1. Introduction:

A real form $G_0$ of a complex semisimple Lie group $G$ has only finitely many orbits in any given compact $G$-homogeneous projective algebraic manifold $Z = G/Q$ and therefore there are open orbits, and a unique closed orbit $\gamma^c ([2],[6])$.

In our paper we study the orbits of the real form $SU (p, q)$ and $SO(p, q)$ when they act on the Grassmannian spaces. In both cases we study the parametrization of the closed orbit and the open orbits which play a role in understanding the geometry of Grassmannian spaces, see ([2],[6] and [8]).

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1. The parametrization of $SU(p, q)$-Orbits in Grassmannian $Gr(r, n)$

1.1 Some basics in Linear Algebra

Let $V$ be a vector space $C^n$. Let $b(v, w), v, w \in V$, be a bilinear form defined $V$ and $h(v, w) = b(v, w)$ is its Hermiation form.

**Definition 2.1.** Any two nonzero vectors $v_1, v_2 \in V$ are orthogonal if they satisfy that

$$h(v_1, v_2) = 0.$$ 

**Definition 2.2.** For any vector $v \in V$ if $h(v, v) > 0$, we call $v$ a positive vector, if $h(v, v) < 0$, we call $v$ a negative vector and if $h(v, v) = 0$, we call $v$ an isotropic vector.
For the basic information in this section and more see [3].

**Definition 2.3.** A set $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an orthonormal basis of $(V, h)$ if
1. $h(v_i, v_j) = 0$, $\forall i, j$ with $i \neq j$,
2. $h(v_i, v_i) = \pm 1$.

**Remark 2.4.** By using Definition 2.3, we can write $V = W' \oplus W$ where $W'$ is a maximal positive subspace of $V$ generated by positive vectors in $\mathcal{B}$, and $W$ is a maximal negative subspace of $V$ generated by negative positive vectors in $\mathcal{B}$. So $W'$ is the orthogonal complement of $W$.

**Definition 2.5.** A subspace $W \subseteq V$ is called nondegenerate if and only if $W \cap W^\perp = \{0\}$, and called maximally degenerate if $W \subset W^\perp$.

**Notation 2.6.** We will denote the Hermitian form in the subspace $W$ to be $h \mid W = h_{W,W}$.

**Definition 2.7.** If $V$ is nondegenerate space of dimension $n$ such that $V = W' \oplus W$, then the signature of $V$ is $\text{sign}(V) = (\dim W, \dim W')$.

**Question 1.** How can we find an orthogonal basis for any subspace $(W, h)$ of dimension $r$?
To answer this question we have three cases:-

**Case 1:-** If $W$ is maximally degenerate subspace, then
$$\forall w_r, w_j \in W, \quad h(w_r, w_j) = 0.$$ 
So we need only to choose $r$-linearly independent vectors and then we finish.

**Case 2:-** If $W$ is nondegenerate, fix a nonisotropic vector $w_1 \in W$. Let $X_1 = Cw_1$, then
$$X_1 \cap X_1^\perp = \{0\}.$$ 
Choose nonisotropic vector $w_2 \in X_1^\perp$ so $h(w_2, w_2) = 0$.

Let $X_2 = \text{span}\{w_1, w_2\}$, then
$$X_2 \cap X_2^\perp = \{0\}.$$ 
Choose nonisotropic vector $w_3 \in X_2^\perp$, so $h(w_3, w_i) = 0 \forall i = 1, 2$.

Assume that we have $r - 1$ nonisotropic orthogonal vectors $w_2, \ldots, w_{r-1}$. Let
$$X_{r-1} = \text{span}\{w_2, \ldots, w_{r-1}\},$$
then
$$X_{r-1} \cap X_{r-1}^\perp = \{0\}.$$ 
Choose nonisotropic vector $w_r \in X_{r-1}^\perp$, so $h(w_r, w_i) = 0 \forall i = 1, \ldots, r - 1$.

Hence we have $r$ orthogonal vectors and these vectors spans $W$.

**Case 3:-** If $W \cap W^\perp \neq \{0\}$ and $W \cap W^\perp \subset W$, then
$$W = Q_t \oplus B_s$$
where $\dim Q_t = t$, $\dim B_s = s$, $t + s = r$ and
$$B_s = W \cap W^\perp = W^\perp$$
and $Q_t \cap Q_t^\perp = \{0\}$.

