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Parametrization of the orbits of the real forms SU (p, q) and SO(p, q) in Grassmannian

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Abstract

Let G be a complex semi-simple Lie group with real form G0. Let Z = G/P be identified with Gr(k, n), the Grassmannian of k planes in Cn. Equivalently, P is a maximal parabolic subgroup defined by the dimension sequence (k, n − k). Consider the action of G0 on the Grassmannian Gr(k, n). It is known that G0 has only finitely many orbits in G/P and therefore it has a unique closed orbit and at least one open orbit ([2],[6]).

In this paper we will prove that the G0-orbits in Gr(k, n) are parameterized by signature, where G0 is SU (p, q) and SO(p, q) a real form of SL(n, C) and SO(p, q) respectively .

Keywords:
Grassmannian, orbit،
degenerate subspace ،
non-degenerate subspace،
isotropic.

1. Introduction:

A real form G_0 of a complex semisimple Lie group G has only finitely many orbits in any given compact G-homogeneous projective algebraic manifold $Z = G/Q$ and therefore there are open orbits, and a unique closed orbit γ^{cl} ([2],[6]).

In our paper we study the orbits of the real form SU (p, q) and $SO(p, q)$ when they acts on the Grassmannian spaces. In both cases we study the parametrization of the closed orbit and the open orbits which play a role in understanding the geometry of Grassmannian spaces, see ([2],[6] and [8]).

1 The parametrization of $SU(p,q)$-Orbits in Grassmannian $Gr(r,n)$

1.1 *Some basics in Linear Algebra*

Let V be a vector space C^n. Let $b(v, w)$, v, w V, be a bilinear form defined V and $h(v, w) = b(v, w)$ is its Hermiation form.$^{\overline{2}}$

Deftnition 2.1. Any two nonzero vectors $v_1, v_2 \in V$ are orthogonal if they satisfy that

$h(v_1, v_2) = 0.$

Deftnition 2.2. For any vector $v \in V$ if $h(v, v) > 0$, we call v a positive vector, if $h(v, v) < 0$, we call v a negative vector and if $h(v, v) = 0$, we call v an isotropic vector.

[2] For the basic information in this section and more see[3].

Definition 2.3. A set $\beta = \{v_1,...,v_n\}$ is an orthonormal basis of (V,h) if

1. $h(v_i,v_j) = 0$, $\forall i,j$ with $i \neq j$,

2. $h(v_i,v_i) = \pm 1$.

Remark 2.4. By using Definition 2.3, we can write V as $V = W^+ \oplus W^-$ where W^+ is a maximal positive subspace of V generated by positive vectors in β, and W^- is a maximal negative subspace of V generated by negative positive vectors in β. So W^- is the orthogonal complement of W^+.

Definition 2.5. A subspace $W \subseteq V$ is called nondegenerate if and only if $W \cap W^\perp = \{0\}$, and called maximally degenerate if $W \subset W^\perp$.

Notation 2.6. *We will denote the Hermiation form in the subspace W to be* $\tilde{h} = h|_{W\times W}$.

Definition 2.7. If V is nondegenerate space of dimension n such that $V = W^+ \oplus W^-$, then the signature of V is $sign(V) = (dimW^-, dimW^+)$.

Question 1. How can we find an orthogonal basis for any subspace $(W,\tilde{h})$ of dimension r?

To answer this question we have three cases:-

Case 1:- If W is maximally degenerate subspace, then

$$\forall w_i, w_j \in W, \qquad \tilde{h}(w_i,w_j) = 0.$$

So we need only to choose r-linearly independent vectors and then we finish.

Case 2:- If W is nondegenerate, fix a nonisotropic vector $w_1 \in W$. Let $X_1 = Cw_1$, then

$$X_1 \cap X_1^{\perp_{\tilde{h}}} = \{0\}.$$

Choose nonisotropic vector $w_2 \in X_1^{\perp_{\tilde{h}}}$ so $\tilde{h}(w_1,w_2) = 0$.

Let $X_2 = span\{w_1,w_2\}$, then

$$X_2 \cap X_2^{\perp_{\tilde{h}}} = \{0\}.$$

Choose nonisotropic vector $w_3 \in X_2^{\perp_{\tilde{h}}}$, so $\tilde{h}(w_3,w_i) = 0$ $\forall i = 1,2$.

Assume that we have $r-1$ nonisotropic orthogonal vectors $w_1,...,w_{r-1}$. Let

$$X_{r-1} = span\{w_1,...,w_{r-1}\},$$

then

$$X_{r-1} \cap X_{r-1}^{\perp_{\tilde{h}}} = \{0\}.$$

Choose nonisotropic vector $w_r \in X_{r-1}^{\perp_{\tilde{h}}}$, so

$$\tilde{h}(w_r,w_i) = 0, \ \forall i = 1,...r-1.$$

Hence we have r orthogonal vectors and these vectors spans W.

