_{i:n}$ be the i^{th} order statistic corresponding to a random sample of size n form the Gompertz-Makeham distribution with parameters $\alpha > 0, \beta > 0$, and $\lambda > 0$. Then the moment-generating of $X_{i:n}$ is*

$$M(t) = \sum_{j=0}^{i-1} i(-1)^{i-j-1} \binom{i-1}{j} \binom{n}{i} e^{(\alpha/\beta)(n-j)}$$

$$\times \left\{ \frac{\alpha}{\beta} E^{0}_{\left(\frac{\lambda(n-j)-t}{\beta}\right)} \left(\frac{\alpha}{\beta}(n-j)\right) + \frac{\lambda}{\beta} E^{0}_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)} \left(\frac{\alpha}{\beta}(n-j)\right) \right\}, \tag{3.5}$$

where $-\infty < t < \infty$.

Proof. The moment-generating function of the i^{th} order statistic $X_{i:n}$, for any real number t, is given by

$$M(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)\ dx = \sum_{j=0}^{i-1} (-1)^{i-j-1} \binom{i-1}{j} \binom{n}{i} I, \tag{3.6}$$

where

$$I = \int_0^\infty i\alpha \left(\frac{\lambda}{\alpha} + e^{\beta x}\right) \exp\{(\alpha/\beta)(n-j)\left(1-e^{\beta x}\right) - (\lambda(n-j)-t)x\}\, dx. \tag{3.7}$$

By changing the variable x of (3.7) into $y = e^{\beta x}$, (3.7) simplifies to,

$$I = \int_1^\infty i(\alpha/\beta) \left(\frac{\lambda}{\alpha} + y\right) e^{-(\alpha/\beta)(y-1)(n-j)} y^{(t-\lambda(n-j))/\beta - 1}\, dy.$$

We write I as the sum of I_1 and I_2, where

$$I_1 = i(\alpha/\beta) \int_1^\infty y^{-(\lambda(n-j)-t)/\beta} e^{-(\alpha/\beta)(n-j)(y-1)}\, dy, \tag{5.8}$$

$$I_2 = i(\lambda/\beta) \int_1^\infty y^{-(1+\lambda(n-j)-t)/\beta} e^{-(\alpha/\beta)(n-j)(y-1)}\, dy. \tag{5.9}$$

Using the integro-exponential function defined in (3.1), we rewrite I_1 and I_2 as

$$I_1 = ie^{(\alpha/\beta)(n-j)}(\alpha/\beta)E^0_{\left(\frac{\lambda(n-j)-t}{\beta}\right)}\left(\frac{\alpha}{\beta}(n-j)\right), \tag{3.10}$$

$$I_2 = ie^{(\alpha/\beta)(n-j)}(\lambda/\beta)E^0_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\left(\frac{\alpha}{\beta}(n-j)\right). \tag{3.11}$$

On substituting for I_1 and I_2 from (3.10) and (3.11), respectively, in (3.6), we get (3.5).

Theorem 3.2 *Let $X_{i:n}$ be the i^{th} order statistic corresponding to a random sample of size n form the Gompertz-Makeham distribution with parameters α, β, and λ. Then the k^{th} derivative of the moment-generating of $X_{i:n}$ with respect to t is*

$$M^{(k)}(t) = \sum_{j=0}^{i-1} \frac{c(i,j,n)k!\beta^{-k}}{(n-j)} e^{(\alpha/\beta)(n-j)} \left\{\frac{\alpha}{\beta}(n-j)E^k_{\left(\frac{\lambda(n-j)-t}{\beta}\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right.$$

$$\left. + \frac{\lambda}{\beta}(n-j)E^k_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right\}, \tag{3.12}$$

where $-\infty < t < \infty$ and

$$c(i,j,n) = i(-1)^{i-j-1}\binom{i-1}{j}\binom{n}{i}.$$

Proof. Note that

$$\frac{\partial^k}{\partial t^k} y^{-\frac{\lambda(n-j)-t}{\beta}} = \left(\frac{1}{\beta}\right)^k \log^k(y) y^{-\frac{\lambda(n-j)-t}{\beta}}.$$

Similarly,

$$\frac{\partial^k}{\partial t^k} y^{-\left(\frac{\lambda(n-j)-t}{\beta}+1\right)} = \left(\frac{1}{\beta}\right)^k \log^k(y) y^{-\frac{\lambda(n-j)-t}{\beta}-1}.$$

$$M^{(k)}(t) = \sum_{j=0}^{i-1} c(i,j,n)\beta^{-k} e^{(\alpha/\beta)(n-j)}(I_3 + I_4), \tag{3.13}$$

where

$$I_3 = (\alpha/\beta)\int_1^{\infty} \log^k(y) e^{-(\alpha/\beta)(n-j)y} y^{-\frac{\lambda(n-j)-t}{\beta}}\, dy, \tag{3.14}$$

$$I_4 = (\lambda/\beta)\int_1^{\infty} \log^k(y) e^{-(\alpha/\beta)(n-j)y} y^{-\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\, dy. \tag{3.15}$$

By (3.2),

$$I_3 = k!\,(\alpha/\beta) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}\right)}\left(\frac{\alpha}{\beta}(n-j)\right), \tag{3.16}$$

$$I_4 = k!\,(\lambda/\beta) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\left(\frac{\alpha}{\beta}(n-j)\right). \tag{3.17}$$

