DECOMPOSITION OF MEASURES ON DIFFERENCE POSETS

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تجزئة المقاييس على مجموعات الفرق المرتبة ترتيباً جزئياً

ملخص نعمم في هذا البحث نتائج معروفة مثل نظرية يوشيدا-هيويت ونظرية ليبيج، والتي كل منهما تعالج تجزئة المقياس الموجب والمنتهي (المحدود) والمُعرف على مجموعة الفرق المرتبة ترتيباً جزئياً ويأخذ القيم في فضاء ريز التام ديدكينياً. وكذلك وفرنا شروطاً كافية لتتمتع مجموعة الفرق المرتبة ترتيباً جزئيا بخاصية جوردان-هان وخاصية جوردان-هان التقريبية

ABSTRACT. We prove generalizations of the Yosida-Hewitt decomposition theorem and the Lebesgue decomposition theorem for positive finitely additive maesures defined on difference posets (generalizing orthomodular posets and orthoalgebras) with values in a Dedekind complete Riesz space. Moreover, we provide some conditions for a difference poset to possess the Jordan-Hahn property and the approximate Jordan-Hahn property.

1 INTRODUCTION

One of the areas of noncommutative measure theory is the study of measures and states on algebraic structures less rich than a $\sigma-$ field which arose from the realization that quantum mechanical events fail to form a $\sigma-$ field. Such structures are, for example, quantum logics (=orthomodular posets), orthoalgebras or more generally difference posets [7, 8, 9, 10, 12].

In the last years many classical decomposition theorems had been proved in the setting of finitely additive measures defined on orthomodular posets and orthoalgebras [2, 3, 4, 14, 15].

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The main results of the this paper are the following: 1) generalizing the Yosida-Hewitt decomposition and the Lebesgue decomposition of finitely additive measures, $\sigma-$additive measures, and completely additive measures on difference posets with values in a Dedekind complete Riesz space [2, 3, 4]. 2) Finding necessary or sufficient conditions for difference posets to possess the Jordan-Hahn property or the approximate Jordan-Hahn property [14, 15].

2 PRELIMINARIES

Let $(L, \leq)$ be a partially ordered set (*poset*), and let D be a nonempty subset of L. If the supremum (resp., the infimum) of D in L exists, it will be denoted by $\bigvee D$ (resp., $\bigwedge D$). In particular, if $D = \{x, y\}$ we write $\bigvee D = x \vee y$ and $\bigwedge D = x \wedge y$.

Definition 2.1 [12, 13] Let $(L, \leq)$ be a poset with a least element 0 and a greatest element 1. Let $\ominus$ be a partially defined binary operation on L such that $b \ominus a$ is defined if and only if $a \leq b$. Then $(L, \leq, \ominus, 0, 1)$ is called a *difference poset* (a DP, for short) if the following conditions are satisfied $\forall a, b, c \in L$:

(DP1) For any $a \in L$, $a \ominus 0 = a$.

(DP2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

The following statements had been proved in [10].

Proposition 2.2 Let a, b, c be elements in a DP L. Then

(i) $a \leq b \leq c \Rightarrow b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.

(ii) $a \leq c$ and $b \leq c \ominus a \Rightarrow (c \ominus a) \ominus b = (c \ominus b) \ominus a$.

(iii) $a \leq b \leq c \Rightarrow a \leq c \ominus (b \ominus a)$ and $(c \ominus (b \ominus a)) \ominus a = c \ominus b$.

Let $(L, \leq, \ominus, 0, 1)$ be a difference poset. For any element $a \in L$ we put

$$a' := 1 \ominus a.$$

Then (i) $a'' = a$; (ii) $a \leq b$ implies $b' \leq a'$. Two elements $a, b \in L$ are *orthogonal*, and we write $a \perp b$ iff $a \leq b'$ (or equivalently $b \leq a'$). The properties of a DP enable us to define a partial binary operation $\oplus : L \times L \to L$ as follows :

$$\text{For every } a, b \in L \text{ with } a \perp b, \text{ put } a \oplus b := (a' \ominus b)' = (b' \ominus a)'. \qquad (2.1)$$

The partial binary operation $\oplus$ on L is commutative and associative. Consequently, if $a, b \in L$ with $a \leq b$, then $\exists c \in L$ such that $c \perp a$ and $b = a \oplus c$. Moreover, $c = b \ominus a$. Very important examples of difference posets are orthomodular posets, orthoalgebras, and effect algebras [6, 8, 10, 12].

Example 2.3 [7, 8] An *orthoalgebra* (OA) is a set L containing two special elements 0,1 and equipped with a partially defined binary operation $\oplus : L \times L \to L$ such that for all $a, b, c \in L$ we have

(OA1) (*Commutativity*) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$;

(OA2) (*Associativity*) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$;

(OA3) (*Orthocomplementation*) For any $a \in L$ there is a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$;

(OA4) (*Consistency*) If $a \oplus a$ is defined, then $a = 0$.

Note that every OA $(L, \oplus, 0, 1)$ becomes a DP if we define $\ominus$ on L by $b \ominus a := (a \oplus b')'$ for all pairs (a, b) with $a \perp b'$, where b' is the unique element in L such that $b \oplus b' = 1$.

By Theorem 1.11 in [12], we conclude that a DP with $0, 1$ and $\oplus$, defined by (2.1), is an orthoalgebra if and only if $a \leq 1 \ominus a$ implies $a = 0$. Therefore, it is not hard to give many examples of DPs which are not orthoalgebras. For example, let H be a Hilbert space, and let $E(H)$ be the set of all self-adjoint operators A on H with $O \leq A \leq I$, where O and I are the zero and identity operators, respectively, on H. The partial order on $E(H)$ is defined by setting $A \leq B$ iff $(Ax, x) \leq (Bx, x)$, $x \in H$, and $C = B \ominus A$ iff $(Bx, x) - (Ax, x) = (Cx, x)$, $x \in H$. It is shown in [10, Example 7] that $(E(H), \leq, \ominus, O, I)$ is a difference poset which is not an orthoalgebra.