Case 3:- If $W \cap W^{\perp_{\tilde{h}}} \neq \{0\}$ and $W \cap W^{\perp_{\tilde{h}}} \subset W$, then

$$W = Q_t \oplus B_s$$

where $dimQ_t = t$, $dimB_s = s$, $t+s = r$ and

$B_s = W \cap W^\perp = W^{\perp_{\tilde{h}}}$ and $Q_t \cap Q_t^{\perp_{\tilde{h}}} = \{0\}$.

From case 1, any s linearly independent vectors $\{v_1,...,v_s\}$ from B_s spans B_s, and from case 2, we can find an orthogonal basis $\{w_1,...,w_t\}$ for Q_t.

Therefore, $\{v_1,...,v_s,w_1,...,w_r\}$ is an orthogonal basis for W.

Example 2.8. *Consider the vector space* $V = \mathbb{C}^6$ *.Let the hermaition form h to be defined as*

$$h(v,w) = b(v,\sigma(w)) = -\sum_{i=1}^{3} v_1\sigma(w_1) + \sum_{i=4}^{6} v_i\sigma(w_i)$$

.

Fix the standard basis $\{e_1,e_2,e_3,e_4,e_5,e_6\}$ *to be the orthonormal basis of* V.
The subspace $W_1 = span\{e_1 + e_6, e_2 + e_5\}$ *is a degenertate subspace with the orthonormal basis* $\{e_1 + e_6, e_2 + e_5\}$ *since* $h(e_1 + e_6, e_2 + e_5) = 0$, $h(e_1 + e_6, e_1 + e_6) = 0$ *and* $h(e_2+e_5, e_2+e_5) = 0$. *On the other hand, the subspace* $W_2 = span\{e_1,e_4\}$ *is nondegerate subspace with the orthonormal basis* $\{e_1,e_4\}$ *since* $h(e_1,e_4) = 0$, $h(e_1,e_1) = -1$ *and* $h(e_4,e_4) = 1$, *and* e_1,e_4 *is an orthonormal basis.*

Define the subspace $W_3 = W_1 \oplus W_2 = span\{e_1 + e_6, e_2 + e_5, e_1, e_4\}$, *by GramSchmidt Orthogonalisation Process we can find orthorormal basis for* W_3 *to be* $W_3 = span\{e_1, e_6, e_4, e_2 + e_5\}$ *which means that we can write* $W_3 = B \oplus Q$ *where* $B = span\{e_2 + e_5\}$ *and* $Q = span\{e_1,e_6,e_4\}$, *where* B *is degenerate subspace and* Q *is non-degenerate subspace.*

2.2 The Orbit Structure of the nondegenerate subspaces

Let (V,h) be the complex nondegenerate vector space C^n of signature (p,q), where $p+q = n$. Let $G = SL(n,C)$, and P be a maximal parabolic subgroup of G. In this case the homogenous space $Z = G/P$ can be identified with the set of all subspaces with dimension r called the Grassmannian $Gr(r,n)$. Define the bilinear form b on V to be

$$b(v,w) = -\sum_{i=1}^{q} v_i w_i + \sum_{i=q+1}^{n} v_i w_i$$

Consider the real form $G_0 = SU(p,q)$ of $SL(n,C)$ where $p + q = n$. The Hermitian form $h : C^n \times C^n \rightarrow C$ defied $SU(p,q)$ is the standard Hermitian form of signature (p,q) defined by

$$h(v,w) = -\sum_{i=1}^{q} v_i \overline{w}_i + \sum_{i=q+1}^{n} v_i \overline{w}_i, \quad \forall v,w \in \mathbb{C}^n$$
,

then $SU(p,q)$ is the group of isometries of V associated to h, that is if $T \in SU(p,q)$, then $h(Tv,Tw) = h(v,w)$. Let us concerned with the action of the real form $SU(p,q)$ on $Gr(r,n)$,

$SU(p,q) \times Gr(r,n) \rightarrow Gr(r,n)$.

By the results given by Wolf in [6], this action has finitely many orbits with a unique closed orbit and an open orbit exists. Here a question arise: How can we parameterize the orbits of this action? In the following sections we prove that the orbits of the above action parameterized by signature.

Definition 2.9. Given a subspace $(W,\tilde{h})$ of (V,h). We define a signature of the subspace W to be $sign(W) = (n^+,n^-,d)$ where n^+ is the dimension of maximal positive subspace of W and n^- is the dimension of maximal negative subspace of W and $d = dim(W \cap W^\perp) = dim(W^{\perp_{\tilde{h}}})$.

