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Shult Sets In a Class Of Lie Incidence Geometries

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مجموعات شولت في مجموعة من هندسات لي الإسقاطية

ملخص نقدم في هذا البحث بيئة عامة والتي بها نعرف بعض المجموعات الجزئية العظمى (والتي أسميناها shult sets) والتي تتكون من جميع النقاط التي تكون أبعد ما يكون عن بعضها البعض ، مثل هذا التعريف يجعل بالإمكان استخدام مفهوم واحد يجمع عدة مفاهيم ومنها مفاهيم Ovoid ، Spread و Projective Spread كما ندرس إمكانية وجود مثل هذه المجموعات من عدمه.

ABSTRACT **In this paper we introduce a general situation for which we define maximal subsets (Shult sets) of points of some geometries of spherical Lie type. These subsets will introduce a uniform theory of ovoids, spreads, projective spreads, and more, all in one theory.**

To be specific, We fix an integer $n \geq 1$, a geometry in E_1 is a single line together with its incident points. In general for $n > 1$, a point-line geometry Γ is in E_n if and only if

(*SS*1) Γ is a gamma space of point-diameter n.

(*SS*2) If $(x_0, x_1, \ldots, x_d)$ is a geodesic of length d, then it can be extended to a geodesic of length n.

(*SS*3) For each point x in Γ; The set of all points of distance at most $n-1$ from x ; $\Delta^*_{n-1}(x)$ is a geometric hyperplane of Γ.

(*SS*4) If x and y are non-collinear points of Γ of distance $d = d_\Gamma(x, y)$, then the convex closure $\langle x, y\rangle$ belongs to E_d.

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We call this system Shult system E_n.

Weak hexagon property. In any geometry Γ we say that the weak hexagon property holds if for any hexagon (x_0, x_1,..., x_5) of external diameter 3; we have: $x_i^{\perp} \cap x_{i+2}^{\perp} \cap x_{i+4}^{\perp} \neq \emptyset$, indices are taken modulo 6, for $i = 0, 1$.

In particular, we find the maximal cardinality (size) of sets O of points in a geometry $\Gamma \in E_n$, $n = 2, 3$, in which the weak hexagon property holds, such that every two points of O are of mutual distance n. In addition, we discuss the existence or non- existence of such sets.

Key word: Ovoid, Spread, Projective spread, half spin geometry, Grassmannians.

1. Introduction

In recent years, many mathematicians have studied the following three problems:

(1) *Ovoids in finite classical polar spaces*, these are maximal subsets of points in a polar space of mutual distance 2 that intersects each maximal singular (or isotropic) subspace in at most one point. The cardinality of an ovoid depends on the type of the underlying polar space and these numbers are called ovoid numbers. For more details see [Tha1], [Tha2], [Tha5], [Sh1], [Sh2], and [Kan1].

(2) *Spreads in finite classical polar spaces*, these are maximal subsets of totally maximal singular (or isotropic) subspaces that partition the set of points of the polar space. Again the cardinality of a spread depends on the type of the underlying polar space and it turned out that these numbers are the ovoid numbers in each polar space. For more details see [Tha3], [Tha4], [Tha5], [Kan2], and [Kan3].

(3) *Projective spreads*, these are collections of projective subspaces that partition the whole projective space. Many papers studied various kinds of projective spreads, namely dividing the projective space into disjoint m-dimensional subspaces. More recently, some authors have studied mixed spreads (partitions using subspaces of different dimensions) see [Bak], [Eis1], [Hir1].

All these maximal subsets may have no apparent common ground, however, in this paper we shall shed some light on a remarkable connection between these three different concepts. We shall introduce one theory to house all these concepts and more by defining the concept of a *Shult set*. Some of the results that we get are already known, however, they are not unified in one theory. In addition, this study can be considered as an alternate approach. Some of the results can be generalized to more comprehensive situations, and to a wide variety of geometries.

To be more specific; we give an upper bound of sets of points called *Shult sets*, these are sets of points of arbitrary geometry, such that the mutual distance between any pair is the diameter of the geometry, i.e., all points of a Shult set are antipodal points. Some of the upper bound of the sizes of Shult sets are the known ovoid numbers such as half spin geometry $D_{2n,2n}(q)$ of even type, and the Grassmann geometry $A_{2n-1,n}(q)$, over finite field of order q. Some of the upper bounds were not known before this study, such as the size of Shult set in $E_{7,1}(q)$.

The existence or non-existence of such Shult sets has to be studied separately.

As a result of our study the sizes of Shult sets of the following four types of geometries; half spin geometry $D_{2n,2n}(q)$ of even type, the Grassmann geometry $A_{2n-1,n}(q)$, over finite field of order q and the exceptional geometry $E_{7,1}(q)$, (through what we have called Shult system E_n, for $n = 3$), in addition to the case of polar spaces ($n = 2$).

This study is independent of the classification given in [At2] for the Shult system.

2. Basic Geometric Definitions

A *point-line geometry* $\Gamma = (P, L)$ is a pair of sets, P is called the set of "*points*" and L is called the set of "*lines*", where members of L are just subsets of P. If p is a point that belongs to a line l we say that p lies on l, or l passes through p, or p is incident with l. If p, q are two points on one line l we say that p and q are collinear and this is denoted by $p \sim q$. $\Gamma = (P, L)$ is called *linear* (*singular*) *space* if each pair of distinct points lies exactly on one line. Γ is called *partial linear* if each pair of points lie on at most one line. A *subspace* of a point-line geometry Γ = (P, L) is a subset X of points together with all lines l in L such that if l has at least two points of X then l lies entirely in X. A *path* of length k from x_0 to x_k is a set of $k+1$ points $x_0, x_1, x_2, ..., x_k$, such that x_i is collinear with x_{i+1}, i = 1, 2, 3, ..., k-1. A *geodesic* is a shortest path between two points. We define the *distance function d;*

$$d: P \times P \to Z$$

by d(x, y) = the length of any geodesic from x to y.