On the other hand, if in the definition of an orthoalgebra, axiom (OA4) is replaced by the weaker axiom

(EA4) $a \oplus 1$ is defined implies $a = 0$,

we obtain the so-called *effect algebra* (EA) which generalizes orthoalgebra. It is shown in [6, Section 4] that EAs and DPs are the same thing.

Definition 2.4 Let L be a DP. A subset $M \subseteq L$ is called an *orthogonal* set if any two elements of M are orthogonal. A subset $M \subseteq L$ is said to be *jointly orthogonal* iff it is an orthogonal set and contained in a Boolean subalgebra. A difference poset L is said to be *locally finite* if every jointly orthogonal subset of L is finite.

Let $\{a_1, \cdots, a_n\} \subseteq L$. Recursively we define for $n \geq 3$

$$a_1 \oplus \cdots \oplus a_n := (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n, \tag{2.2}$$

supposing $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ exist in L. From the associativity of $\oplus$ in a DP we conclude that (2.2) is correctly defined, and we put $a_1 \oplus \cdots \oplus a_n = a_1$ if $n = 1$, and $a_1 \oplus \cdots \oplus a_n = 0$ if $n = 0$. Then for any permutation $(i_1, \cdots, i_n)$ of $(1, \cdots, n)$ and for any k with $1 \leq k \leq n$, we have

$$a_1 \oplus \cdots \oplus a_n = a_{i_1} \oplus \cdots \oplus a_{i_n},$$

$$a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_k) \oplus (a_{k+1} \oplus \cdots \oplus a_n).$$

We say that a finite subset $F = \{a_1, \cdots, a_n\}$ of L is $\bigoplus$*-orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists in L. In this case, we say that F has an $\bigoplus$*-join*, defined as

$$\bigoplus F = a_1 \oplus \cdots \oplus a_n.$$

It is clear that two elements a and b of L are orthogonal, i.e., $a \perp b$, iff $\{a, b\}$ is $\bigoplus$*-orthogonal.*

An arbitrary subset G of L is $\bigoplus$*-orthogonal* if every finite subset F of G is $\bigoplus$*-orthogonal.* If G is $\bigoplus$-orthogonal, then any subset of G is $\bigoplus$-orthogonal. An $\bigoplus$*-orthogonal* subset G of L has an $\bigoplus$*-join* in L if

$$\bigoplus G := \bigvee_{F \in \mathcal{F}(G)} \bigoplus F$$

exists in L (where $\mathcal{F}(G)$ denotes the set of all finite subsets of G). It should be noted that every jointly orthogonal subset of L is $\bigoplus$-orthogonal (but not conversely).

We say that a DP L is a *complete DP* (resp., a σ*-DP*) if for any $\bigoplus$-orthogonal subset (resp., any countable $\bigoplus$-orthogonal subset) of L, there exists the $\bigoplus$-join in L [4, 6].

Definition 2.5 [14] Let L be a DP. A finite set $D \subseteq L$ is called a *difference set* if either D is empty or there exists a strictly increasing sequence $(p_i)_{i=0}^{n}$, $n \geq 1$, in $(L, \leq)$ such that

$$D = \{p_i \ominus p_{i-1} : i = 1, 2, ..., n\}.$$

We say that the sequence $(p_i)_{i=0}^{n}$ yields the difference set D.

It should be noted that a subset of a difference set of L is also a difference set. Moreover, by [14, Lemma 2.9], if D is a nonempty difference set yielded by the strictly increasing sequence $(p_i)_{i=0}^{n}$, $n \geq 1$, then n is the cardinality of D. and D is orthogonal set of nonzero elements. It follows from [14, Theorem 2.15] or [8, Lemma 3.1] that every difference set is jointly orthogonal, and, therefore is $\bigoplus$ −orthogonal.

Now, using Proposition 2.2, it had proved in [10] that if L is a DP, $a, b, c \in L$ and $a \leq b \leq c$, then $(c \ominus b) \oplus (b \ominus a) = c \ominus a$. Using this fact and induction, it can be easily shown that if D is a difference set which is yielded by the strictly increasing sequence $(p_i)_{i=0}^{n}$, $n \geq 1$, then the $\bigoplus$ −joint of D is given by $\bigoplus D = p_n \ominus p_0$.

A non-zero element p of a difference poset L is said to be an *atom* in L if, for elements $q, r \in L$,

$$p = q \oplus r \text{ implies that } q = 0 \text{ or } r = 0.$$

Lemma 2.6 [14, **Lemma** 6.2] Let L be a locally finite difference poset. Then:

(i) For each nonzero element p in L there exists an atom q and an element r in L such that $p = q \oplus r$.

(ii) For each nonzero element p in L there exists a difference set D consisting of atoms such that $p = \bigoplus D$.

(iii) A difference set D consisting of atoms is maximal if and only if $1 = \bigoplus D$.

Definition 2.7 [1] Let $(E, +, ., \leq)$ be a real vector space which is equipped with a partial ordering $\leq$. We say that E is a *Riesz space* if the following axioms are satisfied:

(i) $(E, \leq)$ is a lattice;

(ii) $x, y \in E$ and $x \leq y \Rightarrow x + z \leq y + z$ for all $z \in E$;

(iii) $x, y \in E$ and $x \leq y \Rightarrow \alpha x \leq \alpha y$ holds for all $\alpha \geq 0$.

The set of all positive elements of E will be denoted by E^+ (i.e., $E^+ := \{x \in E : x \geq 0\}$). For any vector x in a Riesz space we define

$$x^+ := x \vee 0; \qquad x^- := (-x) \vee 0; \qquad |x| := x \vee (-x).$$

The elements x^+, x^- and $|x|$ are respectively called the positive part, the negative part and the absolute value of x. For any $x \in E$, we have (i) $x = x^+ - x^-$, (ii) $|x| = x^+ + x^-$ and (iii) $|x| = 0$ iff $x = 0$.