Definition 2.10. Given a subspace $(W,h|_W))$ of signature $sign(W) = (n^+,n^-,d)$ where $dimW = n^+ + n^- + d = k$. A set $\beta = \{v_1,\dots,v_k\}$ is a suborthonormal basis of $(W,h|_W))$ if:

1. $v_1,\dots,v_{n^+}$ are orthonormal positive vectors .
2. $v_{n^++1},\dots,v_{n^++n^-}$ are orthonormal negative vectors .
3. $v_{n^++n^-+1},\dots,v_k$ are linearly independent isotropic vectors and we will define the signature of this basis to be (n^+,n^-,d).

Remark 2.11. Any orthogonal basis of the subspace $(W,\tilde{h})$ has the same signature as the subspace signature.

Proposition 2.12. *Given* $X_1,X_2 \in Gr(r,n)$ *be nondegenerate subspaces such that* $sign(X_1) = sign(X_2)$, *then there exist* $g \in SU(p,q)$ *with* $g(X_1) = X_2$

Proof. Given two nondegenerate subspaces $X_1,X_2 \in Z$ with orthonormal bases $\beta_1 = \{v_1,\dots,v_r\}$ for X_1, and $\beta_2 = \{u_1,\dots,u_r\}$ for X_2. These two bases have the same signature and we can rearrange them to have firstly the positive vectors and then the negative.

Similarly for $X_1^{\perp}, X_2^{\perp} \in Z$ with orthonormal bases

$$\beta_1^{\perp} = \{v_{r+1}, ..., v_n\},\ \beta_2^{\perp} = \{u_{r+1}, ..., u_n\}$$

for $X_1^{\perp}, X_2^{\perp}$ respectively, these two bases $\beta_1^{\perp}, \beta_2^{\perp}$ have the same signature and we can rearrange them to have firstly the positive vectors and then the negative. So we can assume that v_i and u_i are both positive or both negative.

Now, since $V = X_1 \oplus X_1^{\perp} = X_2 \oplus X_2^{\perp}$, we can define a linear map

$T : V \longrightarrow V$ by $T(v_i) = u_i$ for $v_i \in \beta_1$ *and* $u_i \in \beta_2$,

and

$T(v\tilde{}_i) = u\tilde{}_i$

for $\tilde{}v_i \in \beta_1^{\perp}$ and $\tilde{}u_i \in \beta_2^{\perp}$, so $T(X_1) = X_2$.

To show that $h(T(w_i),T(w_j)) = h(w_i,w_j)$ $\forall w_i,w_j \in X_1$, start with bases vectors $v_i,v_j \in X_1$, if i 6= j, then

$$h(T(v_i),T(v_j)) = h(u_i,u_j) = 0 = h(v_i,v_j), \qquad (1)$$

and if $i = j$, then

$$h(T(v_i),T(v_i)) = h(u_i,u_i) = 1 = h(v_i,v_i). \qquad (2)$$

Let $w_1,w_2 \in V$ where $w_1 = \sum_{k=1}^{n} \alpha_k v_k$ and $w_2 = \sum_{t=1}^{n} \gamma_t v_t$, then

$$h(T(w_1),T(w_2)) = h(T(\sum_{k=1}^{n} \alpha_k v_k)\ ,T(\sum_{t=1}^{n} \gamma_t v_t))$$

$$= h(\sum_{k=1}^{n} \alpha_k T(v_k)\ ,\sum_{t=1}^{n} \gamma_t T(v_t))$$

$$= \sum_{k=1}^{n} \sum_{t=1}^{n} \alpha_k \gamma_t h(T(v_k)\ ,T(v_t))$$

$$\text{By (1) and (2)} \qquad = \sum_{k=1}^{n} \sum_{t=1}^{n} \alpha_k \gamma_t h(v_k\ ,v_t)$$

$$= h(\sum_{k=1}^{n} \alpha_k v_k\ ,\sum_{t=1}^{n} \gamma_t v_t) = h(w_1, w_2).$$

Therefore $T \in SU(p,q)$.

Example 2.13. *Let* $G = SU(1,2)$ *and define the non-degenerate subspaces* $X_1 = span\{e_1,e_3\}$ *and* $X_2 = span\{e_1,e_2\}$*. The signature* $sign(X_1) = sign(X_2) = (1,1)$*. Choose the matrix* $g \in SU(1,2)$ *where*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $g.e_1 = e_1$ *and* $g.e_3 = e_2$*, which means that*

$g.X_1 = X_2$.

2.3 The Orbit Structure of the degenerate subspaces

Recall that the signature of the vector space

$V = \mathbb{C}^n$ with respect to the group $SU(p,q)$ is (p,q).