From case 1, any $s$ linearly independent vectors $\{v_1, \ldots, v_s\}$ from $B_s$ spans $B_s$, and from case 2, we can find an orthogonal basis $\{w_1, \ldots, w_r\}$ for $Q_t$.
Therefore, $\{v_1, \ldots, v_s, w_1, \ldots, w_r\}$ is an orthogonal basis for $W$.

**Example 2.8.** Consider the vector space $V = \mathbb{C}^6$. Let the hermitian form $h$ to be defined as

$$h(v, w) = b(v, \sigma(w)) = -\sum_{i=1}^{3} v_1 \sigma(w_1) + \sum_{i=4}^{6} v_i \sigma(w_i)$$.
Fix the standard basis \( \{e_1, e_2, e_3, e_4, e_5, e_6\} \) to be the orthonormal basis of \( V \).

The subspace \( W_1 = \text{span}\{e_1 + e_6, e_2 + e_5, e_3\} \) is a degenerate subpace with the orthonormal basis \( \{e_1 + e_6, e_2 + e_5, e_3\} \)

since \( h(e_1 + e_6, e_2 + e_5) = 0\) and \( h(e_1 + e_6, e_3) = 0 \) and \( h(e_2 + e_5, e_3) = 0 \). On the other hand, the subspace \( W_2 = \text{span}\{e_5, e_4\} \) is nondegenerate subpace with the orthonormal basis \( \{e_5, e_4\} \) since \( h(e_5, e_4) = 0\) and \( h(e_5, e_1) = -1 \) and \( h(e_4, e_1) = 1 \), and \( e_5, e_4 \) is an orthonormal basis.

Define the subspace \( W_3 = W_1 \bigoplus W_2 = \text{span}\{e_1 + e_6, e_2 + e_5, e_3, e_5, e_4\} \), by GramSchmidt Orthogonalisation Process we can find orthonormal basis for \( W_3 \) to be \( W_3 = \text{span}\{e_1 + e_6, e_2 + e_5, e_3\} \) which means that we can write \( W_3 = B \bigoplus Q \) where \( B = \text{span}\{e_2 + e_3\} \) and \( Q = \text{span}\{e_5, e_4\} \), where \( B \) is degenerate subpace and \( Q \) is nondegenerate subpace.

2.2 The Orbit Structure of the nondegenerate subspaces

Let \((V, h)\) be the complex nondegenerate vector space \( C^n \) of signature \((p, q)\), where \( p + q = n \). Let \( G = SL(n, C) \), and \( P \) be a maximal parabolic subgroup of \( G \). In this case the homogenous space \( Z = G/P \) can be identified with the set of all subspaces with dimension \( r \) called the Grassmannian \( Gr(r,n) \). Define the bilinear form \( b \) on \( V \) to be

\[
b(v, w) = -\sum_{i=1}^{q} v_i w_i + \sum_{i=q+1}^{n} v_i w_i\]

Consider the real form \( G_0 = SU(p,q) \) of \( SL(n, C) \) where \( p + q = n \). The Hermitian form \( h : C^n \times C^n \to C \) defined \( SU(p,q) \) is the standard Hermitian form of signature \((p, q)\) defined by

\[
h(v, w) = -\sum_{i=1}^{q} v_i \overline{w_i} + \sum_{i=q+1}^{n} v_i \overline{w_i}, \quad \forall v, w \in C^n
\]

then \( SU(p,q) \) is the group of isometries of \( V \) associated to \( h \), that is if \( T \in SU(p,q) \), then \( h(Tv, Tw) = h(v,w) \).

Let us concerned with the action of the real form \( SU(p,q) \) on \( Gr(r,n) \), \( SU(p,q) \times Gr(r,n) \to Gr(r,n) \).

By the results given by Wolf in [6], this action has finitely many orbits with a unique closed orbit and an open orbit exists. Here a question arise: How can we parameterize the orbits of this action? In the following sections we prove that the orbits of the above action parameterized by signature.

Definition 2.9. Given a subspace \((W, h)\) of \((V, h)\). We define a signature of the subspace \( W \) to be \( \text{sign}(W) = (n^+, n^-, d) \) where \( n^+ \) is the dimension of maximal positive subspace of \( W \) and \( n^- \) is the dimension of maximal negative subspace of \( W \) and \( d = \dim(W \cap W^\perp) = \dim(W^-) \).

Definition 2.10. Given a subspace \((W, h)\) of \((V, h)\). We define \( \text{sign}(W) = (n^+, n^-, d) \) where \( n^+ \) is the dimension of maximal positive subspace of \( W \) and \( n^- \) is the dimension of maximal negative subspace of \( W \) and \( d = \dim(W \cap W^\perp) = \dim(W^-) \).