Therefore,

$$I_3 + I_4 = \frac{k!}{(n-j)}\left\{\frac{\alpha}{\beta}(n-j) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right.$$
$$\left. + \frac{\lambda}{\beta}(n-j) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right\}. \tag{3.18}$$

Therefore,

$$M^{(k)}(t) = \sum_{j=0}^{i-1} \frac{c(i,j,n)k!\beta^{-k}}{(n-j)} e^{(\alpha/\beta)(n-j)} \left\{\frac{\alpha}{\beta}(n-j) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right.$$
$$\left. + \frac{\lambda}{\beta}(n-j) E^k_{\left(\frac{\lambda(n-j)-t}{\beta}+1\right)}\left(\frac{\alpha}{\beta}(n-j)\right)\right\}.$$

Theorem 3.3 *Let $X_{i:n}$ be the i^{th} order statistic corresponding to a random sample from the Gompertz-Makeham distribution with parameters α, β, and λ. Then the k^{th} moment of $X_{i:n}$ is given by*

$$E[X_{i:n}^k] = \frac{\Gamma(k+1)}{\beta^k} \sum_{h=n-i+1}^{n} (-1)^{h-(n-i+1)} \binom{n}{h} \binom{h-1}{n-i} e^{(\alpha/\beta)h} E_{\left(\frac{\lambda h}{\beta}+1\right)}^{k-1} \left(\frac{\alpha h}{\beta}\right), \tag{3.19}$$

where k is a positive integer.

Proof. By letting $t = 0$, (3.12) becomes

$$M^{(k)}(0) = \sum_{j=0}^{i-1} \frac{c(i,j,n)k!\beta^{-k}}{(n-j)} e^{(\alpha/\beta)(n-j)} \left\{ \frac{\alpha}{\beta}(n-j) E_{\left(\frac{\lambda(n-j)}{\beta}\right)}^{k} \left(\frac{\alpha}{\beta}(n-j)\right) \right.$$

$$\left. + \frac{\lambda}{\beta}(n-j) E_{\left(\frac{\lambda(n-j)}{\beta}+1\right)}^{k} \left(\frac{\alpha}{\beta}(n-j)\right) \right\}. \tag{3.20}$$

By (3.4), (3.20) simplifies to

$$M^{(k)}(0) = \sum_{j=0}^{i-1} \frac{c(i,j,n)k!\beta^{-k}}{(n-j)} e^{(\alpha/\beta)(n-j)} E_{((\lambda/\beta)(n-j)+1)}^{k-1} \left(\frac{\alpha}{\beta}(n-j)\right). \tag{3.21}$$

Now, let $h = n - j$, then h goes from $n - i + 1$ to n and

$$E[X_{i:n}^k] = \frac{\Gamma(k+1)}{\beta^k} \sum_{h=n-i+1}^{n} (-1)^{h-(n-i+1)} \binom{n}{h} \binom{h-1}{n-i} e^{(\alpha/\beta)h} E_{\left(\frac{\lambda h}{\beta}+1\right)}^{k-1} \left(\frac{\alpha h}{\beta}\right).$$

Corollary 3.1

$$E[X_{i:n}^k] = \sum_{h=n-i+1}^{n} (-1)^{h-(n-i+1)} \binom{n}{h} \binom{h-1}{n-i} E[Y], \tag{3.22}$$

where $Y \sim GM(h\alpha, \beta, h\lambda)$.

Proof. Equation (3.22) follows from the fact that

$$E[Y] = \frac{\Gamma(k+1)}{\beta^k} e^{(\alpha/\beta)h} E^{k-1}_{\left(\frac{\lambda h}{\beta}+1\right)}\left(\frac{\alpha h}{\beta}\right). \tag{3.23}$$

4 Numerical Example and Conclusion

Numerical Example

As an example of applications, we use (3.19) to compute some moments of some order statistics from the Gompertz-Makeham distribution when the size of the sample is $n = 30$ and parameters $\alpha = 0.00049, \beta = 0.071, \lambda = 0.00092$. This distribution represents the lifetime (in days) of females of Drosophila melanogaster (fruit fly) (see [Promislow and Haselkorn, 2002]).

For the implementation, we used the Wolfram *Mathematica* 11.3. The results are summarized in the following table.

Table 1: The first two moments of $X_{i:30}$ together with the standard deviation

i	$E[X_{i:30}]$	$E[X_{i:30}^2]$	$Stdv(X_{i:30})$
1	14.8423	325.548	10.2594
3	31.8758	1101.45	9.24033
5	41.3657	1765.61	7.38176
7	47.7306	2315.92	6.14094
9	52.5476	2789.54	5.31949
11	56.486	3213.17	4.74392
13	59.8816	3604.39	4.31168
15	62.9283	3975.75	3.97184
17	65.7537	4338.12	3.81703
19	68.4541	4702.35	4.04844

21	71.1154	5076.95	4.42167
23	73.8331	5467.92	4.073
25	76.7432	5901.44	3.45293
27	80.1076	6428.37	3.33874
29	84.7152	7190.21	3.67958

Conclusion

We conclude that the moments of order statistics from the Gompertz-Makeham distribution can be explicitly given in closed forms and can be computed using a computer algebra system such as Wolfram *Mathematica*. In Table 0, we used double machine accuracy during the computations to guarantee that the values of the first two moments and standard deviation of the i^{th} order statistics are to a good extend accurate for different values of i. Then, we approximated the results so that the table become readable.

The moments of order statistics can be used to approximate the parameters of the distribution. They can also be used to characterize distributions.

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