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Existence and Uniqueness of Solution for Hadamard Fractional Sequential Differential Equations

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Abstract
In this article, we obtain sufficient conditions for existence and uniqueness of Hadamard fractional sequential nonlinear differential equations of orders $1 < \alpha < 2$, and $2 < \alpha < 3$. The obtained results are based on Banach and Schauder's fixed point theorems. Some examples are introduced to explain the applicability of the theorems.

Keywords:
Existence and uniqueness,
Hadamard fractional differential equations,
Fixed point theorems.

1. Introduction:

The amazing results of applying the fractional order derivatives in the models of many underlying phenomena attracted the researchers to investigate indepth work about various directions of fractional calculus (see [1], [2], [3] and the references cited therein). Among these investigations, the existence theory of solutions for fractional differential models has gained attentions of many authors. Most of them have focused on using Riemann-Liouville and Caputo derivatives in representing the underlying fractional differential equation (see [5]-[8]). Another kind of fractional derivative is Hadamard type which was introduced in 1892 [9]. This derivative differs from various derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in ([10], [11]). Recently, the existence and uniqueness of solution for fractional differential equations in Hadamard sense was introduced in many faces by several authors (see [12], [13] and references therein). We add in this article a new idea concerning the sequential definition of Hadamard fractional operator with constant coefficients of order less than three.

More precisely, we consider the nonlinear Hadamard fractional differential equations given by

$$\begin{cases} \left({}^{H}D_{a}^{\alpha} + \gamma\, {}^{H}D_{a}^{\alpha-1}\right)x(t) = f\big(t, x(t)\big), 1 < \alpha < 2\,, \\ x(a) = x'(a) = 0\,, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \left({}^{H}D_{a}^{\alpha} + \lambda\, {}^{H}D_{a}^{\alpha-1} + \dfrac{\lambda^2 - 1}{4}\, {}^{H}D_{a}^{\alpha-2}\right)x(t) = g\big(t, x(t)\big), \\ x(a) = x'(a) = x''(a) = 0\,, \end{cases} \tag{1.2}$$

where $2 < \alpha < 3$, $t \in J = [a, T]$, $1 \leq a < T$, $f, g: J \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, and γ and λ are real numbers.

2. Hadamard Fractional Linear Differential Equations:

The fractional derivative due to Hadamard, introduced in 1892 [9], differs from the aforementioned derivatives in the sense that the kernel of the integral in the dentition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in ([10],[11]).
We firstly, present the definitions and some properties of the Hadamard fractional integrals and derivatives.

Definition 2.1 The Hadamard fractional integral of order α for a continuous function f is defined as

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds , \alpha > 0.$$

Definition 2.2 The Hadamard derivative of fractional order $\alpha > 0$ for a continuous function $f : [a, \infty) \to \mathbb{R}$ is defined as

$${}^H D_a^\alpha f(t) = \delta^n J_a^\alpha f(t), \delta = t\frac{d}{dt},$$

for $n-1 < \alpha < n$, $n = [\alpha] + 1$, here $[\alpha]$ denotes the integer part of α.
Let $C(J,\mathbb{R})$ denotes the Banach space of all real valued continuous functions endowed with the norm $\|x\| = \sup\{|x(t)|: t \in J\}$. The Banach space$C_\delta^n(J,\mathbb{R})$ denotes all real valued functions x such that $\delta^n x \in C(J,\mathbb{R})$. It is obvious that if $f \in C(J,\mathbb{R})$, then $J_a^\alpha f \in C(J,\mathbb{R})$, and if $x \in C_\delta^n(J,\mathbb{R})$ then ${}^H D_a^\alpha x \in C(J,\mathbb{R})$.

Lemma 2.3 [10]

1. The equality ${}^H D_a^\alpha x(t) = 0$ is valid if, and only if,

$$x(t) = \sum_{j=1}^{n} c_j \left(\ln\frac{t}{a}\right)^{\alpha-j},$$

where $c_j \in \mathbb{R}$, $(j = 1,2, \dots , n)$, are arbitrary constants, and $x \in C(J,\mathbb{R})$.