Definition 2.11. Given a subspace \((W, h)\) of \((V, h)\) has the same signature as the subspace signature.

Proposition 2.12. Given \( X_1, X_2 \in Gr(r,n) \) be nondegenerate subspaces such that \( \text{sign}(X_1) = \text{sign}(X_2) \), then there exist \( g \in SU(p,q) \) with \( g(X_1) = X_2 \).

Proof. Given two nondegenerate subspaces \( X_1, X_2 \in Z \) with orthonormal bases \( \beta_1 = \{v_1, \ldots, v_r\} \) for \( X_1 \), and \( \beta_2 = \{u_1, \ldots, u_r\} \) for \( X_2 \). These two bases have the same signature and we can rearrange
them to have firstly the positive vectors and then the negative.

Similarly for $X^+_1$, $X^+_2 \in Z$ with orthonormal bases

$$\beta^+_1 = \{v_{r+1}, \ldots, v_n\}, \quad \beta^+_2 = \{u_{r+1}, \ldots, u_n\}$$

for $X^+_1$, $X^+_2$ respectively, these two bases $\beta^+_1, \beta^+_2$ have the same signature and we can rearrange them to have firstly the positive vectors and then the negative. So we can assume that $v_i$ and $u_i$ are both positive or both negative.

Now, since $V = X_1 \oplus X^+_1 = X_2 \oplus X^+_2$, we can define a linear map

$$T : V \rightarrow V \text{ by } T(v_i) = u_i \text{ for } v_i \in \theta_1 \text{ and } u_i \in \theta_2,$$

and

$$T(v^-) = u^-_i$$

for $v_i \in \beta^+_1$ and $u_i \in \beta^+_2$, so $T(X_1) = X_2$.

To show that $h(T(w_i), T(w_j)) = h(w_i, w_j)$ for $w_i, w_j \in X_1$, start with bases vectors $v_i, v_j \in X_1$, if $i \neq j$, then

$$h(T(v_i), T(v_j)) = h(u_i, u_j) = 0 = h(v_i, v_j), \quad (1)$$

and if $i = j$, then

$$h(T(v_i), T(v_j)) = h(u_i, u_i) = 1 = h(v_i, v_i). \quad (2)$$

Let $w_1, w_2 \in V$ where $w_1 = \sum_{k=1}^n \alpha_k v_k$ and

$$w_2 = \sum_{t=1}^n \gamma_t v_t,$$

then

$$h(T(w_1), T(w_2)) = h(T(\sum_{k=1}^n \alpha_k v_k), T(\sum_{t=1}^n \gamma_t v_t))$$

$$= h(\sum_{k=1}^n \alpha_k T(v_k), \sum_{t=1}^n \gamma_t T(v_t))$$

$$= \sum_{k=1}^n \sum_{t=1}^n \alpha_k \gamma_t h(T(v_k), T(v_t))$$

By (1) and (2) $= \sum_{k=1}^n \sum_{t=1}^n \alpha_k \gamma_t h(v_k, v_t)$

$$= h(\sum_{k=1}^n \alpha_k v_k, \sum_{t=1}^n \gamma_t v_t) = h(w_1, w_2).$$

Therefore $T \in SU(p,q)$.

**Example 2.13.** Let $G = SU(1,2)$ and define the non-degenerate subspaces $X_1 = \text{span}(e_1,e_3)$ and $X_2 = \text{span}(e_1,e_2)$. The signature $\text{sign}(X_1) = \text{sign}(X_2) = (1,1)$.

Choose the matrix $g \in SU(1,2)$ where

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $g.e_1 = e_1$ and $g.e_3 = e_2$, which means that $g.X_1 = X_2$.

### 2.3 The Orbit Structure of the degenerate subspaces

Recall that the signature of the vector space $V = \mathbb{C}^n$ with respect to the group $SU(p,q)$ is $(p,q)$.

**Lemma 2.14.** Given any isotropic vector $v \in V$, then $v = v^+ + v^-$ where $v^+, v^-$ are orthogonal positive and negative vectors respectively.