A subspace X is called *convex* if it contains all geodesics between any two points of X. The smallest subspace containing a set X is called the *subspace generated by* X and is denoted by $\langle X \rangle$. A subspace X is called *connected* if for each pair of points there is a path that connects them and lies entirely in X.

If p is a point, we use the following notations:

$p^{\perp} = \{ x \in P \mid x \text{ is collinear to } p\} \cup \{p\}$.

$\Delta_k(p) = \{x \in P \mid x \text{ is at distance } k \text{ from } p\}$.

$\Delta^*_k(p) = \{x \in P \mid x \text{ is of distance at most } k \text{ from } p \}$.

A (*geometric*) *hyperplane* is a subspace that meets every line of the space. A point-line geometry (P, L) is called a *gamma space* if $x^{\perp}$ is a subspace for every point $x \in$ P.

A point-line geometry (P, L) is called *a projective plane* if it satisfies the following three axioms:

(j1) every pair of points lies exactly on one line.

(j2) every pair of lines intersects at exactly one point.

(j3) there are four points such that no three of which are collinear.

A point-line geometry (P, L) is called *a projective space* if it satisfies the following two axioms:

(j1) every pair of points lies exactly on one line.

(j4) every pair of distinct intersecting lines generates a projective plane.

If $\Gamma = (P, L)$ is a point-line geometry; *The singular rank* (or just *the rank* of Γ is the largest integer n for which there is a chain of singular subspaces $\{X_i\}$, $i = 1, 2, ..., n$, such that: $\varnothing = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n$, where $X_i \neq X_j$, $i \neq j$, and if there is no such integer; the *rank* of Γ is infinite.

A *polar space* is a point-line geometry that satisfies the following Buekenhout-Shult axiom:

Each point p that is not on a line l; p is collinear to one or all points of l.

If $\Gamma = (P, L)$ is a point-line geometry; $Rad\,(\Gamma) = \{\, q \in P \mid q$ collinear to p for all $p \in P\}$. *The polar rank* (or just *the rank*) of Γ is the largest integer n for which there is a chain of singular subspaces $\{X_i\}$, $i = 1, 2, ..., n$, such that:

$$Rad(\Gamma) = X_0 \subset X_1 \subset X_2 \subset ... \subset X_n, \text{ where } X_i \neq X_j, \text{ for } i \neq j.$$

If there is no such integer n; the *polar rank* of Γ is defined to be infinite. If Γ is a polar space and $Rad\,(\Gamma) = \varnothing$, then Γ is called *non-degenerate polar space*; otherwise Γ is called *degenerate polar space.*

A point-line geometry is called a *parapolar space of rank* $r + 1$, $r \geq 2$; if it satisfies the following conditions:

(pp1) Γ is a connected gamma space.

(pp2) for every line l; $l^{\perp}$ is not a singular space.

(pp3) for every pair of distinct points x, y of distance 2; $x^{\perp} \cap y^{\perp}$ is empty, point, or non-degenerate polar space of *rank* $r \geq 2$.

If x, y are two points of a parapolar space; (x, y) is called a *special pair* if $x^{\perp} \cap y^{\perp}$ is just one point, and (x, y) is called a *polar pair* if $x^{\perp} \cap y^{\perp}$ is a non-degenerate polar space of *rank* at least 2.

A *strong parapolar space* is a parapolar space in which $x^{\perp} \cap y^{\perp}$ is a polar space for every pair of points distinct x, y of distance 2.

Let p be a point in a point-line geometry $\Gamma = (P, L)$; *Residue of* Γ at p denoted by Γ_p or *Res*(p); is a point-line geometry (P_p, L_p) defined as follows: P_p is the set of all lines containing p; a member of L_p is the set of all lines containing p and contained in a plane (singular space of *rank* 2).

3. Basic previous results

3.1 **Lemma.** *Let* $\Gamma = (P, L)$ *be a polar space. Then the following hold:*

(*i*) *If* l *is a line disjoint from* rad(Γ), *then for each* $p \in l$ *there is some* $q \in P$ *such that* $q^{\perp} \cap l = \{p\}$;

(*ii*) *If* $p^{\perp} \subseteq q^{\perp}$ *and* $p \neq q$, *then* Γ *is a degenerate polar space and either* $q^{\perp}$ *equal* $p^{\perp}$ *or* $q^{\perp}$ *equals* P.

Proof. (*i*) Given $p \in l$ then $p \notin \mathrm{rad}(\Gamma)$, so there is a point $r \notin p^{\perp}$. There is a unique point $s \in l$ with r and s collinear, say via a line m. Choose a point $t \notin s^{\perp}$ and a line n through t and some point $z \in m$. Now p is not collinear to z since p is not collinear to r and $z \neq s$. Then $p^{\perp} \cap n = \{e\}$ for some point e. Here $e \neq z$, and s is not collinear to q as $s \sim z$ and s is not collinear to t. Thus $e^{\perp} \cap l = \{p\}$.

(*ii*) Here $p \sim q$, so there is a line l through p and q, but no point w with $w^{\perp} \cap l = \{p\}$. By (i) $l \cap \mathrm{rad}(\Gamma) \neq \varnothing$, and Γ is degenerate. If (*ii*) fails then $p^{\perp} \subset q^{\perp} \subset \mathrm{rad}(\Gamma)$. But, choosing $r \in l \cap \mathrm{rad}(\Gamma)$ and $s \in q^{\perp} \backslash p^{\perp}$, we have s $\sim q$ and $s \sim r$ but not s $\sim p$, contrary to the definition of Gamma space.