2. $J_a^\alpha J_a^\beta x = J_a^{\alpha+\beta} x, x \in C(J,\mathbb{R})$,
3. $J_a^\alpha\left(c_1 x(t) + c_2 y(t)\right) = c_1 J_a^\alpha x(t) + c_2 J_a^\alpha y(t), x, y \in C(J,\mathbb{R})$, and $c_1, c_2 \in \mathbb{R}$,
4. $J_a^\alpha\, {}^H D_a^\alpha x(t) = x(t) - \sum_{j=1}^n c_j \left(\log\frac{t}{a}\right)^{\alpha-j}$, $c_j \in \mathbb{R}$, $(j = 1,2, \dots , n)$, $x \in C_\delta^n(J,\mathbb{R})$.

The first result is obtaining the solution of the corresponding linear equation of (1.1).

Theorem 2.4 Let $f \in C(J,\mathbb{R})$, and $x \in C_\delta^2(J,\mathbb{R})$. The Hadamard fractional linear differential equation

$$\begin{cases} \left({}^H D_a^\alpha + \gamma\, {}^H D_a^{\alpha-1}\right) x(t) = f(t), 1 < \alpha < 2 , \\ x(a) = x'(a) = 0 , \end{cases} \tag{2.1}$$

has a solution given by

$$x(t) = t^{-\gamma} \int_a^t s^{\gamma-1} J_a^{\alpha-1} f(s) ds. \tag{2.2}$$

Proof. Taking the Hadamard fractional integral J_a^α to both sides of equation (2.1), we obtain

$$J_a^\alpha\left(D_a^\alpha(x(t))\right) + \gamma J_a^\alpha\left(J_a^{\alpha-1} D_a^{\alpha-1}(x(t))\right) = J_a^\alpha\left(f(t)\right).$$

Using Lemma 2.3, implies that

$$\left(x(t) - b_1\left(\ln\frac{t}{a}\right)^{\alpha-1} - b_2\left(\ln\frac{t}{a}\right)^{\alpha-2}\right) + \gamma J_a^1\left(x(t) - d_1\left(\ln\frac{t}{a}\right)^{\alpha-2}\right) = J_a^\alpha\left(f(t)\right),$$

which implies that

$$x(t) - b_1\left(\ln\frac{t}{a}\right)^{\alpha-1} - b_2\left(\ln\frac{t}{a}\right)^{\alpha-2} + \frac{\gamma}{\Gamma(1)} \int_a^t \frac{1}{s}\left(x(s) - d_1\left(\ln\frac{s}{a}\right)^{\alpha-2}\right) ds = J_a^\alpha\left(f(t)\right). \tag{2.3}$$

The initial condition $x(a) = 0$,leads to $b_2 = 0$. Now, taking the first ordinary derivative for equation (2.3),

$$x'(t) + \gamma\frac{x(t)}{t} = \left(b_1(\alpha-1) + \gamma d_1\right)\frac{1}{t}\left(\ln\frac{t}{a}\right)^{\alpha-2} + \frac{1}{t} J_a^{\alpha-1} f(t).$$

The condition $x'(a) = 0$, implies that $b_1(\alpha - 1) + \gamma d_1 = 0$. Let $x(t) = t^{-\gamma} u(t)$, then $x'(t) = t^{-\gamma} u'(t) - \gamma t^{-\gamma-1} u(t)$, hence

$$t^{-\gamma} u'(t) = \frac{1}{t} J_a^{\alpha-1} f(t),$$

accordingly,

$$u'(t) = t^{\gamma-1} J_a^{\alpha-1} f(t). \tag{2.4}$$

Integrating equation (2.4), it follows that

$$u(t) = u(a) + \int_a^t s^{\gamma-1} J_a^{\alpha-1} f(s)\, ds.$$

condition $x(a) = 0$, implies $u(a) = 0$, hence

$$x(t)=t^{-\gamma}\int_a^t s^{\gamma-1}J_a^{\alpha-1}f(s)\,ds.$$

This finishes the proof. ■

Next result is obtaining a solution for the corresponding linear equation of (1.2).

