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INTEGRATION WITH RESPECT TO A NONCONTINUOUS $\oplus$-MEASURE

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التكامل بالنسبة لقياس ⊕ الغير متصل

ملخص لقد قدمت من Z.Rievcanova (ريكانوفا) في المرجع (4)و I.Marinova (مارينوفا) في المرجع (3) قياس خاص يعمم كل من القياس الجمعي المعدود والقياس المعدود العِظَمْ .

نقوم في هذا البحث بدراسة خواص الاتصال لقياسات ⊕ وكذلك التكامل بالنسبة لمثل هذه القياسات، ونبيّن أن الكثير من النتائج الخاصة بهذا التكامل مثل نظرية التقارب (المضطرد) ونظرية فاتو، والتي تم برهانها في المرجع (2) لقياسات ⊕ المتصلة ، يمكن تعميمها لقياسات ⊕ الغير المتصلة.

ABSTRACT: A special measure that generalizes σ-additive and σ-maxitive measures, which is called a $\oplus$-measure, have been introduced by Z. Riečanová [4] and I. Marinová [2]. In this paper, we study the continuity properties of $\oplus$-measures and integration with respect to these measures, and we show that many of the results about this integration such as the monotone convergence theorem and Fatou's lemma, that have been obtained by [2] for continuous $\oplus$-measures, can be generalized to non-continuous $\oplus$-measures.

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1 Introduction

In 1971, N. Shilkert defined a special measure on a ring $\Re$ of sets, which he called $\sigma - \textit{maxitive measure}$, and then he defined integration with respect to this measure [6]. Although they are different, the σ-additive and the σ-maxitive measures have some common properties. Moreover, common properties of both measures and their integration can be performed simultaneously by the help of another special measure called a $\oplus - \textit{measure}$, which will be one of the techniques for studying these common properties [2], [4].

In this paper, we study the continuity of a positive set function defined on a ring $\Re$ of sets and we give conditions which are equivalent to continuity. In fact, we show that a supremeasure, as defined in [4], is a positive set function m on a ring $\Re$ which is continuous from below and $m(\emptyset) = 0$. We also study integration with respect to a $\oplus-$measure and we show that the monotone convergence theorem, which appears in [2] and requires that $(X, \mathcal{B}, m)$ to be a continuous $\oplus-$measure space, can be generalized by proving it for an arbitrary $\oplus-$measure space. We then use this result to derive other results such as Fatou's lemma.

Let $\oplus$ be some binary operation on $[0, \infty]$ with the following axioms: $\forall a, b, c \in [0, \infty], k \geq 0$,

1. $a \oplus b = b \oplus a$.
2. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
3. $k(a \oplus b) = (ka \oplus kb)$.
4. $a \oplus 0 = a, \quad a \oplus \infty = \infty$.
5. If $a \leq b$, then $\quad a \oplus c \leq b \oplus c$.
6. $(a + b) \oplus (c + d) \leq (a \oplus c) + (b \oplus d)$.
7. If $a_n \longrightarrow a,\ b_n \longrightarrow b$, then $a_n \oplus b_n \longrightarrow a \oplus b \quad \forall a_n, b_n, a, b \in [0, \infty]$.

We shall write $\bigoplus_{i=1}^{n} a_i$ for $a_1 \oplus a_2 \oplus ... \oplus a_n$, and $\bigoplus_{i=1}^{\infty} a_i$ for $\sup_n \bigoplus_{i=1}^{n} a_i$.

1.1 Definitions.

Let $\Re$ be a ring of subsets of a nonempty set X, and let $\bigoplus$ be a binary operation on $[0,\infty]$ which satisfies the above properties $1-8$. A set function $m : \Re \longrightarrow [0,\infty]$ is called a $\bigoplus$ *−measure* (see [4]) if

(a) $m(\emptyset) = 0$, and

(b) $m(\bigcup_{i=1}^{\infty} E_i) = \bigoplus_{i=1}^{\infty} m(E_i)$ for each sequence of mutually disjoint sets $\{E_i\}_{i=1}^{\infty}$ in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$.

A set function $m : \Re \longrightarrow [0,\infty]$ is called a $\sigma-$ *maxitive measure* (see [6]) if m satisfies (a) above and if $m(\bigcup_{i=1}^{\infty} E_i) = \sup_i m(E_i)$ for each sequence of mutually disjoint sets $\{E_i\}_{i=1}^{\infty}$ in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$.

Note that a $\oplus-$measure agrees with the $\sigma-$ additive measure when the binary operation $\oplus$ is the ordinary $+$ operation on $[0,\infty]$, and agrees with the $\sigma-$maxitive measure when the operation $\oplus$ is the maximum operation on $[0,\infty]$. It is easy to derive (from axioms (1) – (7) of the $\oplus$ operation and the definition of the $\oplus-$ measure) the following properties:

(a) $a \le a \oplus b$ for all $a, b \in [0,\infty]$.

(b) For all $a, b \in [0,\infty]$, $a \oplus b \le a + b$.

(c) If $a, b, c, d \in [0,\infty]$ are such that $a \le b$ and $c \le d$, then $a \oplus c \le b \oplus d$.

(d) A $\oplus-$ measure is monotone; i.e., $A, B \in \Re$, $A \subseteq B$ implies that $m(A) \le m(B)$.

2 Continuity of a $\oplus$-Measure

2.1 Definition [1].

Let m be a set function from a ring $\Re$ to $[0,\infty]$. Then we say that:

(a) m is *continuous from above* (abbreviated C.F.A.) if whenever $E_n \downarrow E$ in $\Re$ and $\exists\ k \in \mathbf{N}$ such that $m(E_k) < \infty$, we then have $m(E_n) \downarrow m(E)$;

(b) m is *continuous from below* (abbreviated C.F.B.) if whenever $E_n \uparrow E$ in $\Re$, we then have $m(E_n) \uparrow m(E)$;

(c) m is *continuous* if it is both continuous from above and continuous from below.