**Proof.** Assume that $V_1$ is a maximal positive subspace

$$h(w_1, w_1) = 0$$
Parametrization of the orbits of the real forms SU(p, q) and SO(p, q) in Grassmannian

in V then \( V_1 \) is a maximal negative

subspace where \( V_1 \cap V_1^\perp = \{0\} \) because

\( \text{sign}(V) = (p,q,0) \). Then any vector \( x \in V \) is

uniquely represented as \( x = t^+ + t^- \), where \( t^+ \in V_1 \)

and \( t^- \in V_1^\perp \). Therefore any isotropic vector

\( v \in V \) can be written as \( v = v^+ + v^- \)

but

\[
\begin{align*}
  h(w_1, w_2) &= 0 \\
  &= h(v^+ + v^-, \alpha_1 v^+ + \alpha_2 v^-) \\
  &= h(v^+, \alpha_1 v^+) + h(v^-, \alpha_2 v^-) + h(v^+, \alpha_2 v^-) + h(v^-, \alpha_1 v^+) \\
  &= \alpha_1 h(v^+, v^+) + \alpha_2 h(v^-, v^-)
\end{align*}
\]

where \( v^+ \in V_1 \) and \( v^- \in V_1^\perp \).

**Notation 2.15.** In the following lemmas \( E_i \) will be a nondegenerate subspace with \( \text{sign}(E_i) = (1,1,0) \).

**Lemma 2.16.** Given two orthogonal isotropic vectors \( w_1, w_2 \) where \( w_1 = v^+ + v^- \) and \( E_1 = \text{span}(v^+, v^-) \), then \( w_2 \in E_1^\perp \).

**Proof.** Firstly, since

\[
\begin{align*}
  h(v^+, v^-) &= -h(v^-, v^+).
\end{align*}
\]

so \( h(v^+, v^-) = -h(v^-, v^+) \).

Assume that \( w_2 \in E_1 \) then \( w_2 = \alpha_1 v^+ + \alpha_2 v^- \), \( \alpha_1, \alpha_2 \in \mathbb{C} - \{0\} \),

which implies that \( \alpha_1 = \alpha_2 \) and \( w_2 = \alpha_1 w_1 \) which is a contradiction.

Also if \( w_2 = a_1 v^+ + a_2 v^- \), \( \hat{v}^- \in E_1^\perp \),

or \( w_2 = a_1 \hat{v}^+ + a_2 v^- \), \( \hat{v}^+ \in E_1^\perp \),

then \( h(w_1, w_2) \neq 0 \).

Therefore \( w_2 \in E_1^\perp \).

**Lemma 2.17.** Let \( D \) be a degenerate subspace with dimension \( r \), there exist \( r \) subspaces \( E_1, E_2, \ldots, E_r \) such that

\[
E_i \cap E_j = \{0\} \quad 1 \leq i < j \leq r \quad \text{and} \quad D \subseteq E_1 \oplus E_2 \oplus \ldots \oplus E_r.
\]

**Proof.** We will prove it by induction.

**Step 1:** If \( \dim D = 1 \), then \( D = \text{span}(w_1) \) where \( w \) is an isotropic vector, so by Lemma 2.14 \( w_1 = v^+ + v^- \) where \( v^- \perp v^+ \) and \( D \subseteq E_1 = \text{span}(v^+, v^-) \).

**Step 2:** If \( \dim D = 2 \), then \( D = \text{span}(w_1, w_2) \) where \( h(w_1, w_2) = 0 \).

By step 1,

\[
W = \text{span}\{w_1\} \subseteq E_1
\]

and by Lemma 2.16 \( w_2 \in E_1^\perp \), then by Lemma 2.14 there exist \( v^+, \hat{v}^- \in E_1^\perp \) such that \( w_2 = v^+ + v^- \). So we have a nondegenerate subspace \( E_2 = \text{span}(v^+, v^-) \) where \( v_2 \in E_2 \) and \( E_1 \cap E_2 = \{0\} \), then

\[
D \subseteq E_1 \oplus E_2.
\]

**Step 3:** Assume that the lemma is true if \( \dim D < r \).

**Step 4:** If \( \dim D = r \).

Choose any vector \( w \) in \( D \), then \( D = \text{span}(w) \oplus \hat{v}^- \) where \( D^- \) is the orthogonal complement of \( \text{span}(w) \) in \( D \), So \( D^- \) is a subgroup of \( D \) with \( \dim D^- = r - 1 \) and by step 3 there exist \( E_1, E_2, \ldots, E_{r-1} \) such that

\[
E_i \cap E_j = \{0\} \quad 1 \leq i < j \leq r - 1 \quad \text{and} \quad D^- \subseteq E_1 \oplus E_2 \oplus \ldots \oplus E_{r-1}.
\]

By Lemma 2.16, \( w \in E_i^\perp \quad \forall i \), so
$w \in (E_1 \oplus E_2 \oplus \ldots \oplus E_r)^\perp$

again by step 1, $w = v^+ + v^-$ where $v^+ \perp v^-$ and

$\text{span}(w) \subseteq E_r = \text{span}(v^+, v^-)$.