3.2 **Lemma.** (Cooperstein's Theory) [Coo]. *Let* (P, L) *be a parapolar space of thick lines. Then for every pair of points p, q of distance* 2, *the convex closure* $\langle p, q \rangle$ *is a non-degenerate convex polar space whose rank is* $r - 1$. *Moreover all maximal singular subspaces of* $\langle p, q \rangle$ *are projective spaces.*

Such a convex polar space $\langle p, q \rangle$ is called *symplecton*. Convexity of such polar spaces forces the following very important property:

For any point p and any symplecton S of a parapolar space (P, L), $p \notin S$; *the subspace* $p^{\perp} \cap S$ *is a singular subspace of S. Moreover if* S_1, S_2 *are two distinct symplecta then* $S_1 \cap S_2$ *is also a singular subspace.*

Cohen and Cooperstein have characterized some of these Lie incidence geometries in a remarkable couple of theorems, as geometric spaces in terms of parapolar spaces. Here we mention the first of them.

3.3 **Theorem**. (Cohen-Cooperstein, Brouwer-Cohen) *Let* Γ = (P, L) *be a strong parapolar space of rank* $k \geq 3$ *of finite singular rank all of whose lines are thick and in which* $x^{\perp} \cap S$ *is empty, point or maximal singular subspace of S. Then one of the following situations occurs.*

i) (P, L) *is a non-degenerate polar space of rank k.*

ii) $k = 3$ *and* (P, L) *is the Grassmannian space* $A_{n,j}$ *for some* $n, j \in N$ *with* $1 \leq j \leq \frac{1}{2}(n+1) < \infty$.

iii) $k = 3$ *and* (P, L) *is the quotient of a space of type* $A_{2m-1,m}$ (*for some m with* $3 \leq m < \infty$) *by an involutory automorphism with the property that no point of the space is mapped to a point at distance* ≤ 4.

iv) $k = 5$ *and* (P, L) *is the quotient space of type* $D_{n,n}$ *for some* $4 \leq n < \infty$ *by a group of automorphisms such that the identity is the only element mapping a point of the space to a point at distance* ≤ 4.

v) $k = 5$ *and* (P, L) *is the space of type* $E_{6,1}$.

vi) $k = 6$ *and* (P, L) *is the space of type* $E_{7,1}$.

Conversely, the spaces in i)…vi) satisfies the hypotheses if there lines are thick.

A point-line geometry Γ = (P, L), whose points P are the cosets of a maximal parabolic subgroup of Lie type, and whose lines L are the cosets of the parabolic subgroup P'; corresponding to the collection of all nodes in the dynkin diagram adjacent to the unique node corresponding to P, is called *Lie incidence geometry.* In this paper we mark the node of the Dynkin diagram that corresponds to the set of points by P, and we mark the node(s) of the Dynkin diagram that corresponds to the set of lines by

L. These geometries constitute a very important class of point-line geometries that have been under intensive study recently to investigate their properties and to characterize them using points and lines. In this study, we mainly apply all of our results to them.

4. Finite classical polar spaces

Let V be a vector space over a finite field F = GF(q), q is a prime power.

1. *Symplectic Geometry* $W_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a symplectic bilinear form B. In this case n is even, the polar space is of rank ½ n.

2. *Hyperbolic Geometry* $\Omega^+_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a hyperbolic bilinear form B. In this case n is even, the polar space is of rank ½ n.

3. *Elliptic Geometry* $\Omega^-_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a elliptic bilinear form B. In this case n is even, the polar space is of rank ½ n - 1.

4. *Orthogonal Geometry* $\Omega_n(q)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for an orthogonal bilinear form B. In this case n is odd, the polar space is of rank ½ (n-1).

5. *Hermitian Geometry* $H^+_n(q^2)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x\rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y\rangle$ for which $B(x, y) = 0$, for a Hermitian form B. In this case n is even, the polar space is of rank n.

6. *Hermitian Geometry* $H^-_n(q^2)$ is the point-line geometry (P, L), where P is the set of all one dimensional subspaces $\langle x \rangle$ of V for which $B(x, x) = 0$, and L is the set of all two dimensional subspaces $\langle x, y \rangle$ for which $B(x, y) = 0$, for a Hermitian bilinear form B. In this case n is odd, the polar space is of rank n-1

Polar spaces of rank at least 3 have been classified in [BS], [B], [Jp], [BSP]. It turned out that those spaces in which lines contains at least 3 points and rank at least 3 are classical polar spaces.

In a polar space Γ = (P, L) of rank r, let |P| denote the number of points of P, and let Σ(P) be the set of all maximal totally singular subspaces of P; all members of Σ(P) have projective dimension r - 1. For the next Theorems see [Tha5].