2.2 Examples.

(a) In this example, we give a continuous $\oplus$−measure. Let $X = \{1, 2, 3\}$ and $\Re = \mathcal{P}(X)$. Define $m : \Re \longrightarrow [0, \infty]$ by

$$m(A) := \begin{cases} \max\ A, & A \neq \emptyset \\ 0, & A = \emptyset. \end{cases}$$

Then m is σ−maxitive, and thus a $\oplus$−measure where $\oplus$ is the max operation on $[0, \infty]$. It is easy prove that m is continuous.

(b) Let $X = [1, \infty)$, $\Re = \mathcal{P}(X)$, and define a set function $m : \Re \longrightarrow [0, \infty]$ by

$$m(E) := \begin{cases} \sup E & \text{if } E \neq \emptyset \text{ and } E \text{ is bounded} \\ 0 & \text{if } E = \emptyset \\ \infty & \text{if } E \text{ is unbounded.} \end{cases}$$

Then m is a non-continuous σ−maxitive measure.

2.3 Definition[4].

Let $\Re$ be a ring of subsets of a nonempty set X. A set function $m : \Re \to [0, \infty]$ is called *a supremeasure* on $\Re$ if $m(\emptyset) = 0$ and $m(\bigcup_{i=1}^{\infty} E_i) = \sup_n m(\bigcup_{i=1}^{n} E_i)$ for each sequence of mutually disjoint sets in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$.

2.4 Proposition.

A set function $m : \Re \longrightarrow [0, \infty]$ is
(i) C.F.A iff whenever $\{E_i\}_{i=1}^{\infty}$ is a sequence in $\Re$ such that $\bigcap_{i=1}^{\infty} E_i \in \Re$,

and $m(\bigcap_{i=1}^{k} E_i) < \infty$ for some k $\in$ **N**, then we have

$$m(\bigcap_{i=1}^{\infty} E_i) = \inf_n\{m(\bigcap_{i=1}^{n} E_i)\}; \tag{2.1}$$

(ii) C.F.B iff whenever $\{E_i\}_{i=1}^{\infty}$ is a sequence in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$, then we have

$$m(\bigcup_{i=1}^{\infty} E_i) = \sup_n\{m(\bigcup_{i=1}^{n} E_i)\}. \tag{2.2}$$

Proof: (i) $(\Longrightarrow)$: Suppose that m is C.F.A. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\Re$ such that $\bigcap_{i=1}^{\infty} E_i \in \Re$, and $m(\bigcap_{i=1}^{k} Ei) < \infty$ for some $k \in \mathbf{N}$. Set $F := \bigcap_{i=1}^{\infty} E_i$, and $F_n := \bigcap_{i=1}^{n} E_i$, $n \in \mathbf{N}$. Then $F_n \downarrow F$ and $m(F_k) < \infty$; hence $m(F_n) \downarrow m(F)$, and therefore $m(F) = \lim_{n\to\infty} m(F_n) = \inf_n m(F_n)$.

$(\Longleftarrow)$: Suppose that if $\{E_i\}_{i=1}^{\infty}$ is a sequence in $\Re$ such that $\bigcap_{i=1}^{\infty} E_i \in \Re$ and $m(\bigcap_{i=1}^{k} E_i) < \infty$ for some $k \in \mathbf{N}$, then (2.1) holds. One can easily check that m is monotone; i.e., $A \subseteq B$ in $\Re$ implies $m(A) \leq m(B)$. Let $F_n \downarrow F$ in $\Re$ such that $m(F_s) < \infty$ for some s $\in$ **N**. Then $\bigcap_{i=1}^{\infty} F_i = F \in \Re$, and $m(\bigcap_{i=1}^{s} F_i) = m(F_s) < \infty$. Hence, by hypothesis, we have $m(F) = m(\bigcap_{i=1}^{\infty} F_i) = \inf_n m(\bigcap_{i=1}^{n} F_i) = \inf_n m(F_n)$. Since $F_n \downarrow F$ and m is monotone, it follows that $m(F_n) \downarrow$ and

$$m(F) = \inf_n m(F_n) = \lim_{n\to\infty} m(F_n).$$

(ii)$(\Longrightarrow)$: Suppose that m is C.F.B. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$. Set $F := \bigcup_{i=1}^{\infty} E_i$, and $F_n := \bigcup_{i=1}^{n} E_i$, $n \in \mathbf{N}$. Then $F_n \uparrow F$, and hence $m(F_n) \uparrow m(F)$, so that

$$m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(\bigcup_{i=1}^{n} E_i) = \sup_n\{m(\bigcup_{i=1}^{n} E_i)\}.$$

$(\Longleftarrow)$: Suppose that if $\{E_i\}_{i=1}^{\infty}$ is a sequence in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$, then (2.2) holds. It is easily checked that m is monotone. Let $F_n \uparrow F$ in $\Re$. Then $\bigcup_{i=1}^{\infty} F_i = F \in \Re$. Hence, by the hypothesis (2.2), we have $m(F_n) \uparrow$, and $m(F) = \sup_n\{m(\bigcup_{i=1}^{n} F_i)\} = \sup_n m(F_n) = \lim_{n\to\infty} m(F_n)$. ■

Part (a) of the following proposition proves that the second condition of the definition of the supremeasure of [4] is equivalent to the definition of continuity from below.