Lemma 2.14. *Given any isotropic vector* $v \in V$*, then*

$v = v^+ + v^-$ *where* v^+,v^- *are orthogonal positive*

and negative vectors respectively.

Proof. Assume that V_1 is a maximal positive

subspace

$$h(w_1, w_1) \quad = \quad 0$$

$$\begin{aligned} &= h(v^+ + v^-, v^+ + v^-) \\ &= h(v^+, v^+) + h(v^+, v^-) + h(v^-, v^+) + h(v^-, v^-) \\ &= h(v^+, v^+) + h(v^-, v^-), \end{aligned}$$

in V then $V_1^\perp$ is a maximal negative subspace where $V_1 \cap V_1^\perp = \{0\}$ because *sign*(V) = (p,q,0). Then any vector $x \in V$ is uniquely represented as $x = t^+ + t^-$, where $t^+ \in V_1$ and $t^- \in V_1^\perp$. Therefore any isotropic vector $v \in V$ can be written as $v = v^+ + v^-$

but

$$\begin{aligned} h(w_1, w_2) &= 0 \\ &= h(v^+ + v^-, \alpha_1 v^+ + \alpha_2 v^-) \\ &= h(v^+, \alpha_1 v^+) + h(v^+, \alpha_2 v^-) + h(v^-, \alpha_1 v^+) + h(v^-, \alpha_2 v^-) \\ &= \alpha_1 h(v^+, v^+) + \alpha_2 h(v^-, v^-) \end{aligned}$$

where $v^+ \in V_1$ and $v^- \in V_1^\perp$.

Notation 2.15. *In the following lemmas* E_i *will be a nondegerate subspace with sign*(E_i) = (1,1,0).

Lemma 2.16. *Given two orthogonal isotropic vectors* w_1, w_2*, where* $w_1 = v^+ + v^-$ *and* $E_1 = span\{v^+, v^-\}$*, then* $w_2 \in E_1^\perp$.

Proof. Firstly, since

so $h(v^+, v^+) = -h(v^-, v^-)$.

Assume that $w_2 \in E_1$ then $w_2 = \alpha_1 v^+ + \alpha_2 v^-$, $\alpha_1, \alpha_2 \in C - \{0\}$,

which imples that $\alpha_1 = \alpha_2$ and $w_2 = \alpha_1 w_1$ which is a contradiction.

Also if $w_2 = a_1 v^+ + a_2 \hat{v}^-, \quad \hat{v}^- \in E_1^\perp$,

or $w_2 = a_1 \hat{v}^+ + a_2 v^-, \quad \hat{v}^+ \in E_1^\perp$

, then

$h(w_1, w_2) \neq 0.$

Therefore, $w_2 \in E_1^\perp$.

Lemma 2.17. *Let D be a degenerate subspace with dimension r, there exist r subspaces* $E_1, E_2, \ldots, E_r$ *such that*

$E_i \cap E_j = \{0\}$ $1 \leq i < j \leq r$ *and*

$$D \subset E_1 \oplus E_2 \oplus \ldots \oplus E_r.$$

Proof. We will prove it by induction.

Step 1: If dim$D$ = 1, then $D = span\{w_1\}$ where w is an isotropic vector, so by Lemma 2.14 $w_1 = v^+ + v^-$ where $v^+ \perp v^-$ and $D \subset E_1 = span\{v^+, v^-\}$.

Step 2: If dim$D$ =2, then $D = span\{w_1, w_2\}$ where

$h(w_1, w_2) = 0.$

By step 1,

$W = span\{w_1\} \subset E_1$

and by Lemma 2.16 $w_2 \in E_1^\perp$, then by Lemma 2.14 there exist $\hat{}v^+, \hat{v}^- \in E_1^\perp$ such that w2 = v^+ + v^−. So we have a nondegenerate subspace $E_2 = span\{v^{\hat{}+}, v^{\hat{}-}\}$ where $w_2 \in E_2$, and $E_1 \cap E_2 = \{0\}$, then

$D \subset E_1 \oplus E_2$.

Step 3: Assume that the lemma is true if dim$D < r$.

Step 4: If dim$D = r$.

Choose any vector w in D, then D = span{w}⊕D˜ where D˜ is the orthogonal complement of span{w} in D, So D˜ is a subgroup of D with dimD˜ = r − 1 and by step 3 there exist $E_1, E_2, \ldots, E_{r-1}$ such that

$E_i \cap E_j = \{0\}$ $1 \leq i < j \leq r - 1$ and

$$\tilde{D} \subset E_1 \oplus E_2 \oplus \ldots \oplus E_{r-1}.$$

By Lemma 2.16, $w \in E_i^\perp \quad \forall i,$ so

$w \in (E_1 \oplus E_2 \oplus .. \oplus E_{r-1})^{\perp}$

again by step 1, $w = \tilde{v}^{+} + \tilde{v}^{-}$ where $\tilde{v}^{+} \perp \tilde{v}^{-}$ and $span\{w\} \subset E_r = span\{\tilde{v}^{+}, \tilde{v}^{-}\}$.