4.1 **Theorem.** *The number of points of the finite classical polar spaces are given by the following formulae:*

$$|W_{2n}(q)| = (q^{2n} - 1)/(q - 1),$$
$$|\Omega(2n+1, q)| = (q^{2n} - 1)/(q - 1),$$
$$|\Omega^+(2n, q)| = (q^{n-1} + 1)(q^n - 1)/(q - 1),$$
$$|\Omega^-(2n, q)| = (q^{n-1} - 1)(q^n + 1)/(q - 1),$$
$$|H^-(2n+1, q^2)| = (q^{2n+1} + 1)(q^{2n+1} - 1)/(q^2 - 1),$$
$$|H^+(2n, q^2)| = (q^{2n} - 1)(q^{2n} + 1)/(q^2 - 1).$$

4.2 **Theorem.** *The numbers of maximal totally singular subspaces of the finite classical polar spaces are given by the following formulae*:

$$|\Sigma(W_{2n}(q))| = (q + 1)(q^2 + 1) \ldots (q^n + 1),$$
$$|\Sigma(\Omega(2n+1, q))| = (q + 1)(q^2 + 1) \ldots (q^n + 1),$$
$$|\Sigma(\Omega^+(2n, q))| = 2(q + 1)(q^2 + 1) \ldots (q^{n-1} + 1),$$
$$|\Sigma(\Omega^-(2n, q))| = (q^2 + 1)(q^3 + 1) \ldots (q^n + 1),$$
$$|\Sigma(H^-(2n+1, q^2))| = (q^3 + 1)(q^5 + 1) \ldots (q^{2n+1} + 1),$$
$$|\Sigma(H^+(2n, q^2))| = (q + 1)(q^3 + 1) \ldots (q^{2n-1} + 1).$$

Let P be a finite classical polar space of rank r; $r \geq 2$. An *ovoid* O of P is a subset of P, which has exactly one point in common with each maximal totally singular subspace of P. A *spread* S of P is a set of maximal totally singular subspace of P which partitions the set P. Ovoids and spreads may or may not exist, for a survey of the existence or non-existence of known and unknown cases, see [Tha5].

4.3 **Theorem.** *Let O be the ovoid and S be the spread of the finite classical polar space P. Then*

for $P = Sp(2n, q)$; $|O| = |S| = (q^n + 1)$,

for $P = \Omega(2n+1, q)$; $|O| = |S| = (q^n + 1)$,

for $P = \Omega^+(2n, q)$; $|O| = |S| = (q^{n-1} + 1)$,

for $P = \Omega^-(2n, q)$; $|O| = |S| = (q^n + 1)$,

for $P = U^-(2n+1, q^2)$; $|O| = |S| = (q^{2n+1} + 1)$,

for $P = U^+(2n, q^2)$; $|O| = |S| = (q^{2n-1} + 1)$.

The numbers |O| are called *ovoid numbers*.

5. A Family of Convex Spaces (Shult system)

Fix an integer $n \geq 1$, a geometry in E_1 is a single line together with its incident points. In general for $n > 1$, a point-line geometry Γ is in E_n if and only if

(*SS*1) Γ is a gamma space of point-diameter n.

(*SS*2) If $(x_0, x_1, \ldots, x_d)$ is a geodesic of length $d < n$, then it can be extended to a geodesic of length n.

(*SS*3) For each point x in Γ; the set of points of distance at most n-1 from x; $\Delta^*_{n-1}(x)$ is a geometric hyperplane of Γ.

(*SS*4) If x and y are non-collinear points of Γ of distance $d = d_\Gamma(x, y)$, then the convex closure; $\langle x, y\rangle$ belongs to E_d.

We call this system Shult system E_n.

This class of geometries was formed by Ernest Shult in order to provide uniform proofs that Veldkamp planes exist (see [Sh5], [Sh6]) for a large class of strong parapolar spaces, thus avoiding lengthy case-by-case arguments. El-Atrash and Shult have studied this class and characterized some of its geometries (see [At2]).

Clearly, the geometries of E_2 are polar spaces, and so by (*SS*4) every geometry of Γ is a strong parapolar space.

For $n \geq 2$, the classical known examples of geometries in E_n are the following geometries:

(1) All polar spaces.

(2) Grassmann space $A_{2n-1,n}(D)$, where D is a division ring, and $n \geq 3$; symplecta have polar rank three of type $A_{3,2}(D)$.

(3) Half-spin geometries $D_{2n,2n}(K)$, where K is a field, $n \geq 3$; symplecta have polar rank four of type $D_4(K)$.

(4) $E_{7,1}(K)$, where K is a field; symplecta have polar rank six of type $D_6(K)$.

(5) All products of geometries in (1) - (5), symplecta have various polar ranks.

Remark. Near polygons $C_{n,n}$, where $n \geq 2$ satisfy all axioms of E_n except it is a strong parapolar space of rank 2 i.e., symplecta have polar rank two (generalized quadrangles).

Geometries in E_n were classified in [At2] with the help of the following extra condition that is called "*weak hexagon property*".

5.1 Definition. (weak hexagon property) *In any geometry* Γ *we say that weak hexagon property holds if for any hexagon* $(x_0, x_1, \ldots, x_5)$ *of external diameter* 3*; we have:* $x_i^{\perp} \cap x_{i+2}^{\perp} \cap x_{i+4}^{\perp} \neq \emptyset$, *indices are taken modulo* 6, *for* $i = 0, 1$.

Part of the main theorem of El-Atrash and Shult [At2] is the following:

5.2 Theorem. *Suppose that* $\Gamma = (\mathrm{P}, \mathrm{L})$ *is a geometry in* E_n *with thick lines such that*:

(i) *the condition* (WH) *holds*,

(ii) *every symplecton of* Γ *has rank at least* 3,

(iii) Γ *has finite singular rank.*

Then Γ *is one of the following point-line geometries*

(1) *The Grassmannian* $\mathrm{A}_{2n-1,n}(\mathrm{D})$

(2) *The half-spin geometry* $\mathrm{D}_{2n,\,2n}(\mathrm{K})$.

(3) *The exceptional coset geometry* $\mathrm{E}_{7,1}(K)$, *for some field K.*

Note, the classification of the Shult system was among a more general system of axioms.

"Gatedness" played an important roll in the classification of such system, it simply says that certain sets are "gated" with respect to certain points, i.e., the distance has to be measured through the gate. The following is the form of gatedness we use in this paper.