2.5 Proposition.

(a) A set function m on a ring $\Re$ is C.F.B., if $m(\bigcup_{i=1}^{\infty} E_i) = \sup_n\{m(\bigcup_{i=1}^{n} E_i)\}$ for any sequence of mutually disjoint sets $\{E_i\}_{i=1}^{\infty}$ in $\Re$ such that $\bigcup_{i=1}^{\infty} E_i \in \Re$.

(b) If $m : \Re \longrightarrow [0, \infty]$ is a $\oplus-$measure, then

(i) m is C.F.B. [4];

(ii) m need not be continuous (see Example 2.2(b));

(iii) m is C.F.A. iff whenever $E_n \downarrow \emptyset$ in $\Re$ and $m(E_k) < \infty$ for some $k \in \mathbf{N}$, we have $m(E_n) \downarrow 0$ [6].

Proof: (a) Let $\{F_i\}_{i=1}^{\infty}$ be any sequence of sets in $\Re$ such that $\bigcup_{i=1}^{\infty} F_i \in \Re$. Let $Q_1 := F_1$, and for n ≥ 2, let $Q_n := F_n \setminus \bigcup_{i=1}^{n-1} F_i$. Then $\{Q_i\}_{i=1}^{\infty}$ is a disjoint sequence in $\Re$ such that $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} Q_i$, and $\bigcup_{i=1}^{n} Q_i = \bigcup_{i=1}^{n} F_i \quad \forall n \in \mathbf{N}$. Hence

$$m(\bigcup_{i=1}^{\infty} F_i) = m(\bigcup_{i=1}^{\infty} Q_i) = \sup_n m(\bigcup_{i=1}^{n} Q_i) = \sup_n m(\bigcup_{i=1}^{n} F_i),$$

and therefore by Proposition 2.4(ii), m is C.F.B. ■

2.6 Lemma [4].

Let m be a set function that is C.F.B. on a ring $\Re$ such that $m(\emptyset) = 0$, and let $\oplus$ be a binary operation on $[0, \infty]$ satisfying axioms (1) – (7) of Section 1. Then m is a $\oplus-$measure iff $m(A \cup B) = m(A) \oplus m(B) \quad \forall A, B \in \Re$ such that $A \cap B = \emptyset$.

3 Integration

3.1 Definition.

A $\oplus-$*measure space* is a triple $(X, \mathcal{B}, m)$ where $(X, \mathcal{B})$ is a measurable space, and m is a $\oplus-$measure defined on $\mathcal{B}$. In a $\oplus-$measure space, a property P is said to hold *almost everywhere* (abbreviated a.e.) if the set of points where it fails to hold has measure zero.

Throughout this section, unless otherwise stated, we let $(X, \mathcal{B}, m)$ denote a fixed $\oplus$-measure space. We shall first define an integral with respect to m for a non-negative simple function (abbreviated NSF).

3.2 Definition [2].

Let f be a NSF, so that $f = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$, for some measurable disjoint sets E_i and $0 < \alpha_i < \infty$ for $i = 1, 2, 3, ..., n$. We define the *integral* of f with respect to m by

$$\int f dm := \bigoplus_{i=1}^{n} \alpha_i m(E_i)$$

and we say that f is *integrable* iff $\int f dm < \infty$.

3.3 Proposition.

$\int f dm$ is well-defined; i.e., it depends only on f and not on $\alpha_1, \alpha_2, ..., \alpha_n$, and $E_1, E_2, ..., E_n$.

Proof: Suppose that $f = \sum_{k=1}^{n} a_k \chi_{E_k} = \sum_{j=1}^{m} b_j \chi_{F_j}$, where $a_k, b_j \geq 0$, and $\{E_k\}_{k=1}^{n}$, $\{F_j\}_{j=1}^{m}$ are both disjoint in $\mathcal{B}$. Let $a_o := 0$, $b_o := 0$, $E_o := X \setminus \bigcup_{k=1}^{n}$ and $F_o := X \setminus \bigcup_{j=1}^{m} F_j$. Then $f = \sum_{k=0}^{n} a_k \chi_{E_k} = \sum_{j=0}^{m} b_j \chi_{F_j}$.

Claim: $\bigoplus_{k=0}^{n} a_k \; m(E_k) = \bigoplus_{j=0}^{m} b_j \; m(F_j)$.

[To see this, note that $E_k = E_k \cap X = \bigcup_{j=0}^{m} (E_k \cap F_j)$, so $m(E_k) = \bigoplus_{j=0}^{m} m(E_k \cap F_j)$ and hence

$$\bigoplus_{k=0}^{n} a_k \; m(E_k) = \bigoplus_{k=0}^{n} a_k \bigoplus_{j=0}^{m} m(E_k \cap F_j) = \bigoplus_{k=0}^{n} \bigoplus_{j=0}^{m} a_k \; m(E_k \cap F_j).$$

But if $x \in E_k \cap F_j$, then $f(x) = a_k = b_j$, and so $a_k\ m(E_k \cap F_j) = b_j\ m(E_k \cap F_j)$. Therefore

$$\begin{aligned} \bigoplus_{k=0}^{n} a_k\ m(E_k) &= \bigoplus_{k=0}^{n}\bigoplus_{j=0}^{m} b_j\ m(E_k \cap F_j) = \bigoplus_{j=0}^{m} b_j \bigoplus_{k=0}^{n} m(E_k \cap F_j) \\ &= \bigoplus_{j=0}^{m} b_j\ m(\bigcup_{k=0}^{n} E_k \cap F_j) = \bigoplus_{j=0}^{m} b_j\ m(F_j). \quad] \quad \blacksquare \end{aligned}$$

3.4 Proposition [2].

Letf, g be NSF-s. Then we have

(1) $\int (f+g) \leq \int f + \int g$.

(2) If $f \cdot g = 0$, then $\int f + g = \int f \oplus \int g$.

