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Some Properties of Centrality in A Complex Banach Algebra

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Abstract: This paper proves that under some conditions the σ-quasi centrality is preserved under quotient mapping . Also two examples of two complex Banach algebras an given:

(a) Q(A), $Q_\sigma(A)$ and $Q_\rho(A)$ need not be subsets of Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ respectively.

(b) Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ need not be subsets of Q(A), $Q_\sigma(A)$ and $Q_\rho(A)$ respectively.

For a closed subalgebra B of a unital complex Banach algebra A. But we also show that under some conditions Q(A) , $Q_\sigma(A)$ and $Q_\rho(A)$ are subsets of Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ respectively; moreover we show that under some conditions Q(B) is a subset of Q(A).

Where Q(A) is the set of quasi central elements in A, $Q_\sigma(A)$ is the set of σ-quasi central elements in A, and $Q_\rho(A)$ is the set of ρ-quasi central elements in A.

Finally we show that under some conditions $\bigcap_{i=1}^{n} \boldsymbol{Q(B_i)}$ = Q(A), $\bigcap_{i=1}^{n} \boldsymbol{Q_\sigma(B_i)}$ = $Q_\sigma(A)$ and $\bigcap_{i=1}^{n} \boldsymbol{Q_\rho(B_i)}$ = $Q_\rho(A)$, for $\boldsymbol{B_1, B_2, \ldots, B_n}$ as closed subalgebras of a unital complex Banach algebra A.

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1. Introduction

The purpose of this paper is to study under what conditions, the σ-quasi centrality is preserved under the quotient mapping, and to study the relation between the centrality of the complex Banach algebra and the centrality of its closed subalgebras.

Throughout this paper all linear spaces and algebras are assumed to be defined over ¢, the field of complex numbers.

Let A be any complex Banach algebra, then the center of A is denoted by Z(A) = { a∈A : ax = xa for all x∈A }.

In [4] and [5] Rennison defined the set of all quasi central elements in A by Q(A)= $\bigcup_{k\geq 1}$ {a∈A: ||x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈¢ }.

The set of all σ-quasi central elements in A is denoted by $Q_\sigma(A) = \bigcup_{k\geq 1}$ { a∈A : || x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈ $\rho_A(a)$ }, and he has shown that Z(A) ⊆ Q(A) ⊆ $Q_\sigma(A)$.

In [3] Hussein and Asiad define the set of all ρ-quasi central elements in A by $Q_\rho(A) = \bigcup_{k\geq 1}$ { a∈A : || x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈ $\boldsymbol{\sigma_A(a)}$ }, and they have shown that Q(A) ⊆ $Q_\rho(A)$.

2. The σ-quasi center in a quotient Banach algebra

In [5] Rennison shows that " if A is a Banach algebra over the complex field ¢, I is a closed ideal of A that has a bounded right approximate identity and a is quasi central element of A, then I + a is quasi central element in A/I".

In [1] As'ad has shown that under the same conditions the above what? holds for ρ-quasi centrality.

But we don't know that under the same conditions the above holds or not for σ-quasi centrality. In the following theorem a condition is added through which a similar result is achieved. It is a main result of this section.

2.1 Theorem:- Let A be a Banach algebra over the complex field ¢, let I be a closed ideal of A, let I has a bounded right approximate identity and suppose that $\rho_{A/I}(I+a) \subseteq \rho_A(a)$, where a is σ-quasi central element of A. Then I + a is σ-quasi central element in A/I.

Proof

Let $a \in Q_\sigma(A)$, $x \in A$, $\lambda \in \rho_A(a)$ and suppose that $\{e_i\}$ is a right approximate identity in I with $\|e_i\| \le M$ for all I.

Then given ε>0, choose h∈I with ||(λ-a) x-h || < || I + (λ - a) x || +ε, and then choose e_i with $\| h - he_i \| < \varepsilon$.

Since $x e_i (\lambda - a) \in I$, then for all I + x ∈ A/I and $\lambda \in \rho_A(a)$,

$\| (I + x)(\lambda - I - a) \| = \| I + x (\lambda - a) \| \le \| x(\lambda - a) - xe_i (\lambda - a) \|$

$= \| (x - xe_i)(\lambda - a) \| \le k \| (\lambda - a)(x - xe_i) \|$, for some k ≥1

$= k \| (\lambda - a)x - (\lambda - a)xe_i \|$

$\le k \{ \| (\lambda - a)x - h \| + \| h - he_i \| + \| he_i - (\lambda - a)xe_i \| \}$

$< k \{ (M + 1) \| I + (\lambda - a)x \| + (M + 2)\varepsilon \}$.