5.3 Definition. *In any geometry* Γ, *we say that a set* Ω *is gated with respect to a point p not in* Ω, *if there is a point g in* Ω *such that for each* $x \in \Omega$, $d_\Gamma(p, x) = d_\Gamma(p, g) + d_\Omega(g, x)$, *where*

$d_\Gamma(a, b)$ *is the distance between a and b in the geometry* Γ.

$d_\Omega(a, b)$ *is the distance between a and b in the induced geometry on* Ω.

In [Usa1], the authors studied the following type of gatedness, "if $x^\perp \cap S$ is a single point then (x, S) is a gated point symplecton pair".

In the following lemmas we will prove some of the properties of Shult system.

5.4 Lemma. *Suppose* $\Gamma \in \mathrm{E}_n$,

(i) Suppose $Y \in \mathrm{E}_{n\text{-}1}(\Gamma)$ *and that x is at distance* 1 *and at distance n from points of Y. Then Y is gated with respect to point x.*

(ii) Suppose $T \in \mathrm{E}_m(\Gamma)$ $1 \le m < n$ *and x is a point such that there are points a and b in T with* $d_\Gamma(x, a) = n$, $d_\Gamma(x, b) = n - m$. *Then T is gated with respect to x with gate b.*

(iii) If $S \in \mathrm{E}_2(\Gamma)$ *and* $g \in S$, *then there exists a point u with* $d_\Gamma(g, u) = n - 2$ *and S is gated with respect to u.*

Proof can be found in [Sh5].

The following lemma and its proof can be found in [At3], however, for the purpose of completeness the proof will be provided.

5.5 Lemma. *Let* $\Gamma \in \mathrm{E}_3$. *Suppose that the weak hexagon property holds, then for every point x not in a symplecton S we have* $y^\perp \cap S \neq \varnothing$.

Proof. The proof proceeds in three stages. First we prove it under the extra condition that $d(x, y) = 3$, for some point $x \in S$, and then, using this result to prove it in the general case.

Let (y, S) be non-incident point-symplecton pair. Suppose there is a point $x \in S$ such that $d(x, y) = 3$, choose lines $l_1, l_2 \in S$ on the point x such that l_1 is not contained in $l_2^\perp$. Since by (SS3) $\Delta_2^*(y)$ is a hyperplane of Γ, the sets $l_i \cap \Delta_2^*(y)$ consists of single points a_i, $i = 1, 2$, and a_1 is not collinear to a_2. Since Γ is a strong parapolar space, the subspaces $T_i = \langle y, a_i \rangle$ are two symplecta. Since $a_1^\perp \cap T_2$ is a singular subspace and $a_2^\perp \cap y$ is not, it is possible to choose $b_2 \in (a_2^\perp \cap y) \backslash a_1^\perp$. Similarly, as $a_1^\perp \cap y$ it is not the union of two singular subspaces, it is possible to choose $b_2 \in (a_1^\perp \cap y) \backslash (a_2^\perp \cup b_2^\perp)$. Then the closed path $H = (x, a_1, b_1, y, b_2, a_2)$ is a hexagon with external diameter 3. By the weak hexagon property there is a point $z \in a_1^\perp \cap a_2^\perp \cap y^\perp$. Then $z \in \langle a_1, a_2 \rangle \cap y^\perp = S \cap y^\perp$ as desired in the lemma.

Now assume x, y are arbitrary points of $\Gamma \in E_3$. Let S be a symplecton on x. If x is collinear with y there is nothing to prove. Also if $d(x, y) = 3$, we are done by the previous paragraph. Thus, we may assume that $d(x, y) = 2$. By way of contradiction, assume that $y^{\perp} \cap S = \emptyset$. By (SS3) the path (x, a, y) of length 2 can be extended to a path (x, a, y, z) of length 3 in Γ. Thus $d(x, z) = 3$, where $a \in y^{\perp} \cap x^{\perp}$. Then by the previous paragraph there is a point $g \in z^{\perp} \cap S$. Then (z, S) is a gated point-symplecton pair with gate g. Since $d(x, z) = 3$, g is not collinear with x. Now, since $a^{\perp} \cap S$ is a singular subspace containing x and $a \notin g^{\perp}$ then there is a point $b \in g^{\perp} \cap x^{\perp} \cap S \backslash a^{\perp} \cap z^{\perp}$. Now, H' = (x, a, y, z, g, b) is a hexagon with external diameter 3. (see Figure). Thus the weak hexagon property applies to yield a point $q \in y^{\perp} \cap g^{\perp} \cap x^{\perp}$. Clearly, $q \in y^{\perp} \cap S$. This contradicting our assumption. This completes the proof.

5.6 Lemma. *Let $\Gamma \in E_3$. Suppose that the weak hexagon property holds, Let x_0, x_1, x_2 be three points of mutual distance 3. Let S be a symplecton containing x_0. Then $\Delta^*_2(x_1) \cap S \neq \Delta^*_2(x_2) \cap S$.*

Proof. Since the weak hexagon property holds in Γ then by Lemma 5.5, $x_i^{\perp} \cap S \neq \emptyset$, $i = 1, 2$. x_i's are of distance 1, 3 of points of S, then by Lemma (3.1), S is gated with respect to x_i, $i = 1, 2$. Let g_i be the gates i.e., $g_i = x_i^{\perp} \cap S$, $i = 1, 2$. $\Delta^*_2(x_i) \cap S$ is a hyperplane of S and $g_i^{\perp} \cap S = \Delta^*_2(x_i) \cap S$. Since the mutual distance of x_1, x_2 is 3 then $g_1 \neq g_2$. The non-degeneracy of S gives $g_1^{\perp} \cap S \neq g_2^{\perp} \cap S$. Thus, $\Delta^*_2(x_1) \cap S \neq \Delta^*_2(x_2) \cap S$.