3.5 Definition [2].

(a) If $f : X \longrightarrow [0, \infty)$ is a measurable function, then we define

$$\int f dm := \sup\{\textstyle\int g dm : g \leq f,\ g \text{ is NSF}\}$$

and we say that f is *integrable* iff $\int f dm < \infty$.

(b) If $f : X \longrightarrow (-\infty, \infty)$ is a measurable function, and at least one of the functions $f^+ := \max\{f, 0\}$ and $f^- := -\min\{f, 0\}$ is integrable, then we define

$$\int f dm := \int f^+ dm - \int f^- dm$$

and we say that f is *integrable* iff $-\infty < \int f dm < \infty$.

3.6 Remark [2].

For $\sigma-$maxitive measures, N.Shilkert [6] defined the integral of a non-negative measurable function as follows:

$$\int_{sh} f dm = \sup_{a>0} a\, m\{x : f(x) \geq a\}.$$

If a $\oplus-$measure m is a $\sigma-$maxitive measure, then it can be shown (see [2]) that $\int f dm = \int_{sh} f dm$ for each non-negative measurable function f.

3.7 Proposition [2].

Let f, g and h be measurable functions such that $\int f, \int g$ and $\int h$ have meaning; i.e., each belongs to $[-\infty, \infty]$. Then we have the following:

1. If $f \geq 0$, then $\int f \geq 0$.
2. If $f \leq g$, then $\int f \leq \int g$.
3. If $f \leq h \leq g$ and f, g are integrable, then h is integrable.
4. If $A \subseteq B$ in $\mathcal{B}$ and f is a non-negative measurable function, then $\int_A f \leq \int_B f$.
5. If $c \in (-\infty, \infty)$, then $\int cf = c \int f$.

3.8 Proposition.

let f, g and h be measurable functions such that $\int f, \int g$ and $\int h$ have meaning; i.e., each belongs to $[-\infty, \infty]$. Then we have the following:

1. If $E \in \mathcal{B}$ and $m(E) = 0$, then $\int_E f = \int f\chi_E = 0$ for any non-negative measurable function f.
2. For a non-negative measurable function f we have $\int_{A\cup B} f \leq \int_A f + \int_B f$ where $A \cap B = \emptyset$.
3. If f is a non-negative measurable function and $A \in \mathcal{B}$ such that $m(A) = 0$, then $\int_{A\cup B} f = \int_B f \quad \forall B \in \mathcal{B}$ with $A \cap B = \emptyset$.

4. If f is a non-negative measurable function, then $\int f = 0$ iff $f = 0$ a.e.

5. If f is integrable, then f is finite a.e.

6. If $\int f = \int g$, then f need not equal g a.e.

Proof: (1) **Case 1:** Suppose first that $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is a nonnegative simple function. Then we have $0 \le \int_E f = \bigoplus_{i=1}^{n} \alpha_i \; m(A_i \cap E) \le \bigoplus_{i=1}^{n} \alpha_i m(E) = 0$, and therefore $\int_E f = 0$.
Case 2: Suppose that f is a non-negative measurable function. Let g be a NSF such that $g \le f$. Then by case 1, we have that $\int_E g = 0$; therefore $\int_E f = 0$.
Case 3: Suppose that f is a real-valued function. Then, by case 2, we have that $\int_E f^+ = \int_E f^- = 0$, and so $\int_E f = 0$.

(2) **Case 1:** Suppose that f is a NSF. Then, by part (1) of Proposition 3.4, we have

$$\int_{A\cup B} f = \int (f\chi_A + f\chi_B) \le \int f\chi_A + \int f\chi_B = \int_A f + \int_B f.$$

Case 2: Suppose that f is a non-negative measurable function. Let g be a NSF such that $g \le f$. Then, by case 1, we have

$$\int_{A\cup B} g \le \int_A g + \int_B g \le \int_A f + \int_B f.$$

As $g \le f$ was arbitrary NSF, we have $\int_{A\cup B} f \le \int_A f + \int_B f$.

(3) Let $B \in \mathcal{B}$ be such that $A \cap B = \emptyset$. Since f is non-negative, by parts (1),(2) of this proposition and by part (4) of Proposition 3.7 we have that

$$\int_{A\cup B} f \le \int_A f + \int_B f \;\; = \;\; \int_B f \le \int_{A\cup B} f.$$

Therefore $\int_{A\cup B} f = \int_B f$.

(4)($\Longrightarrow$) : Suppose that $\int f = 0$. Let $A_n := \{x : f(x) \geq \frac{1}{n}\}$, $n \in$ N. Then $f \geq \frac{1}{n}\chi_{A_n}$, and hence $m(A_n) \leq n \int f = 0 \quad \forall n$. Thus, as $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} A_n$, we have that

$$0 \leq m(\{x : f(x) > 0\}) = m(\bigcup_{n=1}^{\infty} A_n) \leq \bigoplus_{n=1}^{\infty} m(A_n) = 0,$$

which implies that $m(\{x : f(x) > 0\}) = 0$, and therefore $f = 0$ a.e.

($\Longleftarrow$) : Suppose that $f = 0$ a.e., and let $E = \{x : f(x) > 0\}$. Then $m(E) = 0$; so by part (3) of this proposition, $\int f = \int_{E^c} f = 0$.

(5) Suppose firstly that f is non-negative. Let $E := \{x : f(x) = \infty\}$, and suppose that $m(E) > 0$. Then, by part (4) of Proposition 3.7,

$$\int f = \int_{E \cup E^c} \geq \int_E f \geq n\, m(E) \quad \forall n \in \mathbf{N}.$$

Take the limit as $n \to \infty$ to get that $\int f = \infty$, a contradiction. Hence, f is finite a.e.