A Riesz space E is called *Dedekind complete* (resp., *σ-Dedekind complete*) if, for every nonempty subset (resp., countable subset) B of E that is bounded from above, $\bigvee B$ exists in E. A Riesz space E is *Archimedean* if and only if, given $x, y \in E^+$ such that $nx \leq y$ holds for all $n \in \mathrm{N}$ implies $x = 0$. It should be noted that Dedekind complete Riesz space E is Archimedean [11].

A nonempty subset D of E is *directed downwards* (resp., *upwards*), and we write $D \downarrow$ (resp., $D \uparrow$), if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$, $z \leq y$ (resp., $z \geq x$, $z \geq y$). Two downwards directed sets $\{x_t : t \in T\}$ and $\{y_t : t \in T\}$ indexed by the same index set T are called *equidirected* if, for any $s, t \in T$, there exists $v \in T$ such that $x_v \leq x_s$ and $x_v \leq x_t$ as well as $y_v \leq y_s$ and $y_v \leq y_t$. A similar definition holds for upwards directed sets. If $\{f_t\}$ and $\{g_s\}$ are equidirected, then [11, Theorem 15.8]:

$$\{f_t\} \uparrow f, \{g_s\} \uparrow g \Rightarrow \{f_t + g_s\} \uparrow f + g \tag{2.3}$$

$$\{f_t\} \downarrow f, \{g_s\} \downarrow g \Rightarrow \{f_t + g_s\} \downarrow f + g \tag{2.4}$$

Finally, we say that a net $\{x_\alpha\}$ in E is *order convergent* to an element $x \in E$, and we write $x_\alpha \xrightarrow{o} x$, if there exists a downwards directed net $\{p_\alpha\} \subseteq E$ and $\{p_\alpha\} \downarrow 0$ such that $|x_\alpha - x| \leq p_\alpha \; \forall \alpha$. If $x_\alpha \uparrow x$ (or $x_\alpha \downarrow x$) then $x_\alpha \xrightarrow{o} x$ [1].

3 DECOMPOSITION OF MEASURES

Throughout the rest of this paper, V is assumed to be a Dedekind complete Riesz space, and $L = (L, \leq, ', \ominus, 0, 1)$ is assumed to be a DP for which the partial operation $\oplus : L \times L \to L$ is defined by (2.1). Consider the following binary relation $\leq_n$ on V^L : $\mu_1 \leq_n \mu_2$ iff $\mu_1(a) \leq \mu_2(a)$ for all $a \in L$. Clearly the pair $(V^L, \leq_n)$ is a partially ordered set. An element $\mu \in V^L$ is said to be *positive* if $\mu(a) \geq 0$ for all $a \in L$.

We say that an element $\mu \in V^L$ is a *finitely additive measure* if $\mu(a \oplus b) = \mu(a) + \mu(b)$ whenever $a \oplus b$ is defined in L. Then $\mu(0) = 0$ and $\mu(a') = \mu(1) - \mu(a)$ for all $a \in L$. If μ is positive, then $a \leq b$ in L implies $\mu(a) \leq \mu(b)$ in V. One can easily show that, an element $\mu \in V^L$ is a finitely additive measure iff $\mu(b \ominus a) = \mu(b) - \mu(a)$ whenever $a \leq b$. We define σ−additive and completely additive measures on L as follows.

Definition 3.1 [2, 4] Let L be a complete DP. A mapping $\mu \in V^L$ is said to be a *positive completely additive measure* on L if, for any $\bigoplus$ −orthogonal system $\{a_i : i \in I\}$ in L, we have for any finite subset F of I

$$|\mu(\bigoplus_{i \in I} a_i) - \sum_{i \in F} \mu(a_i)| \leq b_F, \tag{3.1}$$

where $\{b_F\} \downarrow 0$ and $b_{F_1} \leq b_{F_2}$ whenever $F_2 \subseteq F_1$. That is, $\sum_{i\in F} \mu(a_i) \xrightarrow{o} \mu(\bigoplus_{i\in I} a_i)$, and we shall write $\mu(\bigoplus_{i\in I} a_i) = \sum_{i\in I} \mu(a_i)$.

If the index set I in (3.1) is only countable, we say that μ is a *positive $\sigma-$additive measure* (or *a positive countably additive measure*), and we write $\mu(\bigoplus_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} \mu(a_i)$.

Since every Dedekind complete Riesz space is Archimedean, we conclude that $\mu(0) = 0$. In fact, for any finite subset F of I with $|\mu(\bigoplus_{i\in I} a_i) - \sum_{i\in F} \mu(a_i)| \leq b_F$, where $a_i = 0\ \forall i \in I$ and hence $\bigoplus_{i\in I} a_i = 0$, we have that $(card(F) - 1)|\mu(0)| \leq b_F \downarrow 0$, and hence $\mu(0) = 0$.

We denote by $a(L,V)_+$, $\sigma a(L,V)_+$, and $ca(L,V)_+$ the sets of all $\mu \in V_+^L$ which are finitely additive, $\sigma-$additive, and completely additive measures, respectively. It can be shown that $ca(L,V)_+ \subseteq \sigma a(L,V)_+ \subseteq a(L,V)_+$.

It is not hard to prove that a positive additive measure μ on L is $\sigma-$additive or completely additive iff

$$\{\sum_{i=1}^{n} \mu(a_i)\} \uparrow \mu(\bigoplus_{i=1}^{\infty} a_i), \tag{3.2}$$

or

$$\{\sum_{i\in F} \mu(a_i)\}_F \uparrow \mu(\bigoplus_{i\in I} a_i), \tag{3.3}$$

where F runs over all finite subsets of I, whenever $\{a_i : i \in I\}$ is a $\bigoplus-$orthogonal set in L for which $\bigoplus_{i\in I} a_i$ exists in L. Moreover, if μ and ν are elements of $ca(L,V)_+$, then $\mu + \nu \in ca(L,V)_+$, where $(\mu + \nu)(a) := \mu(a) + \nu(a)$, $a \in L$. Indeed, this follows from (3.3) and (2.3).