In an almost similar argument we can prove the following lemma.

5.7 Lemma. *Let $\Gamma \in E_3$. Suppose that the weak hexagon property holds, Let x_0, be a point of distance 3 from the two points x_1, x_2. Let X be a maximal singular subspace containing x_0 and contained in a symplecton $S \in E_2$, such that $S \cap x_1^{\perp} \neq S \cap x_2^{\perp}$. Then*

$$\Delta^*_2(x_1) \cap X \neq \Delta^*_2(x_2) \cap X.$$

Proof. S contains points of distance 1, 3, from the points x_i, i = 1, 2. So, by Lemma , S is gated with respect to x_i. Then there are 2 points $y_i = S \cap x_i^{\perp}$, i = 1, 2. It follows that

$$\Delta^*_2(x_i) \cap S \subset y_i^{\perp} \cap S.$$

The subspace $\Delta^*_2(x_i) \cap X$ is a hyperplane of X, since X is containing x_0, which is of distance 3 from x_1.

Since $y_1 \neq y_2$, S is a non-degenerate polar space, then Lemma (3.1) applies; and we get $y_1^{\perp} \cap S \neq y_2^{\perp} \cap S$. By definition of E_3, $x_0^{\perp} \cap y_2^{\perp}$ is a non-degenerate polar space of rank one less than the rank of S. Then $x_0^{\perp} \cap y_2^{\perp}$ can not be contained in $y_1^{\perp}$. It follows that there are some points that belong to $x_0^{\perp} \cap y_2^{\perp}$ and not in $y_1^{\perp}$. Choose a point a in $y_2^{\perp} \cap X \backslash y_1^{\perp} \cap X$; then $a \in \Delta^*_2(x_2) \cap S \cap X$ and $a \notin \Delta^*_2(x_2) \cap S \cap X$. Therefore, $\Delta^*_2(x_1) \cap X \neq \Delta^*_2(x_2) \cap X$. This completes the proof.

5.8 Lemma. *Let* $\Gamma \in E_n$. *Suppose that* $\Delta^*_k(x) \cap Y \neq \emptyset$ *for every* $Y \in E_{n-k}(\Gamma)$, *Let* x_0*,be a point of distance n from the two points* x_1, x_2. *Let X be a maximal singular subspace containing* x_0 *and contained in* $Y \in E_{n-1}$, *such that* $x_1^{\perp} \cap Y \neq x_2^{\perp} \cap Y$. *Then*

$$\Delta^*_{n-1}(x_1) \cap X \neq \Delta^*_{n-1}(x_2) \cap X.$$

Proof. We will prove it by mathematical induction on n. By Lemma 5.7, it is true for $n = 3$.

Assume that it is true for $n = k - 1$. To show that it is true for $n = k$, let x_1, x_2 be two points of distance k from x_0. Let X be a maximal singular subspace containing x_0. Let Y $\in E_{k-1}$ be a convex subspace containing X, such that $x_1^{\perp} \cap Y \neq x_2^{\perp} \cap Y$. We need to prove that

$$\Delta^*_{k-1}(x_1) \cap X \neq \Delta^*_{k-1}(x_2) \cap X.$$

Let x_1 y_1 , y_2 , …, $y_k = x_0$ be a path of length k from x_1 to x_0. Let Z be the convex subspace $\langle y_2, x_0 \rangle$.

Since $x_2^{\perp} \cap Y \neq \emptyset$, let the unique point $z_1 = x_2^{\perp} \cap Y$ and let

$$z_2 = z_1^{\perp} \cap Z = \Delta_2(x_2) \cap Z.$$

If $y_2 \neq z_2$ then we by inductive hypothesis,

$$\Delta^*_{k-2}(y_1)\cap Z\cap X \neq \Delta^*_{k-2}(z_1)\cap Z\cap X.$$

Gatedness of the pairs (y_1, Z), (z_1, Z), implies that

$$\Delta^*_{k-2}(y_1)\cap Z\cap X = \Delta^*_{k-1}(x_1)\cap Y\cap X, \text{ and}$$

$$\Delta^*_{k-2}(z_1)\cap Z\cap X = \Delta^*_{k-1}(x_2)\cap Y\cap X.$$

Choose a point

$$a \in \Delta^*_{k-2}(z_1)\cap X\cap Z \backslash \Delta^*_{k-2}(y_1)\cap X\cap Z;$$

then

$$a \in \Delta^*_{k-1}(x_2)\cap X\cap Z \text{ and } a \notin \Delta^*_{k-1}(x_1)\cap Z\cap X,$$

since otherwise, a would be of distance $k - 1$ and it belongs to $Z \subset Y$ therefore by gatedness it must be of distance $k - 2$ to y_1 that contradicts our choice of a. Thus

$$\Delta^*_{n-1}(x_1)\cap X \neq \Delta^*_{n-1}(x_2)\cap X.$$

If $y_2 = z_2$ then we will choose a different path, this is possible because $y_1^{\perp}$ is not contained in $z_1^{\perp}$, since $\Delta^*_{k-2}(x_0)\cap\langle y'_1 , y_1\rangle\cap Z$ is a hyperplane of the symplecton $S = \langle y'_1 , y_1\rangle$, where (z_1, y_1, y'_1) is a path of length 3. This completes the proof.

Remark. The condition " $\Delta^*_k(x)\cap Y \neq \varnothing$ *for every* $Y \in E_{n-k}(\Gamma)$" although it seems strong, it is satisfied in all geometries belonging to the class E_n having the weak hexagon property.