Since $D = \text{span}(w) \oplus D^\perp$, then

$D \subseteq E_1 \oplus E_2 \oplus \ldots \oplus E_r$.

**Proposition 2.18.** Given $Y_1, Y_2 \in \text{Gr}(r,n)$

be degenerate subspaces, i.e.

$\text{sign}(Y_1) = \text{sign}(Y_2) = (0,0,r)$,

then there exist $g \in \text{SU}(p,q)$ with $g(Y_1) = Y_2$

Proof. Assume we have two degenerate

subspaces $Y_1, Y_2$. By Lemma 2.17 there

exist r subspaces

$E_1, E_2, \ldots, E_r$

such that $E_i \cap E_j = \{0\}$, $1 \leq i < j \leq r$, and

$Y_1 \subseteq E_1 \oplus E_2 \oplus \ldots \oplus E_r$

where $E_i = \text{span}(v^+_i, v^-_i)$, then we have 2r

orthogonal vectors of $V$,

$\beta_1 = \{v^+_1, v^-_1, \ldots, v^+_r, v^-_r\}$.

it, namely $\beta_2$, where it has $(p - r)$ positive vectors and

$(q - r)$ negative vectors. We can rearrange the vectors

in $\beta_1$ to have the positive vectors firstly, i.e.

$\hat{\beta}_2 = \{u^+_1, u^-_1, \ldots, u^+_r, u^-_r\}$.

Finally, we can define a linear map

$g : V \to V$ by $g(v^+_i) = u^+_i, \ g(v^-_i) = u^-_i, \forall i$,

then $g(Y_1) = Y_2$, and by using the same method we use

in the proof of Proposition 2.12

$h(g(w_1), g(w_2)) = h(w_1, w_2)$.

Therefore $g \in \text{SU}(p,q)$.

**Example 2.19.** Let $G = \text{SU}(1,2)$ and define the subspaces

$Y_1 = \text{span}(e_1 + e_3)$ and $Y_2 = \text{span}(e_1 + e_2)$. The signature

$\text{sign}(Y_1) = \text{sign}(Y_2) = (0,0,1)$. Choose the matrix $g \in \text{SU}(1,2)$ where

$$g = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$$
such that \( g.e_1 = e_1 \) and \( g.e_3 = e_2 \), which means that \( g.Y_1 = Y_2 \).

2.4 General Result

In this section we will prove our main theorem.

**Theorem 2.20.** The \( SU(p,q) \)-orbit in \( Gr(r,n) \) are parameterized by signature. That is given \( W_1, W_2 \in Gr(r,n) \) there exist \( g \in SU(p,q) \) with \( g(W_1) = W_2 \) if and only if \( \text{sign}(W_1) = \text{sign}(W_2) \).

**Proof.** Let \( W_1, W_2 \in Gr(r,n) \), then

\[
W_1 = Q_1 \oplus B_1 \quad \text{and} \quad W_2 = Q_2 \oplus B_2
\]

where \( Q_1, Q_2 \) are nondegenerate subspaces and

\[
B_1 = W_1 \cap W_1^\perp \quad \text{and} \quad B_2 = W_2 \cap W_2^\perp.
\]

By Proposition 2.12, there exist \( g \in SU(p,q) \) such that

\[
g(Q_1) = Q_2 \quad \text{and} \quad g(Q_1^\perp) = Q_2^\perp.
\]

But \( g(B_1) = \tilde{B}_1 \subseteq Q_2^\perp \), by Proposition 2.18 there exist \( \tilde{g} \in SU(p,q) \) such that \( \tilde{g}(B_1) = B_2 \). So we can define our map \( \psi : V \rightarrow V \) as

\[
\psi = (Id \oplus \tilde{g}^*) \circ g,
\]

where \( Id \) is the identity matrix, then

\[
\psi(W_1) = W_2.
\]

Since \( Id, g, g^* \) are all in \( SU(p,q) \), then \( \psi \in SU(p,q) \).