An element $\mu \in a(L,V)_+$ is said to be *weakly purely additive* if

$$\eta \leq_n \mu, \ \ \eta \in ca(L,V)_+ \Rightarrow \eta = 0. \tag{3.4}$$

If (3.4) holds for $\eta \in \sigma a(L,V)_+$, μ is said to be *purely additive*. An element $\mu \in \sigma a(L,V)_+$ is said to be *purely $\sigma-$additive*, if it satisfies (3.4).

The next result is a noncommutative version of the Yosida-Hewitt decomposition theorem for Dedekind complete Riesz space-valued measures on a complete difference poset. This result generalizes that one in [2] and [3] to the more general setting of difference posets.

Theorem 3.2 Let $\mu \in a(L,V)_+$ where L is a complete DP. Then μ can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in ca(L,V)_+$, and η is a positive weakly purely additive measure on L.

Proof. Define $\Gamma_\mu = \{\gamma \in ca(L,V)_+ : \gamma \leq_n \mu\}$. Since $0 \in \Gamma_\mu$, Γ_μ is nonempty. Let $C = \{\gamma_j\}$ be a chain in Γ_μ with respect to the natural ordering $\leq_n$, and define

$$\gamma_o(c) := \bigvee_j \gamma_j(c), \quad c \in L.$$

Since $0 \leq \gamma_j(c) \leq \gamma_j(1) \leq \mu(1)$ and V is Dedekind complete, $\gamma_0(c)$ is defined correctly on L. Moreover, γ_o is finitely additive. Indeed, let $a \oplus b$ be defined in L. Then $\gamma_j(a) \uparrow \gamma_o(a)$, $\gamma_j(b) \uparrow \gamma_o(b)$. Also, it can be shown that $\{\gamma_j(a)\}$ and $\{\gamma_j(b)\}$ are equidirected. By (2.3), we conclude that

$$\gamma_o(a \oplus b) = \gamma_j(a \oplus b) \uparrow = (\gamma_j(a) + \gamma_j(b)) \uparrow = \gamma_j(a) \uparrow + \gamma_j(b) \uparrow = \gamma_o(a) + \gamma_o(b).$$

From the definition of γ_0, we conclude that $\{\gamma_o(c) - \gamma_i(c)\} \downarrow 0$ uniformly in $c \in L$. To see this, let $c \in L$. Then $\gamma_j(c) \uparrow$, hence $-\gamma_j(c) \downarrow$ and therefore, $\{\gamma_o(c) - \gamma_j(c)\} \downarrow$. Moreover,

$$0 = \gamma_o(c) - \bigvee_j \gamma_j(c) = \bigwedge(\gamma_o(c) - \gamma_j(c)),$$

which implies that $\{\gamma_o(c) - \gamma_j(c)\} \downarrow 0$. Now, we shall show that $\gamma_o \in ca(L,V)_+$. Let $\{a_i\}_{i\in I}$ be a $\oplus$-orthogonal system in L with $a = \bigoplus_{i\in I} a_i$. Then for any finite subset F of I, we have

$$0 \leq \gamma_o(a) - \sum_{i\in F} \gamma_o(a_i) = \gamma_o(a \ominus (\bigoplus_{i\in F} a_i))$$

$$= (\gamma_o(a \ominus (\bigoplus_{i\in F} a_i)) - \gamma_j(a \ominus (\bigoplus_{i\in F} a_i))) + (\gamma_j(a \ominus (\bigoplus_{i\in F} a_i)),$$

where $\{\gamma_o(a \ominus (\bigoplus_{i\in F} a_i)) - \gamma_j(a \ominus (\bigoplus_{i\in F} a_i))\} \downarrow 0$. Thus letting $p_j := \gamma_o(a \ominus (\bigoplus_{i\in F} a_i)) - \gamma_j(a \ominus (\bigoplus_{i\in F} a_i))$ and $b_F^j := \gamma_j(a \ominus (\bigoplus_{i\in F} a_i))$, we obtain

$$0 \leq \gamma_o(a) - \sum_{i\in F} \gamma_o(a_i) \leq p_j + b_F^j \quad \forall F \in \mathcal{F}(I),$$

where $\{p_j\} \downarrow 0$ and, by complete additivity of γ_j , $\{b_F^j\}_F \downarrow 0$ for each fixed j. This yields

$$0 \leq \gamma_o(a) - \bigvee_F \sum_{i\in F} \gamma_o(a_i) \leq p_i \downarrow 0,$$

and therefore $\gamma_o(a) = \sum_{i\in I} \gamma_o(a_i)$.

Since $\gamma_o \leq_n \mu$, γ_o is a majorant of C in Γ_μ. It follows from Zorn's Lemma that Γ_μ contains a maximal element ξ. Then $\xi \in ca(L,V)_+$ and $\xi \leq_n \mu$. Let

$\eta := \mu - \xi$. Clearly $\eta \in a(L,V)_+$. To finish the proof, it remains to show that η is weakly purely additive. Let $\gamma \in ca(L,V)_+$ be such that $\gamma \leq_n \eta = \mu - \xi$. Then $\gamma + \xi \in ca(L,V)_+$ and $\gamma + \xi \leq_n \mu$. Hence $\gamma + \xi \in \Gamma_\mu$ and therefore, the maximality of ξ in Γ_μ implies $\gamma = 0$. □

Theorem 3.3 Let $\mu \in \sigma a(L,V)_+$ where L is a complete DP. Then μ can be expressed as a sum $\mu = \xi + \eta$, where $\xi \in ca(L,V)_+$, and η is a purely $\sigma-$additive measure on L.