6. Shult sets

6.1 Definition. *A subset S of a geometry of diameter n is called a partial Shult set if the distance between any pair of points of S is n (antipodal points). If the maximum (to be specified later) is achieved then S is called a Shult set.*

Examples. (trivial) Let Γ be the following geometries:

1. (Quadrangle) Points are the four corners of any rectangle, lines are the sides that consists only of the two points. Diameter of this geometry is 2. Shult set in this case is the end points of any diagonal.

2. (Cube) Points are the eight corners of any cube. Lines are the sides. Diameter of this geometry is 3. Shult set in this case is the end points of any diagonal.

Classical Examples.

(1) (Generalized quadrangle) Points are the fifteen doupletons (sets of 2 points) of the 6-point set $\Omega = \{1, 2, 3, 4, 5, 6\}$, lines are the fifteen 2-2-2-set partitions. I.e.,

The points of this quadrangle are in the following table:

1. {1,2},	6. {2,3},	11. {3,5},
2. {1,3},	7. {2,4},	12. {3,6},
3. {1,4},	8. {2,5},	13. {4,5},
4. {1,5},	9. {2,6},	14. {4,6},
5. {1,6},	10. {3,4},	15. {5,6}

The lines of this quadrangle are in the following table:

1. {1,2}-{3,4}-{5,6},	6. {1,3}-{2,6}-{5,4},	11. {1,5}-{2,4}-{3,6},
2. {1,2}-{3,5}-{4,6},	7. {1,4}-{3,2}-{5,6},	12. {1,5}-{2,6}-{3,2},
3. {1,2}-{3,6}-{5,4},	8. {1,4}-{3,5}-{2,6},	13. {1,6}-{3,4}-{5,2},
4. {1,3}-{2,4}-{5,6},	9. {1,4}-{3,6}-{5,2},	14. {1,6}-{3,5}-{4,2},
5. {1,3}-{2,5}-{4,6},	10. {1,5}-{3,4}-{2,6},	15. {1,6}-{3,2}-{5,4}.

This geometry is a polar space of rank 2. As can be seen easily every line contains 3 points and every point lies on 3 lines. Diameter of this geometry is 2. This example was introduced by Sylvester (1861-1884). He called points *duads*, and lines *synthemes.* Shult set in this case is an ovoid of the generalized quadrangle and it must contain 5 points of mutual distance 2, one example of such set is {{1,2}, {1,3},{1,4}, {1,5}, {1,6}}.

The dual of example 3 is also a generalized quadrangle and an ovoid in this case is called a spread which consists of 5 lines no two are concurrent (intersecting). One example of a spread is the following {{1,2}-{3,4}-{5,6}, {1,3}-{2,5}-{4,6}, {1,4}-{3,5}-{2,6}, {1,5}-{2,4}-{3,6}, {1,6}-{3,2}-{5,4}}.

(2) Let Γ be the polar space of rank *r*. Γ has diameter 2. Thus, in this case, partial Shult-set is a partial ovoid in the underlying polar space. Conversely, every partial ovoid in the underlying polar space produces a partial Shult-set. The cardinality is the ovoid number of the polar space.

(3) Let Γ be the half-spin geometry $D_{2n+1,2n+1}(q)$; points of $D_{2n+1,2n+1}(q)$ are maximal totally singular (isotropic) subspaces of the underlying hyperbolic quadric (or bilinear form) of one class of odd dimension $2n+1$. The diameter of this geometry is *n*. The distance $d_\Gamma(\Sigma_1, \Sigma_2) = ½ (2n+1 - \text{dimension } \Sigma_1\cap\Sigma_2)$, i.e., the distance is half of the co-dimension of the intersection in each of the subspaces. Therefore, $d(\Sigma_1, \Sigma_2) = n$ iff $\text{dimension}(\Sigma_1\cap \Sigma_2) = 1$.

(4) In the case of half-spin geometries $D_{2n,2n}(q)$; points of $D_{2n,2n}(q)$ are maximal totally singular (isotropic) subspaces of the underlying hyperbolic quadric (or bilinear form) of one class. The distance $d_\Gamma(\Sigma_1, \Sigma_2)$ of two points Σ_1, Σ_2 of the half-spin geometry $D_{2n,2n}(q)$ is the dimension of the intersection $\Sigma_1\cap\Sigma_2$ of the two spaces Σ_1, Σ_2. Therefore, $d(\Sigma_1, \Sigma_2) = n$ iff $\text{dimension}(\Sigma_1\cap \Sigma_2) = 0$. Thus, in this case, partial Shult-set is a partial spread in the underlying hyperbolic geometry. Conversely, every partial spread in the underlying hyperbolic geometry produces a partial Shult-set. Thus the existence or non-existence of Shult sets in this case depends on the existence or non-existence of spreads. Moreover, the cardinality of Shult sets is the ovoid number in hyperbolic underlying geometry.

(5) In the case of Grassmann geometries $A_{2n-1,n}(q)$; points of $A_{2n-1,n}(q)$ are all *n*-subspaces of the underlying projective space $PG(2n-1, q)$. The distance $d(\Sigma_1, \Sigma_2)$ of two points (*n*-dimensional subspaces) Σ_1,

Σ_2 of the Grassmann geometry $A_{2n-1,n}(q)$ is the dimension of the intersection $\Sigma_1 \cap \Sigma_2$ of the two spaces Σ_1, Σ_2. Therefore, $d(\Sigma_1, \Sigma_2) = n$ iff dimension$(\Sigma_1 \cap \Sigma_2) = 0$. Thus in this case, partial Shult-set is a partial spread of the projective space. Conversely, every partial projective spread in the underlying projective geometry produces a partial Shult-set. Thus the existence or non-existence of Shult sets in this case depends on the existence or non-existence of projective spreads. Moreover, the cardinality of Shult sets is the number $(q^{2n} - 1)/(q^n - 1) = q^n + 1$.