2. THE PARAMETRIZATION OF

**SO(\( P,Q \))-ORBITS IN**

**ISOTROPIC GRASSMANNIAN** \( Z_k \)

In this section we will prove that \( SO(p,q) \)-orbits in \( Z_k \) are parameterized by signature, where \( Z_k \) is the isotropic Grassmannian.

Consider the semisimple Lie group \( G = SO(n,C) \) where \( G_0 = SO(p,q) \), with complex bilinear form defined by

\[
b(v, w) = - \sum_{i=1}^{p} v_i w_i + \sum_{i=p+1}^{n} v_i w_i
\]

then the Hermitian form which defines the real form is

\[
h(v, w) = b(v, \bar{w})
\]

\( \), so \( G_0 \) is the subgroup of operators \( T \) in \( G \) satisfy \( T = \bar{T} \).

Let \( (V, h) \) be the complex nondegenerate vector space of signature \( (p,q) \). Define \( Z_k \) to be the isotropic Grassmannian which is the set of all-isotropic \( k \)-planes in \( C^n \) where

\[
1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor.
\]

Consider the action of the real form \( SO(p,q) \) on the flag manifold \( Z_k \)

\[
SO(p,q) \times Z_k \rightarrow Z_k
\]

then \( SO(p,q) \) has a unique closed orbit and finitely many open orbits on \( Z_k \). In the following sections we will proof that the orbits of this action parameterized by signature.
3.1 Orbit structure

In this sections we will prove that the $SO(p,q)$-orbits in $Z_d$ are parameterized by signature $\text{sign}(W) = (n^*, n^-, d, r)$ by using our previous results about $SU(p,q)$.

**Definition 3.1.** Given a subspace $(W, h | W)$ of $(V, h)$, We define a signature of $W$ to be $\text{sign}(W) = (n^*, n^-, d, r)$ where $n^*$ is the dimension of maximal positive subspace of $W$ and $n^-$ is the dimension of maximal negative subspace of $W$ and

$$d = \dim(W \cap W^\perp) = \dim(W^\perp h | W)$$

and $r = \dim(W \cap W)$.

We used this signature to parameterize the $SO(p,q)$ orbits while $SU(p,q)$ orbits parameterized by only $(n^*, n^-, d)$.

**Remark 3.2.** Given a non degenerate space with orthogonal basis $\theta$, then we can write $V$ as $V = W^* \oplus W^\perp$ where $W^*$ is a maximal positive subspace of $V$ generated by positive vectors in $\theta$, and $W^\perp$ is a maximal negative subspace of $V$ generated by negative vectors of $\theta$. We can extend each of these bases to get bases for $W^*, W^\perp$, $D$ respectively. We can define a basis of $(W, h | W)$, namely an ideal basis, to be as following:

**Definition 3.3.** A subspace $W$ is said to be of maximal reality if $W = W^*$ and $W = W_k \oplus W_k$ where $W_k \subset R$.

**Lemma 3.4.** Any subspace of maximal reality has a real basis.

**Proof.** Given a subspace $X$ of maximal reality then $X = X'$, i.e., $X = X_k \oplus X_k$ where $X_k$ is a real subspace, so the basis of $X_k$ is basis of $X$ but the basis of $X_k$ is real that mean we can find a real basis $\theta$ for $X$ where $u = u \forall u \in \theta$.

If $(W, h | W)$ is of signature $\text{sign}(W) = (n^*, n^-, d, r)$, then $W = W^* \oplus W^\perp \oplus D$ where $W^*$ is a maximal positive subspace with $\dim W^* = n^*$, $W$ is a maximal negative subspace with $\dim W^\perp = n^-$ and $D = W \cap W^\perp$ with $\dim D = d$. In this cases $W^+ \cap W^\perp$, and $W^- \cap W^\perp$ and $D \cap D$ all of them have real bases $\theta_1, \theta_2$ and $\theta_3$ respectively. We can extend each of these bases to get bases for $W^*, W^\perp$, $D$ respectively. So we can define a basis of $(W, h | W)$, namely an ideal basis, to be as following:

**Definition 3.5.** Given a $k$-subspace $(W, h | W)$ of signature $\text{sign}(W) = (n^*, n^-, d, r)$ where $\dim W = n^* + n^- + d = k$. A set $\theta = \{v_1, \ldots, v_n\}$ is an ideal basis of $(W, h | W)$ if:

1. $v_1, \ldots, v_n$ are orthonormal positive vectors with $r_1$ vectors of them are real.
2. $v_{n+1}, \ldots, v_{n+k}$ are orthonormal negative vectors with $r_2$ vectors of them are real.
3. $v_{n+k+1}, \ldots, v_k$ are linearly independent vectors with $r_3$ vectors of them are real.
4. $r_1 + r_2 + r_3 = r$ and we will define the signature of this basis to be $(n^*, n^-, d, r)$.