Since $D = span\{w\} \oplus \tilde{D}$, then

$D \subset E_1 \oplus E_2 \oplus .. \oplus E_r$.

Proposition 2.18. *Given $Y_1, Y_2 \in Gr(r,n)$ be degenerate subspaces, i.e.*

$$sign(Y_1) = sign(Y_2) = (0,0,r),$$

then there exist $g \in SU(p,q)$ with $g(Y_1) = Y_2$

Proof. Assume we have two degenerate subspaces Y_1, Y_2. By Lemma 2.17 there exist r subspaces

$E_1, E_2, ..., E_r$

such that $E_i \cap E_j = \{0\}$, $1 \leq i < j \leq r$, and

$Y_1 \subset E_1 \oplus E_2 \oplus .. \oplus E_r$,

where $E_i = span\{v_i^{+}, v_i^{-}\}$, then we have $2r$ orthogonal vectors of V,

$\beta_1 = \{v_1^{+}, v_1^{-}, ..., v_r^{+}, v_r^{-}\}$.

it, namely $\hat{\beta}_1$, where it has $(p - r)$ positive vectors and $(q - r)$ negative vectors. We can rearrange the vectors in $\hat{\beta}_1$ to have the positive vectors firstly, i.e.

$$\hat{\beta}_1 = \{v_{r+1}^{+}, ..., v_p^{+}, v_{r+1}^{-}, ..., v_q^{-}\}.$$

Similarly, By Lemma 2.17 there exist r subspaces $\tilde{E}_1, \tilde{E}_2, ..., \tilde{E}_r$ such that

$\tilde{E}_i \cap \tilde{E}_j = \{0\}$ $1 \leq i < j \leq r$

and

$Y_2 \subset \tilde{E}_1 \oplus \tilde{E}_2 \oplus .. \oplus \tilde{E}_r$,

where $\tilde{E}_i = span\{u_i^{+}, u_i^{-}\}$, then we have $2r$ orthogonal vectors of V,

$\beta_2 = \{u_1^{+}, u_1^{-}, ..., u_r^{+}, u_r^{-}\}$.

Let

$$Q_2 = span\{u_1^{+}, u_1^{-}, ..., u_r^{+}, u_r^{-}\},$$

then $V = Q_2 \oplus Q_2^{\perp}$, so we can extend β_2 to a basis for V be adding the basis of $Q_2^{\perp}$ to it, namely $\hat{\beta}_2$, where it has (p − r) positive vectors, and (q − r) negative vectors and we can rearrange the vectors in $\hat{\beta}_2$ to have the positive vectors firstly i,e.

$$\hat{\beta}_2 = \{u_{r+1}^{+}, ..., u_p^{+}, u_{r+1}^{-}, ..., u_q^{-}\}.$$

Finally, we can define a linear map

$g : V \rightarrow V$ by $g(v_i^{+}) = u_i^{+}$, $g(v_i^{-}) = u_i^{-} \forall i,j$,
then $g(Y_1) = Y_2$, and by using the same method we use in the proof of Proposition 2.12

$$h(g(w_1), g(w_2)) = h(w_1, w_2).$$

Therefore $g \in SU(p,q)$.

Example 2.19. *Let $G = SU(1,2)$ and define the subspaces $Y_1 = span\{e_1 + e_3\}$ and $Y_2 = span\{e_1 + e_2\}$. The signature $sign(X_1) = sign(X_2) = (0,0,1)$. Choose the matrix $g \in SU(1,2)$ where*

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $g.e_1 = e_1$ *and* $g.e_3 = e_2$*, which means that*

$g.Y_1 = Y_2$.

2.4 General Result

In this section we will prove our main theorem .

Theorem 2.20. *The* $SU(p,q)$*-orbit in* $Gr(r,n)$ *are parameterized by signature. That is given* $W_1, W_2 \in Gr(r,n)$ *there exist* $g \in SU(p,q)$ *with* $g(W_1) = W_2$ *if and only if* $sign(W_1) = sign(W_2)$*.*

Proof. Let $W_1, W_2 \in Gr(r,n)$, then

$W_1 = Q_1 \oplus B_1$ and $W_2 = Q_2 \oplus B_2$

where Q_1, Q_2 are nondegenerate subspaces and $B_1 = W_1 \cap W_1^{\perp}$ and $B_2 = W_2 \cap W_2^{\perp}$.