Secondly, suppose that f is arbitrary function. Since $f = f^+ - f^-$ is integrable, f^+, f^- are both integrable; hence by the above argument, f^+ and f^- are both finite a.e.; therefore, f is finite a.e.

(6) Recall Example 2.2(b), and take $f := \chi_{[1,2]}$, $\quad g := \chi_{[\frac{3}{2},2]}$. Then

$$2 = m([1,2]) = \int f = \int g = m([\frac{3}{2}, 2]),$$

but $m(\{x : f(x) \neq g(x)\}) = m([1, \frac{3}{2})) = \frac{3}{2} \neq 0$. ∎

3.9 Lemma.

Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint sequence in $\mathcal{B}$ and $\{f_j\}_{j=1}^{n}$ be a finite sequence of non-negative measurable functions. Then

$$\bigoplus_{i=1}^{n} (\sup_k \int_{\bigcup_{j=1}^{k} A_j} f_i) = \sup_k (\bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{k} A_j} f_i).$$

Proof: By monotonicity of the integral (Proposition 3.7(2)), and for a fixed $i \in \{1, 2, ..., n\}$, we have that the sequence $\{\int_{\bigcup_{j=1}^{k} A_j} f_i\}_{k=1}^{\infty}$ is increasing, so

$$\sup_k \int_{\bigcup_{j=1}^{k} A_j} f_i = \lim_{k\to\infty} \int_{\bigcup_{j=1}^{k} A_j} f_i.$$

Therefore, by using axiom (7) of the $\oplus$ operation and since the sequence $\{\bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{k} A_j} f_i\}_{k=1}^{\infty}$ is increasing, we have

$$\begin{aligned}\bigoplus_{i=1}^{n}(\sup_k \int_{\bigcup_{j=1}^{k} A_j} f_i) &= \bigoplus_{i=1}^{n}(\lim_{k\to\infty} \int_{\bigcup_{j=1}^{k} A_j} f_i) \\ &= \lim_{k\to\infty}(\bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{k} A_j} f_i) = \sup_k(\bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{k} A_j} f_i). \quad \blacksquare\end{aligned}$$

The following theorem generalizes Theorem 2 in [2].

3.10 Theorem.

Let f be a non-negative measurable function on $(X, \mathcal{B}, m)$. Define a set function $\nu_f : \mathcal{B} \longrightarrow [0, \infty)$ by

$$\nu_f(E) = \int_E f dm = \int f\chi_E \, dm.$$

Then ν_f is a $\oplus-$measure.

Proof: According to Lemma 2.6, it suffices to prove the following: (i) $\nu_f(\emptyset) = 0$; (ii) ν_f is C.F.B.; and (iii) $\nu_f(A \cup B) = \nu_f(A) \oplus \nu_f(B)$ $\forall A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$.

(i) This follows from Proposition 3.8(1), since $m(\emptyset) = 0$.

(ii) Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in $\mathcal{B}$. The proof that ν_f is C.F.B. is realized in three steps.

Step 1: $f = a\chi_A$ for some $a > 0$, and $A \in \mathcal{B}$. In this case, by continuity of m from below, we have

$$\begin{aligned}\nu_f(\bigcup_{i=1}^{\infty} E_i) &= \int a\chi_{\bigcup_{i=1}^{\infty}(A\cap E_i)} = a\ m(\bigcup_{i=1}^{\infty} A\cap E_i) = a \sup_k m(\bigcup_{i=1}^{k} A\cap E_i) \\ &= \sup_k a\ m(\bigcup_{i=1}^{k} A\cap E_i) = \sup_k \int a\chi_{A\cap(\bigcup_{i=1}^{k} E_i)} \\ &= \sup_k \int_{\bigcup_{i=1}^{k} E_i} f = \sup_k \nu_f(\bigcup_{i=1}^{k} E_i).\end{aligned}$$

Step 2: $f = \sum_{i=1}^{n} a_i\chi_{A_i}$ where $a_i > 0$, and $\{A_i\}_{i=1}^{n}$ is pairwise disjoint in $\mathcal{B}$. In this case, set $f_i := a_i\chi_{A_i}$, $i = 1, 2, ..., n$, so that $f_i.f_j = 0$ for $i \neq j$. Using part (2) of Proposition 3.4, induction, Lemma 3.9 and Step 1 above, we obtain

$$\begin{aligned}\nu_f(\bigcup_{j=1}^{\infty} E_j) &= \int_{\bigcup_{j=1}^{\infty} E_j} \sum_{i=1}^{n} f_i = \bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{\infty} E_j} f_i = \bigoplus_{i=1}^{n} (\sup_k \int_{\bigcup_{j=1}^{k} E_j} f_i) \\ &= \sup_k (\bigoplus_{i=1}^{n} \int_{\bigcup_{j=1}^{k} E_j} f_i) = \sup_k (\int_{\bigcup_{j=1}^{k} E_j} \sum_{i=1}^{n} f_i) \\ &= \sup_k \int_{\bigcup_{j=1}^{k} E_j} f = \sup_k \nu_f(\bigcup_{j=1}^{k} E_j).\end{aligned}$$

Step 3: f is a non-negative measurable function. In this case, let g be a NSF such that $g \leq f$. Using Step 2 and Proposition 3.7(2), we get

$$\nu_g(\bigcup_{j=1}^{\infty} E_j) = \sup_k \nu_g(\bigcup_{j=1}^{k} E_j) \leq \sup_k \nu_f(\bigcup_{j=1}^{k} E_j) \leq \nu_f(\bigcup_{j=1}^{\infty} E_j).$$

Taking the supremum over all NSF-s $g \leq f$, we obtain

$$\nu_f(\bigcup_{j=1}^{\infty} E_j) \leq \sup_k \nu_f(\bigcup_{j=1}^{k} E_j) \leq \nu_f(\bigcup_{j=1}^{\infty} E_j),$$

and therefore $\nu_f(\bigcup_{j=1}^{\infty} E_j) = \sup_k \nu_f(\bigcup_{j=1}^{k} E_j)$.