Proof. It is identical with the proof of Theorem 3.2 after changing $a(L,V)_+$ to $\sigma a(L,V)_+$. □

Let W be another Riesz space, and let $\mu \in a(L,V)_+$, $\lambda \in a(L,W)_+$ be given. We say that (i) μ is $\lambda-$*continuous*, and we write $\mu \ll \lambda$, if for every $\epsilon > 0$, $\epsilon \in V_+$, there exists $\delta > 0$, $\delta \in W_+$ such that $a \in L$, $\lambda(a) < \delta \Rightarrow \mu(a) < \epsilon$. (ii) μ is $\lambda-$*singular*, and write $\mu \perp \lambda$, if whenever $\gamma \in a(L,V)_+$, $\gamma \ll \lambda$ and $\gamma \leq_n \mu$, whenever $\gamma = 0$. Note that if $\mu \leq_n \lambda$, then $\mu \ll \lambda$; hence $\mu \perp \lambda$ implies $\mu \wedge \lambda = 0$ in $a(L,V)_+$. Moreover, if $\mu_1, \mu_2 \in a(L,V)_+$ with $\mu_1 \ll \lambda$ and $\mu_2 \ll \lambda$, then $\mu_1 + \mu_2 \ll \lambda$.

In what follows, we present two Lebesgue-type decompositions which generalize those in [3, 4] to the more general setting of difference posets.

Theorem 3.4 Let $\mu \in a(L,V)_+$, let W be a Riesz space, and let $\lambda \in a(L,W)_+$. Then there exist two elements $\xi, \eta \in a(L,V)_+$ such that $\mu = \xi + \eta$, where $\xi \ll \lambda$ and $\eta \perp \lambda$.

Proof. Define $\Gamma_\mu = \{\gamma \in a(L,V)_+ : \gamma \ll \lambda \text{ and } \gamma \leq_n \mu\}$. Since $0 \in \Gamma_\mu$, Γ_μ is nonempty. Let $C = \{\gamma_i\}$ be a chain in Γ_μ with respect to the natural ordering $\leq_n$. As in the proof of Theorem 3.2, we can define $\gamma_o(c) := \bigvee_i \gamma_i(c)$, $c \in L$, and prove that $\gamma_o \in a(L,V)_+$. Moreover, $\gamma_o \leq_n \mu$.

To establish that $\gamma_o \in \Gamma_\mu$, we must show that $\gamma_o \ll \lambda$. Let $\epsilon \in V_+$ and $\gamma_i \in C$ be given. Since $\gamma_i \ll \lambda$, there exists $\delta \in W_+$ such that $a \in L$ and $\lambda(a) < \delta$ implies $\gamma_i(a) < \epsilon$. Then

$$\gamma_o(a) = (\gamma_o(a) - \gamma_i(a)) + \gamma_i(a) < p_i + \epsilon,$$

where $p_i := \gamma_o(a) - \gamma_i(a)$ and $\{p_i\} \downarrow 0$ because $\{\gamma_o(c) - \gamma_i(c)\} \downarrow 0$ uniformly in $c \in L$. Therefore $\gamma_o(a) \leq \epsilon$. Thus γ_o is a majorant of C in Γ_μ. It follows from Zorn's Lemma that Γ_μ contains a maximal element ξ. Hence $\xi \ll \lambda$ and $\xi \leq_n \mu$. Let $\eta := \mu - \xi$. Clearly, $\eta \in a(L,V)_+$. To finish the proof, it remains to show that $\eta \perp \lambda$. Let $\gamma \in a(L,V)_+$ be such that $\gamma \leq_n \eta = \mu - \xi$ and $\gamma \ll \lambda$. Then

$\gamma \dotplus \xi \in a(L,V)_+$, $\gamma \dotplus \xi \leq_n \mu$ and $\gamma \dotplus \xi \ll \lambda$. So $\gamma \dotplus \xi \in \Gamma_\mu$ and the maximality of ξ implies $\gamma = 0$. □

Theorem 3.5 Let μ, $\lambda \in ca(L,V)_+$ where L is a complete DP. Then there exist two elements $\xi, \eta \in ca(L,V)_+$ such that $\mu = \xi \dotplus \eta$, where $\xi \ll \lambda$ and $\eta \wedge \lambda = 0$.

Proof. Define $\Gamma_\mu = \{\gamma \in ca(L,V)_+ : \gamma \ll \lambda \text{ and } \gamma \leq_n \mu\}$. Applying the proofs of Theorem 3.2 and Theorem 3.4, we obtain a maximal element ξ in Γ_μ which belongs to $ca(L,V)_+$ and $\xi \ll \lambda$. Let $\eta := \mu - \xi$. Then, $\eta \in ca(L,V)_+$. It remains to show that $\eta \wedge \lambda = 0$. In fact, 0 is a minorant of the set $\{\eta, \lambda\}$ in $ca(L,V)_+$. Let $\gamma \in ca(L,V)_+$ be such that $\gamma \leq_n \eta$ and $\gamma \leq_n \lambda$. Then $\gamma \dotplus \xi \leq_n \mu$. Since $\gamma \leq_n \lambda$ implies $\gamma \ll \lambda$, and since $\xi \ll \lambda$, we have $\gamma \dotplus \xi \ll \lambda$. Thus $\gamma \dotplus \xi \in \Gamma_\mu$. By maximality of ξ, we get $\gamma = 0$. □

Remark 3.6 We did not prove the uniquness of the decompositions for measures on orthoalgebras or difference posets. Recently G.T. Rüttimann [16] and A. De Simone [5] have given some conditions that ensure the uniqueness of decompositions on an orthomodular poset.