(6) In the case of dual polar spaces; points are maximal totally singular subspaces. The distance $d_\Gamma(\Sigma_1, \Sigma_2)$ of two points Σ_1, Σ_2 is the co-dimension of the intersection $\Sigma_1 \cap \Sigma_2$ of the two spaces Σ_1, Σ_2 in any of the two subspaces. Therefore, $d(\Sigma_1, \Sigma_2) = n =$ dimension(Σ_1) iff dimension$(\Sigma_1 \cap \Sigma_2) = 0$. Thus, in this case, partial Shult-set is a partial spread in the underlying polar space. Conversely, every partial spread in the underlying polar space produces a partial Shult set. Thus, the cardinality of Shult set in this case is the ovoid number of the underlying polar space.

It is nearly impossible to determine the cardinality of Shult sets in arbitrary geometries, however, in some classes of geometries like the ones we are concerned with here, such as polar spaces, half-spin geometries $D_{2n,\ 2n}(q)$, Grassmann spaces $A_{2n-1,n}(q)$, the exceptional geometry $E_{7,1}(q)$, the cardinality will be determined.

6.2 Theorem. *Let $\Gamma \in E_n$. Let H be a Shult set in Γ. Then $|H| \leq q^n + 1$, where $(q^{n+1}-1)/(q-1)$, is the cardinality of the maximal singular subspace in any symplecton in Γ.*

Proof. By the last Lemma 5.7 the number of points in a Shult set can not be more than the number of hyperplanes in a maximal singular subspace of Γ containing a fixed point. Since the number of hyperplanes in an n+1-dimensional vector space is $(q^{n+1}-1)/(q-1)$, and the number of hyperplanes in an n+1-dimensional vector space containing a 1-dimensional subspace is $(q^n - 1)/(q-1)$. Therefore the difference is q^n. Hence $|H| \leq q^n + 1$.

Application

Independent of the classification of the class E_n, we have found the sizes of Shult sets, in all geometries that belong to the class E_n.

First, geometries in E_2 are polar spaces, therefore, Shult sets in these case are ovoids, thus, the sizes in the finite classical cases are exactly the ovoid numbers, as shown in Theorem 4.3 which can be considered as a corollary of Theorem 6.2.

The class E_3 has a very special geometry $E_{7,1}(q)$, for which the size of Shult set is determined. The following table has the corresponding sizes of Shult sets (Table 1).

The class E_n, $n > 3$, has only the two families of geometries $D_{2n,2n}(q)$, $A_{2n-1,n}(q)$

Type of parapolar space	*Upper bound of Shult-set*
$D_{2n,2n}(q)$	$(q^{2n-1} + 1)$
$A_{2n-1,n}(q)$	$(q^{n} + 1)$
$E_{7,1}(q)$	$(q^{5} + 1)$

Table 1

6.3 Corollary.

(i) *Shult set in $D_{2n,2n}(q)$ exist iff the underlying hyperbolic polar space $\Omega^{+}(2n, q)$ has a spread, $n \geq 3$.*

(ii) *Shult set in $A_{2n-1,n}(q)$ always exist.*

(iii) *Shult set in polar spaces iff ovoid exist.*

Proof. Straight forward.

Example of Shult set in $A_{5,3}(2)$:

We are looking for a total of 9 of 3-dimensional subspaces that partition the projective space PG(5, 2). They can be represented as follows by taking the standard basis:

s_1= {(1,0,0,0,0,0), (0,1,0,0,0,0), (1,1,0,0,0,0), (0,0,1,0,0,0), (1,0,1,0,0,0), (0,1,1,0,0,0), (1,1,1,0,0,0)}

s_2= {(0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,1,1,0), (0,0,0,0,0,1), (0,0,0,1,0,1), (0,0,0,0,1,1), (0,0,0,1,1,1)}

s_3= {(1,0,0,1,0,0), (0,1,0,0,1,0), (1,1,0,1,1,0), (0,0,1,0,0,1), (1,0,1,1,0,1), (0,1,1,0,1,1), (1,1,1,1,1,1)}

s_4= {(0,1,0,1,0,0), (0,0,1,0,1,0), (0,1,1,1,1,0), (1,0,1,0,0,1), (1,1,1,1,0,1), (1,0,0,0,1,1), (1,1,0,1,1,1)}

s_5= {(1,1,0,1,0,0), (0,1,1,0,1,0), (1,0,1,1,1,0), (1,0,0,0,0,1), (0,1,0,1,0,1), (1,1,1,0,1,1), (0,0,1,1,1,1)}

s_6= {(0,0,1,1,0,0), (1,0,1,0,1,0), (1,0,0,1,1,0), (1,1,1,0,0,1), (1,1,0,1,0,1), (0,1,0,0,1,1), (0,1,1,1,1,1)}

s_7= {(1,0,1,1,0,0), (1,1,1,0,1,0), (0,1,0,1,1,0), (1,1,0,0,0,1), (0,1,1,1,0,1), (0,0,1,0,1,1), (1,0,0,1,1,1)}

s_8= {(0,1,1,1,0,0), (1,0,0,0,1,0), (1,1,1,1,1,0), (0,1,0,0,0,1), (0,0,1,1,0,1), (1,1,0,0,1,1), (1,0,1,1,1,1)}

s_9= {(1,1,1,1,0,0), (1,1,0,0,1,0), (0,0,1,1,1,0), (0,1,1,0,0,1), (1,0,0,1,0,1), (1,0,1,0,1,1), (0,1,0,1,1,1)}.

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