(iii) Let $A, B \in \mathcal{B}$ such that $A \cap B = \emptyset$.

Case 1: $\nu_f(A) \oplus \nu_f(B) = \infty$. In this case, either $\nu_f(A) = \infty$ or $\nu_f(B) = \infty$; so $\nu_f(A \cup B) = \infty$.

Case 2: $\nu_f(A) \oplus \nu_f(B) < \infty$. In this case, both $\nu_f(A)$ and $\nu_f(B)$ are finite. Let g be a NSF such that $g \leq f$. Noting that $g\chi_A.g\chi_B = 0$ and using Proposition 3.4(2), monotonicity of the integral and axiom (5) of the $\oplus$ operation, we obtain

$$\begin{aligned} \nu_g(A \cup B) &= \int_{A \cup B} g = \int (g\chi_A + g\chi_B) = \int g\chi_A \oplus \int g\chi_B \\ &\leq \int f\chi_A \oplus \int f\chi_B = \nu_f(A) \oplus \nu_f(B). \end{aligned}$$

Taking the supremum over all NSF-s $g \leq f$, we get

$$\nu_f(A \cup B) \leq \nu_f(A) \oplus \nu_f(B). \tag{3.1}$$

On the other hand, let $\epsilon > 0$ be given. Then there exist NSF-s $g, h \leq f$ such that

$$\int_A f < \int_A g + \frac{\epsilon}{2} \quad \text{and} \quad \int_B f < \int_B h + \frac{\epsilon}{2}.$$

We may assume without loss of generality that $g = h$. So, using property (6) of the $\oplus$ operation, we have

$$\begin{aligned} \nu_f(A) \oplus \nu_f(B) &\leq (\int_A h + \frac{\epsilon}{2}) \oplus (\int_B h + \frac{\epsilon}{2}) \leq (\int_A h \oplus \int_B h) + (\frac{\epsilon}{2} \oplus \frac{\epsilon}{2}) \\ &\leq (\int h\chi_A \oplus \int h\chi_B) + \epsilon. \end{aligned}$$

Since $(h\chi_A).(h\chi_B) = 0$, Proposition 3.4(2) yields

$$\begin{aligned} \nu_f(A) \oplus \nu_f(B) &\leq \int (h\chi_A + h\chi_B) + \epsilon = \int h\chi_{(A \cup B)} + \epsilon \\ &\leq \int f\chi_{(A \cup B)} + \epsilon = \nu_f(A \cup B) + \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we get

$$\nu_f(A) \oplus \nu_f(B) \leq \nu_f(A \cup B). \tag{3.2}$$

Now (3.1) and (3.2) imply that ν_f is finitely $\oplus$-additive. ■

3.11 Theorem.

Let $f, f_n \ (n = 1, 2, ...)$ be NSF-s such that $f_n \uparrow f$. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof: Since $f_1 \leq f_2 \leq ... \leq f$, $\int f_1 \leq \int f_2 \leq ... \leq \int f$, and hence the sequence $\{\int f_n\}_{n=1}^{\infty}$ is increasing and bounded above by $\int f$. Therefore

$$\int f \geq \lim_{n\to\infty} \int f_n. \tag{3.3}$$

For the reverse inequality, let $\epsilon \in (0, 1)$, and suppose that $f = \sum_{i=1}^{k} a_i \chi_{A_i}$ for positive numbers $a_1, a_2, ..., a_k$, and pairwise disjoint measurable sets $A_1, A_2, ..., A_k$. For each $n \in \mathbf{N}$ and each $i \in \{1, 2, ..., k\}$, define $A_i^n = \{x \in A_i : f_n(x) \geq (1 - \epsilon)a_i\}$. Then $A_i^n \in \mathcal{B} \ \ \forall n \in \mathbf{N}$ and $\forall i \in \{1, 2, ..., k\}$.

Claim: The sequence $\{A_i^n : i = 1, 2, ..., k; \ n \in \mathbf{N}\}$ satisfies the following:

(1) For fixed $n \in \mathbf{N}$, we have that $A_1^n, A_2^n, ..., A_k^n$ are disjoint.

(2) For fixed $i \in \{1, 2, ..., k\}$, we have that $\{A_i^n\}_{n=1}^{\infty}$ is non-decreasing.

(3) For each $i \in \{1, 2, ..., k\}$, $A_i = \bigcup_{n=1}^{\infty} A_i^n$.

[(1) Fix $n \in \mathbf{N}$, and suppose that for some $i \neq j$, $A_i^n \cap A_j^n \neq \emptyset$. This implies that $A_i \cap A_j \neq \emptyset$, which is a contradiction.

(2) Fix $i \in \{1, 2, ..., k\}$, and let $n < m$, so that $f_n \leq f_m$. Let $x \in A_i^n$. Then $x \in A_i$, and $f_n(x) \geq (1-\epsilon)a_i$. Hence $x \in A_i$, and $f_m(x) \geq (1-\epsilon)a_i$; which implies that $x \in A_i^m$, and therefore $A_i^n \subseteq A_i^m$.