**Example 3.6.** Let $G = SU(2,3)$ with the hermaition form $h$ defined as

$$h(v, w) = b(v, \sigma(w)) = -\sum_{i=1}^{2} v_i \sigma(w_i) + \sum_{i=3}^{5} v_i \sigma(w_i).$$

Define the subspaces $W = \text{span}\{e_1, e_2, e_3 + ie_4\}$ with signature $\text{sign}(W) = (2, 1, 1, 3)$. The basis $\{e_1, e_2, e_3 + ie_4\}$ is called ideal basis since $h(e_1, e_1) = -1$ and $h(e_2, e_2) = h(e_3, e_3) = 1$ and $h(e_2 + ie_4, e_2 + ie_4) = 0$

and this basis has the same signature as the subspace $W$.

**Lemma 3.7.** If $W \in Z_d$, then $W \cap W^\perp \subset W \cap W^\perp$.

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Proof. If \( v \in W \cap W \), then \( v \in \tilde{W} \cap W \) and

\[ h(v, \tilde{v}) = 0 = h(v, v), \]

so \( v \in W \cap W^\perp \). Therefore, \( W \cap \tilde{W} \subset W \cap W^\perp \).

**Theorem 3.8.** The \( SO(p,q) \)-orbits in \( Z_k \) are parameterized by signature \((n^+, n^-, d, r)\), i.e., given \( Y_1, Y_2 \in Z_k \) there exist \( g \in SO(p,q) \) with \( g(Y_1) = Y_2 \) if and only if

\[ \text{sign}(Y_1) = \text{sign}(Y_2). \]

Proof. Given \( Y_1, Y_2 \in Z_k \) such that

\[ \text{sign}(Y_1) = \text{sign}(Y_2) = (n^+, n^-, d, r), \]

then

\[ \dim(Y_1 \cap Y_1^\perp) = \dim(Y_2 \cap Y_2^\perp) = r. \]

Let \( \theta_1 \) be the real basis of \( Y_1 \cap Y_1^\perp \) and \( \theta_2 \) be the real basis of \( Y_2 \cap Y_2^\perp \), then

\[ u_i = u_i^+ + u_i^- \]

and by using the same procedure in the proof of Proposition 2.18 we can define \( T \) as

\[ T(u_i^+) = v_i^+ \text{ and } T(u_i^-) = v_i^-, \]

and then extend \( T \) by defining it in the other vectors similarly as in Theorem 2.20. This implies that

\[ T(Y_1) = Y_2 \text{ and } T(Y_1 \cap Y_1^\perp) = Y_2 \cap Y_2^\perp. \]

If \( F_1 = Y_1 \cap Y_1^\perp \) and \( F_2 = Y_2 \cap Y_2^\perp \), then \( F_1 = \overline{F_1} \) and \( F_2 = \overline{F_2} \), which implies that

\[ T(F_1) = F_2 = \overline{F_2} = \overline{T(F_1)} = \overline{T(F_1)} = T(F_1). \]

Therefore \( T(F_1) = \overline{T(F_1)} \) if and only if \( T = \overline{T} \). Hence \( T \in SO(p,q) \).

### 3.2 The closed \( SO(p,q) \)-Orbit in \( Z_k \)

In this section we will describe the signature of the closed \( SO(p,q) \)-Orbit in \( Z_k \) with a comparison between this closed orbit and the closed \( SU(p,q) \)-orbit.

**Proposition 3.9.** The closed \( SO(p,q) \)-orbit in \( Z_k \) is the set of all degenerate subspaces with maximal reality. i.e. with signature \((0,0,k)\).

Proof. By theorem 3.8 \( SO(p,q) \) acts transitively on this set.

Define \( Z^d \) to be the set of all subspaces of maximal reality in \( Z_k \). Let \( T \) be the closed \( SU(p,q) \)-orbit in \( Z \), then the set \( \tilde{T} = T \cap Z^d \) is closed in \( Z^d \). If \( O \) is the set of all degenerate subspaces with maximal reality then \( O = \tilde{T} \cap Z_k \), so \( O \) is closed in \( Z_k \).