4 THE JORDAN-HAHN DECOMPOSITION

The present section is concerned with a relaxed form of the Jordan-Hahn decomposition in the noncommutative setting of difference posets and orthoalgebras. In this section, we take $V = \mathbb{R}$. By a measure on a DP L, we mean an element $\mu \in \mathbb{R}^L$ which is finitely additive measure on L. A measure μ on L is said to be *bounded* if the image $\mu(L)$ of L under the map μ is a bounded subset of $\mathbb{R}$, and it is said to be *positive* if $\mu(p) \geq 0$ for all elements p in L. We denote by $ba(L)$ the linear subspace of all bounded measures on L and by $a_+(L)$ the subset of $\mathbb{R}^L$ of all positive measures on L. Notice that $a_+(L)$ is a cone in $\mathbb{R}^L$ (i.e., (i) $a_+(L) + a_+(L) \subseteq a_+(L)$, (ii) $\mathbb{R}_+ a_+(L) \subseteq a_+(L)$ and (iii) $a_+(L) \cap -a_+(L) = \{0\}$). The subspace $a_+(L) - a_+(L)$ of $\mathbb{R}^L$ is denoted by $J(L)$. An element of $J(L)$ is called a *Jordan measure*.

Lemma 4.1
(i) The positive cone $a_+(L)$ is a subset of the linear space $ba(L)$.
(ii) $J(L) \subseteq ba(L)$.

Proof. (i) Let μ be a positive measure on L and let p, q be elements in L with $p \leq q$. Then there exists an element $r \in L$ such that $p \perp r$ and $q = p \oplus r$.

Hence, $\mu(p) \leq \mu(p) + \mu(r) = \mu(p \oplus r) = \mu(q)$. Therefore, for all elements $p \in L$, $0 = \mu(0) \leq \mu(p) \leq \mu(1)$.

(ii) It follows directly from (i). □

For an element $\mu \in ba(L)$, we define μ^+, μ^- and $\|\mu\|$ as follows :

$$\mu^+(p) := \bigvee_{q \leq p} \mu(q), \qquad \mu^-(p) := -\bigwedge_{q \leq p} \mu(q), \qquad \|\mu\| := \bigvee_{p \in L} |\mu(p)|.$$

Notice that μ^- is equal to $(-\mu)^+$ and that μ^+ is a super-additive positive functional on L. Indeed, let p, q be elements in L such that $p \perp q$. If r, s are elements of L with $r \leq p$ and $s \leq q$, then $r \perp s$ and $r \oplus s \leq p \oplus q$. It follows that $\mu(r \oplus s) \leq \mu^+(p \oplus q)$, and by additivity of μ we conclude that

$$\mu(r) + \mu(s) \leq \mu^+(p \oplus q).$$

Therefore, $\mu^+(p) + \mu^+(q) \leq \mu^+(p \oplus q)$.

A difference poset L is said to have the *Jordan-Hahn property* (JHP) if for each $\mu \in ba(L)$ there exist elements $\nu, \xi \in a_+(L)$ and an element $p \in L$ such that

$$\mu = \nu - \xi \quad \text{ and } \quad \nu(p') = 0 = \xi(p).$$

A difference poset L is said to have the *approximate Jordan-Hahn property* (AJHP) if for each $\mu \in ba(L)$ there exist elements $\nu, \xi \in a_+(L)$ such that

(i) $\mu = \nu - \xi$, and

(ii) for each $\epsilon > 0$ there is $p \in L$ with $\nu(p') \leq \epsilon$, $\xi(p) \leq \epsilon$.

If L has the Jordan-Hahn property, then it has the approximate Jordan-Hahn property. Moreover, the two properties coalesce provided that L is finite.

The following result gives a necessary and sufficient condition for a difference poset to possess the approximate Jordan-Hahn property. This result generalizes Theorem 2.1 in [15].

Theorem 4.2 Let L be a difference poset. Then L has the approximate Jordan-Hahn property if and only if for each element μ of $ba(L)$ satisfying

$$\mu(p) \leq 1 \quad \text{ for all } p \in L, \tag{4.1}$$

there exists an element ν of $a_+(L)$ such that

$$\mu(q) \leq \nu(q) \leq 1 \quad \text{ for all } q \in L. \tag{4.2}$$

Proof : ($\Rightarrow$): Let $\mu \in ba(L)$ and suppose that $\mu(p) \leq 1 \quad \forall p \in L$. Since L has the AJHP, there exist elements $\nu, \xi \in a_+(L)$ such that $\mu = \nu - \xi$ and for every $\epsilon > 0$ there exists $p \in L$ such that $\nu(p'), \xi(p) \leq \epsilon$. Then for all $\epsilon > 0$,

$$\begin{aligned} 1 \geq \mu(p) = \nu(p) - \xi(p) &= \nu(1) - \nu(p') - \xi(p) \\ &= \nu(1) - [\nu(p') + \xi(p)] \geq \nu(1) - 2\epsilon; \end{aligned}$$

and, therefore, $\nu(1) \leq 1$. It follows that $\nu(q) = \nu(1) - \nu(q') \leq \nu(1) \leq 1 \; \forall q \in L$.

($\Leftarrow$): Let $\mu \in ba(L)$ be a non-zero element. Since $\mu(q) \leq \|\mu\|$ for all $q \in L$, we have

$$(\frac{\mu}{\|\mu\|})(q) \leq 1 \quad \forall q \in L.$$

By hypothesis, there exists an element ν of $a_+(L)$ with $(\frac{\mu}{\|\mu\|})(q) \leq \nu(q) \leq 1 \forall q \in L$. Set $\quad \xi := \nu - \frac{\mu}{\|\mu\|}$. Then $\xi \in a_+(L)$, $\mu = \|\mu\|\nu - \|\mu\|\xi$, and $\|\mu\|\nu$, $\|\mu\|\xi \in a_+(L)$. Let $\epsilon > 0$ be given. Then, there exists $p \in L$ such that

$$\mu(p) \geq \|\mu\| - \frac{\epsilon}{2} \quad \text{or} \quad \mu(p) \leq \frac{\epsilon}{2} - \|\mu\|. \tag{4.3}$$

Now, from above and (4.3), we have

$$\|\mu\|\xi(p) = \|\mu\|\nu(p) - \mu(p) \leq \|\mu\| - \mu(p) \leq \frac{\epsilon}{2} < \epsilon, \text{ and}$$

$$\begin{aligned} \|\mu\|\nu(p') \leq \|\mu\| &\leq \frac{\epsilon}{2} - \mu(p) \leq \frac{\epsilon}{2} + \|\mu\|\xi(p) - \|\mu\|\nu(p) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} - \|\mu\|\nu(p) = \epsilon - \|\mu\|\nu(p) \leq \epsilon; \end{aligned}$$

which completes the proof that L has the AJHP. $\square$

A positive measure $\mu \in a_+(L)$ is called a *probability measure* if $\mu(1)$ is equal to 1. We denote by $\Omega(L)$ the collection of all probability measures on L. A difference poset L is said to be *unital* if for each non-zero element p in L, there exists an element μ of $\Omega(L)$ which evaluates to one on p.