By Proposition 2.12, there exist $g \in SU(p,q)$ such that

$g(Q_1) = Q_2$ and $g(Q_1^{\perp}) = Q_2^{\perp}$. But $g(B_1) = \tilde{B}_1 \subseteq Q_2^{\perp}$, by Proposition 2.18

there exist $\hat{g} \in SU(p,q)$ such that $\hat{g}(\tilde{B}_1) = B_2$. So we can define our map $\psi : V \dashrightarrow V$ as

$$\psi = (Id \oplus \hat{g}) \circ g,$$

where *Id* is the identity matrix, then

$\psi(W_1) = W_2$. Since $Id, g, \hat{g}$ are all in $SU(p,q)$,

then $\psi \in SU(p,q)$.

2 THE PARAMETRIZATION OF *SO*(*P*,*Q*)-ORBITS IN ISOTROPIC GRASSMANNIAN Z_K

In this section we will prove that $SO(p,q)$-orbits in Z_k are parameterized by signature, where Z_k is the isotropic Grassmannian.

Consider the semisimple Lie group $G = SO(n,\mathbb{C})$ where $G_0 = SO(p,q)$, with complex bilinear form defined by

$$b(v,w) = -\sum_{i=1}^{p} v_i w_i + \sum_{i=p+1}^{n} v_i w_i$$

then the Hermitian form which defines the real form is

$$h(v,w) = b(v,\bar{w})$$

, so G_0 is the subgroup of operators T in G satisfy $T = \bar{T}$. Let (V,h) be the complex nondegenerate vector space of signature (p,q). Define Z_k to be the isotropic Grassmannian which is the set of all-isotropic k- planes in $\mathbb{C}^n$ where

$1 \le k \le \lfloor \frac{n}{2} \rfloor$.

Consider the action of the real form $SO(p,q)$ on the flag manifold Z_k,

$SO(p,q) \times Z_k \dashrightarrow Z_k$

then $SO(p,q)$ has a unique closed orbit and finitely many open orbits on Z_k. In the following sections we will proof that the orbits of this action parameterized by signature.

3.1 Orbit structure

In this sections we will prove that the *SO*(*p*,*q*)-orbits in Z_k are parameterized by signature *sign*(*W*) = (n^+,n^-,d,r) by using our previous results about *SU*(*p*,*q*).

Definition 3.1. Given a subspace $(W,h|_W)$ of (*V*,*h*), We define a signature of *W* to be *sign*(*W*) = (n^+,n^-,d,r) where n^+ is the dimension of maximal positive subspace of *W* and n^- is the dimension of maximal negative subspace of *W* and

$$d = \dim(W \cap W^{\perp}) = \dim(W^{\perp_{h|_W}})$$

and $r = \dim(W \cap W)$.

We used this signature to parameterize the *SO*(*p*,*q*) orbits while *SU*(*p*,*q*) orbits parameterized by only (n^+,n^-,d).

Remark 3.2. Given a non degenerate space with orthogonal basis *β*, then we can write *V* as $V = W^+ \oplus W^-$ where W^+ is a maximal positive subspace of *V* generated by positive vectors in *β*, and W^- is a maximal negative subspace of *V* generated by negative positive vectors in *β*, and W^- is the orthogonal complement of W^+.

Definition 3.3. A subspace *W* is said to

be of maximal reality if *W* = *W* and

$W = W_R \oplus iW_R$ where $W_R \subset$R .

Lemma 3.4. *Any subspace of maximal reality has a real basis.*

Proof. Given a subspace *X* of maximal reality then

$X = \bar{X}$, i.e., $X = X_R \oplus iX_R$ where X_R is a real subspace , so the basis of X_R is basis of *X* but the basis of X_R is real that mean we can find a real basis *β* for X where $u = \bar{u}\ \forall u \in \beta$.

If $(W,h|_W))$ is of signature *sign*(*W*) = (n^+,n^-,d,r), then $W = W^+ \oplus W^- \oplus D$ where W^+ is a maximal positive subspace with $\dim W^+ = n^+$, W^- is a maximal negative subspace with $\dim W^- = n^-$ and $D = W \cap W^{\perp}$ with $\dim D = d$. In this cases $W^+ \cap \overline{W^+}$, and $W^- \cap \overline{W^-}$ and $D \cap \overline{D}$ all of them have real bases β_1,β_2 and β_3 respectively. We can extend each of these bases to get bases for W^+,W^-, *D* respectively. So we can define a basis of $(W,h|_W)$, namely an ideal basis, to be as following:

Definition 3.5. Given a *k*−subspace $(W,h|_W)$ of signature *sign*(*W*) = (n^+,n^-,d,r) where $\dim W = n^+ + n^- + d = k$. A set $\beta = \{v_1,\dots,v_n\}$ is an ideal basis of $(W,h|_W))$ if:

1. $v_1,\dots,v_{n^+}$ are orthonormal positive vectors with r_1 vectors of them are real.
2. $v_{n^+ +1},\dots,v_{n^+ + n^-}$ are orthonormal negative vectors with r_2 vectors of them are real.
3. $v_{n^+ + n^- +1},\dots,v_k$ are linearly independent vectors with r_3 vectors of them are real.
4. $r_1 + r_2 + r_3 = r$ and we will define the signature of this basis to be (n^+,n^-,d,r).

Example 3.6. *Let G* = *SU*(2,3) *with the hermaition form h defined as*

$$h(v,w) = b(v,\sigma(w)) = -\sum_{i=1}^{2} v_1\sigma(w_1) + \sum_{i=3}^{5} v_i\sigma(w_i).$$

Define the subspaces W = *span*$\{e_1,e_4,e_5,e_2+ie_3\}$ *with signature sign*(*W*) = (2,1,1,3). *The basis* $\{e_1,e_4,e_5,e_2+ie_3\}$ *is called ideal basis since* $h(e_1,e_1) = -1$ *and* $h(e_4,e_4) = h(e_5,e_5) = 1$ *and* $h(e_2+ie_3,e_2+ie_3) = 0$

and this basis has the same signature as the subspace W.

Lemma 3.7. *If* $W \in Z_k$ *then*

$$W \cap \overline{W} \subset W \cap W^{\perp} .$$

Proof. If $v \in W \cap W$, then $v \in \overline{W} \cap W$ and

$$b(v, \overline{v}) = 0 = h(v, v),$$

so $v \in W \cap W^{\perp}$. Therefore, $W \cap \overline{W} \subset W \cap W^{\perp}$.

Theorem 3.8. *The $SO(p,q)$-orbits in Z_k are parameterized by signature (n^+,n^-,d,r). i,e., given Y_1, $Y_2 \in Z_k$ there exist $g \in SO(p,q)$ with $g(Y_1) = Y_2$ if and only if $sign(Y_1) = sign(Y_2)$.*

Proof. Given $Y_1, Y_2 \in Z_k$ such that $sign(Y_1) = sign(Y_2) = (n^+,n^-,d,r)$, then

$$\dim(Y_1 \cap \overline{Y_1}) = \dim(Y_2 \cap \overline{Y_2}) = r.$$

Let β_1 be the real basis of $Y_1 \cap \overline{Y_1}$ and β_2 be the real basis of $Y_2 \cap \overline{Y_2}$, then
$u_i = u_i \forall u_i \in \beta_1$, $1 \le i \le r$ and
$v_i = v_i \forall v_i \in \beta_2$, $1 \le i \le r$. Since

$$Y_i \cap \overline{Y_i} \subset Y_i \cap Y_i^{\perp}$$

for $i \in \{1,2\}$, then by Lemma 2.14,

$u_i = u^+_i + u^-_i$ and by using the same procedure in the proof of Proposition 2.18 we can define T as

$$T(u^+_i) = v_i^+ \text{ and } T(u^-_i) = v_i^-,$$

and then extend T by defining it in the other vectors similarly as in Theorem 2.20. This implies that

$$T(Y_1) = Y_2 \quad \text{and} \quad T(Y_1 \cap \overline{Y_1}) = Y_2 \cap \overline{Y_2}.$$

If $F_1 = Y_1 \cap \overline{Y_1}$ and $F_2 = Y_2 \cap \overline{Y_2}$, then $F_1 = \overline{F_1}$ and $F_2 = \overline{F_2}$, which implies that

$$T(F_1) = F_2 = \overline{F_2} = \overline{T(F_1)} = \overline{T}(\overline{F_1}) = \overline{T}(F_1).$$

Therefore $T(F_1) = \overline{T}(F_1)$ if and only if $T = \overline{T}$. Hence $T \in SO(p,q)$.

3.2 *The closed* **SO(p,q)***-Orbit in* $\mathbf{Z_k}$

In this section we will describe the signature of the closed SO(p,q)-Orbit in Z_k with a comparison between this closed orbit and the closed $SU(p,q)$-orbit.

Proposition 3.9. *The closed $SO(p,q)$-orbit in Z_k is the set of all degenerate subspaces with maximal reality. i.e. with signature* $(0,0,k,k)$.

Proof. By theorem 3.8 $SO(p,q)$ acts transitively

on this set.