### 3.3 Open \( SO(p,q) \)-Orbits in \( Z_k \)

In this section we will describe the signature of open \( SO(p,q) \)-Orbits in \( Z_k \) with a comparison between this open orbits and open \( SU(p,q) \)-orbits.

**Proposition 3.10.** Open \( SO(p,q) \)-orbits in \( Z_k \) are parametrized by the signature \((n^+, n^-, 0, 0)\).

Proof. Let \( D^\sim \) be an open \( SU(p,q) \)-orbit in \( Z \), then the set \( D = D^\sim \cap Z_k \) is open in \( Z_k \), and by Lemma 3.7

\[ D = D^\sim \cap Z_k \]

is the set of all nondegenerate subspaces of minimal reality \((r = 0)\), i.e of signature \((n^+, n^-, 0, 0)\).

**Theorem 3.11.** Each \( SU(p,q) \) open orbit contains a unique \( SO(p,q) \) open orbit.

Proof. By the proof of Proposition 3.10 if \( D^\sim \) is open orbit of \( SU(p,q) \) then \( D = D^\sim \cap Z_k \) is open of \( SO(p,q) \) in \( Z_k \).
Remark 3.12. Each $SU(p,q)$ open orbit has a nonempty intersection with $Z_k$.

3.4 Examples

Example 3.13. Let $Z = P_0((C))$ then $Z_1 = \{ x \in P_0(C) : -x^2 + \sum_{i=3}^6 x_i^2 = 0 \}$. Consider the action of $SO(3,4)$ on $Z_1$, then the open orbit of $SO(3,4)$ on $Z_1$ is

$$Z_k\cap P_0(\mathbb{R}) := \{ x \in P_0(\mathbb{R}) : h(x,x) = b(x,x) = 0 \}$$

and open orbits of $SO(3,4)$ in $Z_1$ are

$$D_1 = SO(3,4).(e_1 - ie_2) \subset D^+$$

where $D^+$ is an open orbit of $SU(3,4)$ on $Z$.

$$D_2 = SO(3,4).(e_4 - ie_5) \subset D^-$$

where $D^-$ is an open orbit of $SU(3,4)$ on $Z$.

Example 3.14. Let $Z = Gr(2,7)$, then $Z_2 = \{ x \in Gr(2,7) : x = b - isotropic \}$. Consider the action of $SO(3,4)$ on $Z_2$, then the open orbit of $SO(3,4)$ on $Z_2$ is

$$O := \{ x \in Gr(2,7) : x = a degenerate b-isotropic subspace \}$$

and open orbits of $SO(3,4)$ in $Z_2$ are

$$D_{1,1} = SO(3,4).<e_1-ie_2,e_4-ie_5> \subset \tilde{D}_{1,1}$$

where $\tilde{D}_{1,1}$ is an open orbit of $SU(3,4)$ on $Z$.

$$D_{0,2} = SO(3,4).<(e_4-ie_5),(e_6-ie_7) \subset \tilde{D}_{0,2}$$

where $\tilde{D}_{0,2}$ is an open orbit of $SU(3,4)$ on $Z$.

CONCLUSION

The signature of the subspaces plays an important role of parametrization the $G_0$-orbits. In this paper we proved that the $G_0$-orbits are parametrized by the signature of the subspaces in the orbit where $G_0 = SU(p,q)$ and $G_0 = SO(p,q)$. In the future studies, we can use this parametrization to understand the geometry of the Grassmannian spaces and any flag manifold.

REFERENCES


الملخص:

تعرف المجموعة $G$ على أنها إحدى مجموعات لي الكلاسيكية المركبة بحيث يرمز $G_0$ للمجموعة الحقيقية للGrupo. ليكن الفراغ $G/P$ هو فراغ جزئي على الفراغ $G_0$. بالنسبة للجبرية من الفراغ $Gr(k,n)$، فإن هذه المجموعة تقسم الفراغ $G_0$ إلى مجموعة من الفراغات الخضر من الفراغ $Gr(k,n)$، ثم تقسم الفراغ $G_0$ إلى مجموعة من الفراغات الخضر من الفراغ $Gr(k,n)$ وتجميع الفراغات الخضر من الفراغ $Gr(k,n)$ للفضاء من الفراغات الخضر من الفراغ $Gr(k,n)$. هذه المجموعة يتم تصنيفها بواسطة الهدف المحدد $O$ بناءً على فضاء المتجهات الرئيسية للنموذج $SP(2,p,2q)$ و $SO(p,q)$ لكل من المجموعات هيئتي$O$.