(3) Fix $i \in \{1, 2, ..., k\}$. Since $A_i^n \subseteq A_i \ \ \forall n \in \mathbf{N}$, $\bigcup_{n=1}^{\infty} A_i^n \subseteq A_i$. For the reverse inclusion, let $x \in A_i$. Then $f(x) = a_i$, and hence there exists $n_o \in \mathbf{N}$ such that $f_{n_o}(x) \geq (1 - \epsilon)a_i$ (since otherwise, we would have $f_n(x) < (1 - \epsilon)a_i \ \ \forall n \in \mathbf{N}$, which would imply that $a_i = f(x) = \lim_{n\to\infty} f_n \leq (1 - \epsilon)a_i$, a contradiction). It follows that $x \in$

$A_i^{n_o} \subseteq \bigcup_{n=1}^{\infty} A_i^n$, and therefore $A_i \subseteq \bigcup_{n=1}^{\infty} A_i^n$. This completes the proof of (3), and hence the claim.]

Now, by parts (2) and (3) of the above claim, we have for fixed $i \in \{1, 2, ..., k\}$ that $A_i^n \uparrow A_i$ as $n \longrightarrow \infty$. Since m is C.F.B., $m(A_i^n) \uparrow m(A_i)$. Set $g_n := \sum_{i=1}^{k}(1-\epsilon)a_i\chi_{A_i^n}$, $n \in \mathbf{N}$. Since, by (1) of the above claim, the sets $A_1^n, A_2^n, ..., A_k^n$ are pairwise disjoint for each $n \in \mathbf{N}$, g_n is well-defined NSF. Moreover, $g_n \leq f_n \ \forall n$, since for $x \in X$, $g_n(x) = 0 \leq f_n(x)$ or $g_n(x) = (1-\epsilon)a_i$ for some $i \in \{1, 2, ..., k\}$, so that $x \in A_i^n$, and hence $f_n(x) \geq (1-\epsilon)a_i$. It follows that $\lim_{n\to\infty} \int g_n \leq \lim_{n\to\infty} \int f_n$. But, by axioms (3) and (7) of the $\oplus$ operation, we have

$$\begin{aligned}\lim_{n\to\infty} \int g_n &= \lim_{n\to\infty} \bigoplus_{i=1}^{k}(1-\epsilon)a_i m(A_i^n) = (1-\epsilon) \lim_{n\to\infty} \bigoplus_{i=1}^{k} a_i m(A_i^n) \\ &= (1-\epsilon) \bigoplus_{i=1}^{k} a_i \lim_{n\to\infty} m(A_i^n) = (1-\epsilon) \bigoplus_{i=1}^{k} a_i \, m(A_i) = (1-\epsilon) \int f.\end{aligned}$$

Since $\epsilon \in (0, 1)$ was arbitrary, it follows that

$$\int f \leq \lim_{n\to\infty} \int f_n. \tag{3.4}$$

Now (3.3) and (3.4) yield that $\int f = \lim_{n\to\infty} \int f_n$. ■

As a consequence of Theorem 3.11, we obtain Theorems 4 and 5 of [2], since every integrable NSF is a NSF, and every continuous $\oplus-$measure is a $\oplus-$measure. Theorem 3.11 can be improved in the following sense.

3.12 Theorem.

Let f be a non-negative measurable function, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of NSF-s such that $f_n \uparrow f$. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof: Since $f_1 \leq f_2 \leq ... \leq f$, $\int f_1 \leq \int f_2 \leq ... \leq \int f$, and hence

$$\lim_{n\to\infty} \int f_n \leq \int f.$$

For the reverse inequality, it suffices to prove that $\int g \leq \lim_{n\to\infty} \int f_n$ for every NSF g such that $g \leq f$. So let g be such a function. For each $n \in \mathbf{N}$, define

$$h_n := \min\{g, f_n\}.$$

Then $h_n \leq f_n \ \ \forall n$, and $\lim_{n\to\infty} h_n = g$. Since $\{f_n\}_{n=1}^{\infty}$ is nondecreasing, we must have that $\{h_n\}_{n=1}^{\infty}$ is nondecreasing. Hence, by Theorem 3.11, we have that

$$\int g = \lim_{n\to\infty} \int h_n \leq \lim_{n\to\infty} \int f_n. \quad \blacksquare$$

3.13 Theorem (Monotone Convergence Theorem).

Let $(X, \mathcal{B}, m)$ be a $\oplus$−measure space and let $f, f_n \ \ (n = 1, 2, ...)$ be non-negative measurable functions such that $f_n \uparrow f$ a.e. Then

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proof: Firstly, suppose that $f_n(x) \uparrow f(x) \ \ \forall x \in X$. Then $\int f_1 \leq \int f_2 \leq ... \leq \int f$, and hence

$$\lim_{n\to\infty} \int f_n \leq \int f. \tag{3.5}$$

Next, using Proposition 11.7 of [5], we choose for each $n \in \mathbf{N}$ a nondecreasing sequence of NSF-s $\{g_i^n\}_{i=1}^{\infty}$ such that $g_i^n \uparrow f_n$ as $i \longrightarrow \infty$. Define $h_n := \max\{g_n^1, g_n^2, ..., g_n^n\}$.

Claim: The sequence $\{h_n\}_{n=1}^{\infty}$ satisfies the following :

(1) $\{h_n\}_{n=1}^{\infty}$ is nondecreasing;

(2) $h_n \leq f_n \ \ \forall n$; and

(3) $f = \lim_{n\to\infty} h_n$.