Since ε is arbitrary and that $\rho_{A/I}(I + a) \subseteq \rho_A(a)$, we obtain for all I + x ε A/I and $\lambda \in \rho_{A/I}(I + a)$,

$\|(I + x)(\lambda - I - a)\| \le k(M + 1)\| I + (\lambda - a)x\| = k(M+1)\|(\lambda - I - a)(I + x)\|$.

Hence I + a is a σ-quasi central element in A/I.

3. Relation ship between centrality of Banach algebra and centrality of its closed subalgebras

The following examples show:

(a) Q(A), $Q_\sigma(A)$ and $Q_\rho(A)$ need not be subsets of Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ respectively .

(b) Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ need not be subsets of Q(A), $Q_\sigma(A)$ and $Q_\rho(A)$ respectively . Where B is a closed subalgebra of a unital complex Banach algebra A.

3.1 Example: There is a unital complex Banach algebra A and a closed subalgebra B, with

(a) *$a \in Q(A)$, but $a \notin Q(B)$* .

(b) *$a \in Q_\sigma(A)$ but $a \notin Q_\sigma(B)$* .

(c) *$a \in Q_\rho(A)$ but $a \notin Q_\rho(B)$*.

Construction:-

Let A = { β = $\begin{pmatrix} x & y \\ 0 & w \end{pmatrix}$: x,y, w ∈¢ } and define

|| β || = max { | x | + | y | , | w | }, which makes A as a unital complex Banach algebra .

Let B = { α = $\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$: x,y∈¢ } which is a closed subalgebra of A , and

let a = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then a ∈Z(A) , and so a ∈Q(A) , a∈ $Q_\sigma(A)$ and a∈ $Q_\rho(A)$.

But it is clear that a does not belong to any of Q(B), $Q_\sigma(B)$ or $Q_\rho(B)$, because a∉B.

3.2 Example: There is a unital complex Banach algebra A and a closed subalgebra B, with

(a) *a∈ Q(B) but a∉ Q(A).*

(b) $a \in Q_\sigma(B)$ *but* $a \notin Q_\sigma(A)$.

(c) $a \in Q_\rho(B)$ *but* $a \notin Q_\rho(A)$.

Construction

Let X be a Banach space over ¢, and let BL(X) be the space of all bounded linear operators on X with pointwise addition and scalar multiplication but the product as a composition, then BL(X) becomes a complex Banach algebra with unity I under the norm $\|T\| = \sup_{\substack{x\in X \\ \|x\|=1}} \| Tx \|$.

Let A = { β = $\begin{pmatrix} f & g \\ 0 & h \end{pmatrix}$: f, g, h ∈BL(X) } and define

|| β || = max { || f || + || g || , || h || }, which makes A as a unital complex Banach algebra.

Let B = { α = $\begin{pmatrix} f & g \\ 0 & f \end{pmatrix}$: f,g∈BL(X) } which is a closed subalgebra of A,

and let a = $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$, then a ∈Z(B), and so a ∈Q(B) , a∈ $Q_\sigma(B)$ and a∈ $Q_\rho(B)$.

Then $\sigma_A(a)$ = { $\lambda \in ¢$: $\begin{pmatrix} \lambda I - I & I \\ 0 & \lambda I - I \end{pmatrix}^{-1}$ does not exist } = {1}.

Now take x = $\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$, then ||(λ-a)x|| = $\left\| \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \right\|$ = 0, and

|| x(λ-a) || = $\left\| \begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \right\|$ = 1.

Hence there exist x=$\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}$ ∈A, and λ=1∈ $\sigma_A(a)$,with ||x(λ-a)||> k||(λ-a)x || for all k ≥ 1.

So that a = $\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \notin Q_\rho(A)$ and so a∉ Q(A).

But $\sigma_A(a)$ is a countable set, then by [3, Corollary] a∉ $Q_\sigma(A)$.

The following theorem is one of the main results in this section: It shows that under some conditions Q(A), $Q_\sigma(A)$ and $Q_\rho(A)$ are subsets of Q(B), $Q_\sigma(B)$ and $Q_\rho(B)$ respectively .

3.3 Theorem: Let A be a complex Banach algebra with unity e, B a closed subalgebra of A and e, a∈B.

(a) *If a∈Q(A) , then a∈ Q(B).*

(b) *If a∈ $Q_\sigma(A)$, then a∈ $Q_\sigma(B)$.*

(c) *if a∈* $Q_\rho(A)$, $\rho_A(a)$ is connected , then *a*∈ $Q_\rho(B)$.