Remark 4.3 Let L be a difference poset, and let $H(\Omega(L))$ and $H_+(\Omega(L))$ denote the linear hull and the positive hull of $\Omega(L)$, respectively. The set $\Omega(L)$ is said to have the JHP$_R$, the Jordan-Hahn property in the sense of Rüttimann [14] if for each $\mu \in H(\Omega(L))$, there exist elements $\nu, \xi \in H_+(\Omega(L))$ and an element $p \in L$ such that

$$\mu = \nu - \xi \quad \text{and} \quad \nu(p') = 0 = \xi(p).$$

Note that, if L has the JHP, then $\Omega(L)$ has the JHP$_R$. In fact, we note that $\Omega(L)$ is a convex subset of itself and $J(L)$ (resp., $a_+(L)$) coincides with $H(\Omega(L))$

(resp., $H_+(\Omega(L))$). Moreover, by Lemma 4.1, every Jordan measure is bounded. It follows that $\Omega(L)$ has the JHP_R.

Theorem 4.4 [14, **Corollary 5.3**] Let L be a unital orthoalgebra and let $\Delta(L)$ be a convex set of probability measures on L. If $\Delta(L)$ has JHP_R then L is locally finite.

The next result relates local finiteness of a difference poset to the Jordan-Hahn property.

Theorem 4.5 Let L be a unital difference poset. If L has the Jordan-Hahn property, then L is locally finite.

Proof. Apply Remark 4.3 and Theorem 4.4 (which is also valid for unital difference posets) to $\Delta(L) = \Omega(L)$. □

The difference poset $\mathrm{Sch}_{20} = \{0, a, b, c, d, e, f, g, k, h, a', b', c', d', e', f', g', k', h', 1\}$ is due to C. Schindler [15]. Figure 4.1(a) below gives its Greechie-diagram where each line represents a block (more precisely an eight-element Boolean algebra). Note that Sch_{20} is unital and locally finite, but fails to have the JHP. To see this, let the bounded measure μ be given as in Figure 4.1(b). It clearly satisfies condition (4.1). The only element $\nu \in \mathbf{R}^{\mathbf{L}}$ which satisfies (4.2) and which is a positive measure on every block in Sch_{20} is given in Figure 4.1(c). Notice that ν is not a measure on Sch_{20} (becasues, $\nu(c) + \nu(d) = 0 \neq 1 = \nu(g') = \nu(c \oplus d)$).

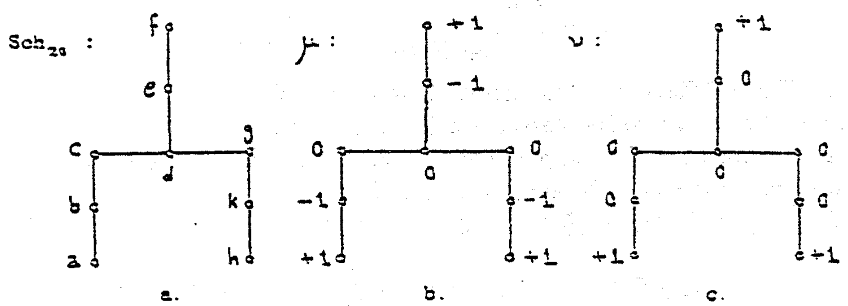

Figure 4.1

Let L be a locally finite difference poset. We denote by $A(L)$ the collection of all atoms in L and by $O(L)$ the collection of all maximal difference sets consisting of atoms. By Lemma 2.6, $A(L)$ and $O(L)$ are not empty, and the pair $(A(L), O(L))$ is called the *atom-hypergraph* of L. A locally finite difference poset is said to satisfy the *outer point condition* if for every element E in $O(L)$ there exists an element $p \in A(L)$ such that, for all elements F in $O(L)$,

$$p \in F \quad \Leftrightarrow \quad E \;=\; F \tag{4.4}$$

The following theorem is a generalization of Theorem 3.1 of [15], which appears without proof, to difference posets.

Theorem 4.6 Let L be a locally finite difference poset. If L satisfies the outer point condition, then it has the approximate Jordan-Hahn property.

Proof. Let μ be a bounded measure on L such that $\mu(p) \leq 1$ for all $p \in L$. For each element $E \in O(L)$ we define a scalar t_E by

$$t_E := \sum_{p \in E} \mu^+(p).$$

Then, as μ^+ is a super-additive positive functional on L we have $0 \leq t_E \leq 1$. Also for each element $E \in O(L)$, we select, using the outer point condition, $p_E \in A(L)$ which satisfies condition (4.4), and define a map $\omega : A(L) \to \mathbf{R}$ as follows

$$\omega(p) := \begin{cases} 1 - t_E + \mu^+(p_E), & \text{if } p = p_E \text{ for some } E \in O(L) \\ \mu^+(p), & \text{otherwise.} \end{cases}$$