[(1) For each $n \in \mathbf{N}$, we have

$h_{n+1} = \max\{g_{n+1}^1, g_{n+1}^2, ..., g_{n+1}^{n+1}\} \geq \max\{g_{n+1}^1, g_{n+1}^2, ..., g_{n+1}^n\} \geq \max\{g_n^1, g_n^2, ...,$

(2) For $n \in \mathbf{N}$ and $x \in X$, we have

$$\begin{aligned} h_n(x) &= \max\{g_n^1(x), g_n^2(x), ..., g_n^n(x)\} = g_n^k(x) \qquad \text{for some } k \in \{1,2,...,n\} \\ &\leq f_k(x) \leq f_n(x). \end{aligned}$$

(3) From (2) we have that $h_n \leq f_n \ \ \forall n$, and so $\lim_{n\to\infty} h_n \leq \lim_{n\to\infty} f_n = f$. For the reverse inequality, fix $k \in \mathbf{N}$. Then we have that $g_n^k \leq h_n$ for each $n \in \mathbf{N}$ such that $k \leq n$. Thus $f_k = \lim_{n\to\infty} g_n^k \leq \lim_{n\to\infty} h_n$; and, therefore, $f = \lim_{k\to\infty} f_k \leq \lim_{n\to\infty} h_n$. This completes the proof of (3), and hence the claim.] Now h_n is NSF $\forall n$, and $h_n \uparrow f$, so by Theorem 3.12, we have that

$$\int f = \lim_{n\to\infty} \int h_n \leq \lim_{n\to\infty} \int f_n. \tag{3.6}$$

From (3.5) and (3.6), we have that $\int f = \lim_{n\to\infty} \int f_n$.

Secondly, suppose that $f_n \uparrow f$ a.e., and let $E := \{x : f(x) \neq \lim_{n\to\infty} f_n(x)\}$. Then $m(E) = 0$ and $f_n\chi_{E^c} \uparrow f\chi_{E^c}$. Hence, using Proposition 3.8(3) and the first part of the proof, we have that

$$\int f = \int_{E^c} f = \int f\chi_{E^c} = \lim_{n\to\infty} \int f_n\chi_{E^c} = \lim_{n\to\infty} \int_{E^c} f_n = \lim_{n\to\infty} \int f_n. \quad \blacksquare$$

As a consequence of Theorem 3.13, we obtain Theorem 6 of [2], since every continuous $\oplus-$measure is a $\oplus-$measure. It should be noted that Theorem 3.13 is a generalization of the standard monotone convergence theorem (see [1] and [5]), since every $\sigma-$additive measure is a $\oplus-$measure.

3.14 Theorem (Fatou's Lemma).

Let $(X, \mathcal{B}, m)$ be a $\oplus-$measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions such that $f_n \longrightarrow f$ a.e. Then

$$\int f \leq \underline{\lim} \int f_n.$$

Proof: For each $n \in \mathbf{N}$, set $g_n := \inf_{k \geq n} f_k$. Then g_n is measurable, $g_n \uparrow \underline{\lim} f_n = f$ a.e., and $g_n \leq f_n\ \forall n$. Hence, using the monotone convergence theorem(M.C.T.) and monotonicity of the integral, we have that

$$\int f = \lim_{n\to\infty} \int g_n = \underline{\lim} \int g_n \leq \underline{\lim} \int f_n. \quad \blacksquare$$

By generalizing M.C.T.(Theorem 6 of [2]), the following theorem generalizes Theorem 7 of [2] for arbitrary $\oplus-$measure spaces.

3.15 Theorem.

Let $(X, \mathcal{B}, m)$ be a $\oplus-$measure space and let f and g be non-negative measurable functions on X. Then

$$\int f \oplus g = \int f \oplus \int g.$$

Proof: It is basically the proof of Theorem 7 of [2]. $\blacksquare$

3.16 Corollary.

Let $(X, \mathcal{B}, m)$ be a $\oplus-$measure space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on X. Then

$$\int \bigoplus_{n=1}^{\infty} f_n = \bigoplus_{n=1}^{\infty} \int f_n.$$

Proof: Define

$$g := \bigoplus_{i=1}^{\infty} f_i \quad \text{and} \quad g_n := \bigoplus_{i=1}^{n} f_i, \quad n \in \mathbf{N}.$$

Clearly, both g and g_n are non-negative measurable functions, and $g_n \uparrow g$. So, by using M.C.T. and Theorem 3.15, we have that

$$\int \bigoplus_{i=1}^{\infty} f_i = \int g = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \int f_i = \bigoplus_{i=1}^{\infty} \int f_i. \quad \blacksquare$$

The Lebesgue dominated convergence theorem provides a difference between integration with respect to a $\sigma-$additive measure and integration with respect to a $\sigma-$maxitive measure. While this result is always true when the measure is $\sigma-$additive(see [1] and [5]), it need not hold for non-continuous $\sigma-$maxitive measures, as the following example shows.

3.17 Example.

Recall Example 2.2(b). We have that $X = [1,\infty)$, $\Re = \mathcal{P}(X)$, and $m : \Re \longrightarrow [0,\infty]$ defined by

$$m(E) := \begin{cases} \sup E & \text{if } E \neq \emptyset \text{ and } E \text{ is bounded} \\ 0 & \text{if } E = \emptyset \\ \infty & \text{if } E \text{ is unbounded} \end{cases}$$

is a $\sigma-$maxitive measure. Note that m is not continuous. Let $E_n = (2 - \frac{1}{n}, 2)$, and define $f(x) := 0$, $f_n(x) := \chi_{E_n}(x)$, and $g(x) := \chi_{[1,2]}(x)$ $\forall x \in X$. Clearly, g is a non-negative integrable function, and $| f_n | \leq g$ $\forall n \in \mathbf{N}$. Moreover, $f_n(x) \longrightarrow f(x)$ $\forall x \in X$. But $\int f_n = m(E_n) = 2$ $\forall n$, while $\int f = 0$. Therefore, $\int f \neq \lim_{n\to\infty} \int f_n$.

We conclude this paper with the following

3.18 Open Question.

Does the Lebesgue dominated convergence theorem hold true for a continuous $\oplus-$measure space?

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