Proof

(i) Let a be any element in Q(A), then there exists k ≥ 1 such that,
|| x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈¢, then
|| x (λ - a) || ≤ k || (λ - a) x || for all x∈B and all λ∈¢ .
But a∈B, then a∈ Q(B) .

(ii) Let a∈ $Q_\sigma(A)$, then there exists k ≥ 1 such that,
|| x (λ - a) || ≤ k || (λ - a) x || for all x∈A and all λ∈ $\rho_A(a)$.
But a∈B, the unital closed subalgebra of A, then by [6, Theorem 10.18], $\sigma_A(a) \subseteq \sigma_B(a)$, and so $\rho_B(a) \subseteq \rho_A(a)$.
Hence there exists k ≥ 1 such that,

$\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in \rho_B(a)$, that means $a \in Q_\sigma(B)$.

(iii) Let $a \in Q_\rho(A)$, then there exists $k \geq 1$ such that,
$\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in A$ and all $\lambda \in \sigma_A(a)$.
But $a \in B$, the unital closed subalgebra of A and $\rho_A(a)$ is connected, then by [6, corollary pp.239], $\sigma_A(a) = \sigma_B(a)$.
Hence the result follows.

Corollary: If A is a complex Banach algebra with unity e and $\Im = \{B : B \subseteq A\}$ is a collection of closed subalgebras of A and a, e $\in \bigcap_{B \in \Im} B$, then

(a) *If* $a \in Q(A)$, *then* $a \in \bigcap_{B \in \Im} Q(B)$.

(b) *If* $a \in Q_\sigma(A)$, then $a \in \bigcap_{B \in \Im} Q_\sigma(B)$.

(c) *If* $a \in Q_\rho(A)$ and $\rho_A(a)$ is connected, *then* $a \in \bigcap_{B \in \Im} Q_\rho(B)$.

Proof :
Follows directly from Theorem 3.3 above.
The following proposition shows that under some conditions we can have Q(B) can be a subset of Q(A) .

3.4 Proposition: Let A be a complex Banach algebra with unity e, and let B be any dense subalgebra of A , then $Q(B) \subseteq Q(A)$.

Proof Let a be any element in Q(B), then there exists $k \geq 1$ such that,
$\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x \in B$ and all $\lambda \in ¢$.
But for any fixed $x \in A$, there exists a sequence (x_n) of elements of B such that $\lim x_n = x$, then
$\| x_n (\lambda - a) \| \leq k \| (\lambda - a) x_n \|$ for all n and all $\lambda \in ¢$.
Then by the continuity of the norm, we have $k \geq 1$ such that
$\| x(\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $\lambda \in ¢$.
But x is arbitrary in A, then $a \in Q(A)$.
Hence $Q(B) \subseteq Q(A)$.

In the following theorem we give sufficient conditions are given to get $\bigcap_{i=1}^{n} \boldsymbol{Q(B_i)}$ = Q(A), $\bigcap_{i=1}^{n} \boldsymbol{Q_\sigma(B_i)}$ = $Q_\sigma(A)$ and $\bigcap_{i=1}^{n} \boldsymbol{Q_\rho(B_i)}$ = $Q_\rho(A)$, for $\boldsymbol{B_1, B_2, \ldots, B_n}$ as closed subalgebras of a unital complex Banach algebra A.

3.5 Theorem: If A is a complex Banach algebra with unity e and $\boldsymbol{B_1, B_2, \ldots, B_n}$ are closed subalgebras of A such that $\bigcup_{i=1}^{n} \boldsymbol{B_i}$ = A and a, e$\in \bigcap_{i=1}^{n} \boldsymbol{B_i}$, then

(a) *$a \in \bigcap_{i=1}^{n} \boldsymbol{Q(B_i)}$ if only if $a \in Q(A)$.*

(b) *If* $\rho_A(a)$ is connected *then, $a \in \bigcap_{i=1}^{n} \boldsymbol{Q_\sigma(B_i)}$ only if $a \in Q_\sigma(A)$.*

(c) *If* $\rho_A(a)$ is connected *then, $a \in \bigcap_{i=1}^{n} \boldsymbol{Q_\rho(B_i)}$ only if $a \in Q_\rho(A)$.*

Proof

We prove (ii) and omit the similar proofs of (i) and (iii).

By the corollary of Theorem 3.3 above a$\in \bigcap_{i=1}^{n} \boldsymbol{Q_\sigma(B_i)}$, if a$\in Q_\sigma(A)$.