Then $\omega(p) \geq 0$ and $\mu(p) \leq \mu^+(p) \leq \omega(p)$ for all $p \in A(L)$. Define an element μ_ω in $\mathbf{R}^L$ by

$$\mu_\omega(p) := \begin{cases} 0, & \text{if } p = 0 \\ \sum_{q \in N} \omega(q), & \text{if } p \neq 0 \end{cases}$$

where N is a difference set consisting of atoms such that $\bigoplus N$ is equal to p. We claim that μ_ω is a positive measure on L which extends ω. To see this, let p, q be nonzero elements in L with $p \perp q$ and let M, N be difference sets consisting of atoms such that $p = \bigoplus M$ and $q = \bigoplus N$. Then, $M \bigcup N$ is a difference set and $\bigoplus(M \bigcup N) = (\bigoplus M) \oplus (\bigoplus N) = p \oplus q$. Since $M \bigcap N$ is empty,

$$\begin{aligned} \mu_\omega(p \oplus q) &= \sum_{r \in M \cup N} \omega(r) = \sum_{r \in M} \omega(r) + \sum_{r \in N} \omega(r) \\ &= \mu_\omega(p) + \mu_\omega(q). \end{aligned}$$

Note that $\mu_\omega(p) = \omega(p)$ for all $p \in A(L)$, since $p = \bigoplus\{p\}$. Moreover, $\mu_\omega(p) \geq 0$ because $\omega(p) \geq 0$ for all $p \in L$. Finally, it remains to show that, for every $F \in O(L)$, $\sum_{p \in F} \omega(p) = 1$. To this end, we have two cases to consider. If F is a singleton set, then $F = \{p_F\}$ and $\sum_{p\in F}\omega(p) = \omega(p_F) = 1$. The other case is that F has more than one element. In this case, we have

$$\begin{aligned}\sum_{p\in F}\omega(p) &= \omega(p_F) + \sum_{p\in F\setminus\{p_F\}}\omega(p)\\ &= 1 - t_F + \mu^+(p_F) + \sum_{p\in F\setminus\{p_F\}}\mu^+(p)\\ &= 1 - t_F + \sum_{p\in F}\mu^+(p) = 1 - t_F + t_F = 1.\end{aligned}$$

From the above we conclude μ_ω is a positive measure and $\mu(p) \le \mu_\omega(p) \le 1$ for all $p \in L$. Therefore the assertion follows from Theorem 4.2. □

The difference posets (in fact, orthoalgebras) G_{14} (which is due to Greechie, see [8, 9]) and J_{18} (which is due to Janowitz see [9]) are locally finite. Figure 4.2 gives their atom-hypergraph (see [8, 9]) where atoms are represented by points in the plane and a collection of points connected by a smooth line-segment forms a maximal difference set of atoms.

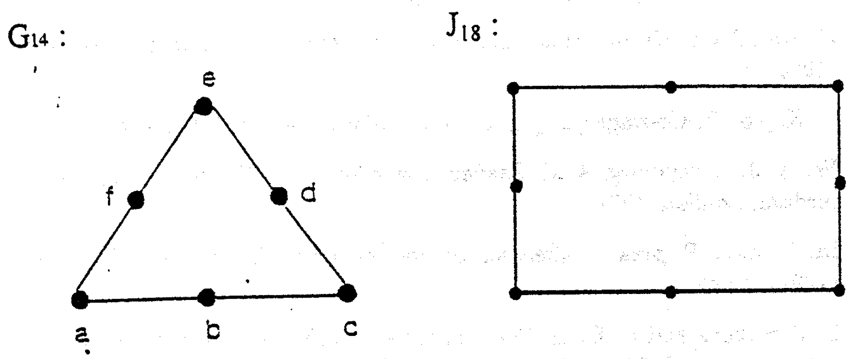

Figure 4.2

Notice that G_{14} and J_{18} satisfy the outer point condition and therefore, by Theorem 4.6, possess the approximate Jordan-Hahn property.

References

[1] D. Aliprautis, O. Burkinshaw, *Positive Operators*, Academic Press, Inc. (1985).

[2] P. De Lucia, A. Dvurečenskij, Yosida-Hewitt decompositions of Riesz space-valued measures on orthoalgebra, *Tatra Mount. Math. Publ.* **2** (1993), 229-239.

[3] P. De Lucia and P. Morales, Noncommutative decomposition theorems in Riesz space, *Proceedings of A.M.S.* **120** (1994),193-202.

[4] A. Dvurečenskij, B. Riečan, Decompositions of measures on orthoalgebras and difference posets, *Inter. J. Theor. Phys.* **33** (1994), 1387-1402.

[5] A. De Simone, Decomposition theorems in orthomodular posets: the problem of uniqueness, *Proceedings of A.M.S.* **126** (1998),2919-2926.

[6] D. J. Foulis, M. K. Bannett, Effect algebras and unsharp quantum logics, *Found. Phys.* **24** (1994), 1325-1346.

[7] D. J. Foulis, R. J. Greechie and G. T. Rüttimann, Filters and supports in orthoalgebras, *Inter. J. Theor. Phys.* **31** (**5**) (1992), 789-807.

[8] E. D. Habil, *Orthoalgebras and Noncommutative Measure Theory*, Ph.D. dissertation, Kansas State University (1993).

[9] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London/Now York, (1983).

[10] F. Kôpka, F. Chovanec, D-posets, *Math. Slovaca* **44** (1994), 21-34.

[11] W. A. J. Luxemburg, A. C. Zaanen, *Riesz Space I,*" North-Holland", Amsterdam, London, 1971.

[12] M. Navara, P. pták , Difference posets and orthoalgebras, BUSEFAL **69** (1997), 64-69.

[13] Z. Riečanová and D. Bršel, Cotraexamples in difference posets and orthoalgebras, *Inter. J. Theor. Phys.* (**33**) (1994), 133-141.

[14] G. T. Rüttimann, The approximate Jordan-Hahn decomposition, *Canad. J. Math.* **41** (1989), 1124-1146.

[15] G. T. Rüttimann, The Jordan-Hahn property, *Proceeding of the First Winter School on Measure Theory* (Slovak Academy of Sciences). Eds. A. Dvurečenskij, S. Pullmanova, Bratislava, 1988, 138-145.

[16] G. T. Rüttimann, Weakly purely finitely additive measure, *Canad. J. Math.* **46** (1994), 872-885.