Define Z^R to be the set of all subspaces of maximal reality in Z_k. Let T be the closed $SU(p,q)$-orbit in Z, then the set $\tilde{T} = T \cap Z^R$ is closed in Z^R. If O is the set of all degenerate subspaces with maximal reality then $O = \tilde{T} \cap Z_k$, so O is closed in Z_k.

3.3 *Open* **SO(p,q)***-Orbits in* $\mathbf{Z_k}$

In this section we will discribe the signature of open $SO(p,q)$-Orbits in Z_k with a comparison between this open orbits and open $SU(p,q)$-orbits.

Proposition 3.10. *Open $SO(p,q)$-orbits in Z_k are parametrized by the signature* $(n^+, n^-, 0, 0)$.

Proof. Let $\tilde{D}$ be an open $SU(p,q)$-orbit in Z, then the set $D = \tilde{D} \cap Z_k$ is open in Z_k, and by Lemma 3.7 $D = \tilde{D} \cap Z_k$ is the set of all nondegenerate subspaces of minimal reality ($r = 0$), i.e of signature $(n^+,n^-,0,0)$.

Theorem 3.11. *Each $SU(p,q)$ open orbit contains a unique $SO(p,q)$ open orbit.*

Proof. By the proof of Proposition 3.10 if $\tilde{D}$ is open orbit of $SU(p,q)$ then $D = \tilde{D} \cap Z_k$ is open of $SO(p,q)$ in Z_k.

Remark 3.12. Each $SU(p,q)$ open orbit has a nonempty intersection with Z_k.

3.4 Examples

Example 3.13. *Let* $Z = P_6((C))$ *then* $Z_1 = \{x \in P_6(\mathbb{C}) : -\sum_{i=1}^{2} x_i^2 + \sum_{i=3}^{6} x_i^2 = 0\}$. *Consider the action of* $SO(3,4)$ *on* Z_1, *then the closed orbit of* $SO(3,4)$ *on* Z_1 *is*

$$Z_k \cap P_6(\mathbb{R}) := \{x \in P_6(\mathbb{R}) : h(x,x) = b(x,x) = 0\}$$

and open orbits of $SO(3,4)$ *in* Z_1 *are*

$$D_1 = SO(3,4).(e_1 - ie_2) \subset D^+$$

where D^+ *is an open orbit of* $SU(3,4)$ *on* Z.

$$D_2 = SO(3,4).(e_4 - ie_5) \subset D^-$$

where D^- *is an open orbit of* $SU(3,4)$ *on* Z.

Example 3.14. *Let* $Z = Gr(2,7)$, *then* $Z_2 = \{x \in Gr(2,7) : x \text{ is } b-isotropic\}$. *Consider the action of* $SO(3,4)$ *on* Z_2, *then the closed orbit of* $SO(3,4)$ *on* Z_2 *is*

$O := \{x \in Gr(2,7) : x$ *is a degenerate b–isotropic subspace*$\}$

and open orbits of $SO(3,4)$ *in* Z_2 *are*

$$D_{1,1} = SO(3,4). < (e_1 - ie_2), (e_4 - ie_5) > \subset \tilde{D}_{1,1}$$

where $\tilde{D}_{1,1}$ *is an open orbit of* $SU(3,4)$ *on* Z.

$$D_{0,2} = SO(3,4). < (e_4 - ie_5), (e_6 - ie_7) > \subset \tilde{D}_{0,2}$$

where $\tilde{D}_{0,2}$ *is an open orbit of* $SU(3,4)$ *on* Z.

CONCLUSION

The signature of the subspaces plays an important role of parametrization the G_0orbits. In this paper we proved that the G_0-orbits are parametrized by the signature of the subspaces in the orbit where $G_0 = SU(p,q)$ and $G_0 = SO(p,q)$. In the future studies, we can use this parametrization to understand the geometry of the Grassmannian spaces and any flag manifold.

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الملخص:

تعرف المجموعة G على انها احدى مجموعات لي الكلاسيكية المركبة بحيث يرمز للمجموعة الحقيقي لها بالرمز Go . ليكن الفراغ G/P هو فراغ جرازمانيين Gr(k,n) من الدرجة k ، و المعرف بانه مجموعة الفراغات الجزئية من الفراغ من الدرجة k . نفرض ان Go تؤثر على الفراغ Gr(k,n) فإ ن هذه المجموعة تقسم الفراغ Gr(k,n) إلى مجموعة منتهية من المدارات الغير متقطعة منها المفتوحة وواحدة فقط منها مغلقة توبولوجيا. في هذا البحث قمنا بعمل تصنيف لهذه المدارات بناءا على إشارة المتجهات الرئيسة و المسمى بال signature of flag domains لكل من المجموعتين الحقيقيتين SO(p,q) و SP(2p,2q) .