Conversely suppose that a$\in \bigcap_{i=1}^{n} \boldsymbol{Q_\sigma(B_i)}$, then a$\in \boldsymbol{Q_\sigma(B_i)}$ for all i.

But a$\in \boldsymbol{Q_\sigma(B_i)}$ means that there exists $\boldsymbol{k_i} \geq 1$ such that

$\| x(\lambda - a) \| \leq \boldsymbol{k_i} \| (\lambda - a) x \|$ for all x$\in \boldsymbol{B_i}$ and all $\lambda \in \boldsymbol{\rho_{B_i}(a)}$.

Since $\rho_A(a)$ is connected then by [6, corollary pp.239], $\rho_A(a)$ = $\boldsymbol{\rho_{B_i}(a)}$ for all i.

But $\bigcup_{i=1}^{n} \boldsymbol{B_i}$ =A, then there exists k= max { $\boldsymbol{k_1, k_2, \ldots, k_n}$ }$\geq 1 \|x(\lambda - a)\| \leq k \| (\lambda - a) x \|$ for all x$\in$A and all $\lambda \in \rho_A(a)$. Hence a$\in Q_\sigma(A)$.

3.6 Proposition: Let A be a unital complex Banach algebra and k ≥ 1. Then

(a) If $a \in Q_\rho(k, A) \cap A^{-1}$ then $a^{-1} \in Q_\rho(k \| a \| \| a^{-1} \|, A)$.

(b) If a $\in Q_\rho(k, A)$ and b$\in A^{-1}$ then $b^{-1}ab \in Q_\rho(k \| b \|^2 \| b^{-1} \|^2, A)$.

Proof

(a) Let a $\in Q_\rho(k,A)\cap A^{-1}$, then $\| x(\lambda - a)\| \le k\|(\lambda - a)x\|$ for all x∈A and all $\lambda\in\sigma_A(a)$, and $a^{-1}\in$A and so $0\notin\sigma_A(a)$.

Note that $\lambda\in\sigma_A(a)$ only if $(\lambda a)^{-1}(\lambda^{-1}-a^{-1})^{-1}$ does not exist only if $(\lambda^{-1}-a^{-1})^{-1}$ does not exist only if $\lambda^{-1}\in\sigma_A(a^{-1})$.

Now for any $\lambda\in\sigma_A(a)$ and all x∈A we have

$\| x(\lambda^{-1}-a^{-1})\| = \| x(\lambda - a)(\lambda a)^{-1}\| \le \| x(\lambda - a)\| \,|\lambda^{-1}|\,\|a^{-1}\|$

$\le k\|(\lambda - a)x\|\,|\lambda^{-1}|\,\|a^{-1}\| = k\|\lambda a(a^{-1}-\lambda^{-1})x\|\,|\lambda^{-1}|\,\|a^{-1}\|$

$\le k\|(\lambda^{-1}-a^{-1})x\|\,\|a\|\,\|a^{-1}\|$.

Therefore, $\| x(\mu - a^{-1})\| \le k\|a\|\,\|a^{-1}\|\,\|(\mu - a^{-1})x\|$ for all x∈A and all $\mu\in\sigma_A(a^{-1})$. This means that $a^{-1}\in Q_\rho(k\|a\|\|a^{-1}\|, A)$.

(b) Let a $\in Q_\rho(k,A)$ and b$\in A^{-1}$ then,

$\sigma_A(b^{-1}ab) = \{\lambda\in ¢ : (\lambda - b^{-1}ab)^{-1}$ does not exist $\}$

$= \{\lambda\in ¢ : b^{-1}(\lambda - a)^{-1}b$ does not exist $\} = \sigma_A(a)$.

Now for any $\lambda\in\sigma_A(b^{-1}ab) = \sigma_A(a)$ and all x∈A we have

$\| x(\lambda - b^{-1}ab)\| = \| x b^{-1}(\lambda - a) b\| \le \| b^{-1}bx\, b^{-1}(\lambda - a)\|\;\|b\|$

$\le \|b^{-1}\|\;\|bx\, b^{-1}(\lambda - a)\|\;\|b\|$

$\le k\|(\lambda - a)bx b^{-1}\|\,\|b\|\,\|b^{-1}\| = k\|b(\lambda - b^{-1}ab)x b^{-1}\|\,\|b\|\,\|b^{-1}\|$

$\le k\|b\|^2\|b^{-1}\|^2\|(\lambda - b^{-1}ab)x\|$.

Therefore, $b^{-1}ab\in Q_\rho(k\|b\|^2\|b^{-1}\|^2, A)$.

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