Theorem 2.5 Let $f\in C(J,\mathbb{R})$, and $x\in C_\delta^3(J,\mathbb{R})$. The Hadamard fractional linear differential equation

$$\begin{cases}\left({}^HD_a^\alpha+\lambda\,{}^HD_a^{\alpha-1}+\dfrac{\lambda^2-1}{4}\,{}^HD_a^{\alpha-2}\right)x(t)=g(t),2<\alpha<3,\\ x(a)=x'(a)=x''(a)=0\,,\end{cases}\quad(2.5)$$

has a solution given by

$$x(t)=t^{\frac{-(1+\lambda)}{2}}\int_a^t(t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}g(s)\,ds.\qquad(2.6)$$

Proof. Taking the Hadamard fractional integral J_a^α to both sides of equation (2.5), we obtain

$$J_a^\alpha\big(D_a^\alpha(x(t))\big)+\lambda J_a^1\big(J_a^{\alpha-1}D_a^{\alpha-1}(x(t))\big)+\frac{\lambda^2-1}{4}J_a^2\big(J_a^{\alpha-2}D_a^{\alpha-2}(x(t))\big)=J_a^\alpha\big(g(t)\big).$$

Lemmas 2.3 would imply,

$$\begin{aligned}&\left(x(t)-b_1\left(\ln\frac{t}{a}\right)^{\alpha-1}-b_2\left(\ln\frac{t}{a}\right)^{\alpha-2}-b_3\left(\ln\frac{t}{a}\right)^{\alpha-3}\right)\\&+\lambda J_a^1\left(x(t)-d_1\left(\ln\frac{t}{a}\right)^{\alpha-2}-d_2\left(\ln\frac{t}{a}\right)^{\alpha-3}\right)\\&+\frac{\lambda^2-1}{4}J_a^2\left(x(t)-w_1\left(\ln\frac{t}{a}\right)^{\alpha-3}\right)\\&=J_a^\alpha\big(g(t)\big),\end{aligned}$$

that implies

$$\begin{aligned}&x(t)-b_1\left(\ln\frac{t}{a}\right)^{\alpha-1}-b_2\left(\ln\frac{t}{a}\right)^{\alpha-2}-b_3\left(\ln\frac{t}{a}\right)^{\alpha-3}\\&+\lambda\int_a^t\frac{1}{s}\left(x(s)-d_1\left(\ln\frac{s}{a}\right)^{\alpha-2}-d_2\left(\ln\frac{s}{a}\right)^{\alpha-3}\right)ds\\&+\frac{\lambda^2-1}{4}\int_a^t\left(\ln\frac{t}{s}\right)\frac{1}{s}\left(\mathrm{x(s)}-w_1\left(\log\frac{s}{a}\right)^{\alpha-3}\right)ds=J_a^\alpha\big(g(t)\big).\end{aligned}$$

The condition $x(a)=0$, implies that $b_3=0$. Taking the first ordinary derivative, we obtain

$$\begin{aligned}&x'(t)-b_1\frac{(\alpha-1)}{t}\left(\ln\frac{t}{a}\right)^{\alpha-2}-b_2\frac{(\alpha-2)}{t}\left(\ln\frac{t}{a}\right)^{\alpha-3}\\&+\lambda\left(\frac{x(t)}{t}-\frac{d_1}{t}\left(\ln\frac{t}{a}\right)^{\alpha-2}-\frac{d_2}{t}\left(\ln\frac{t}{a}\right)^{\alpha-3}\right)\\&+\frac{\lambda^2-1}{4t}\int_a^t\frac{1}{s}\left(x(s)-w_1\left(\ln\frac{s}{a}\right)^{\alpha-3}\right)ds\\&=\frac{1}{\mathrm{t}}J_a^{\alpha-1}\big(g(t)\big).\end{aligned}\qquad(2.7)$$

Since $x'(a)=x(a)=0$, then$b_2(\alpha-2)+\lambda d_2=0$. Multiplying equation (2.7) by t, and then taking the derivative again, we have

$$\begin{aligned}&tx'(t)+(1+\lambda)x'(t)+\left(\frac{\lambda^2-1}{4}\right)\frac{x(t)}{t}\\&=\left(b_1(\alpha-1)(\alpha-2)+\lambda d_1(\alpha-2)+\frac{\lambda^2-1}{4}w_1\right)\frac{1}{t}\left(\ln\frac{t}{a}\right)^{\alpha-3}\\&+\frac{1}{\mathrm{t}}J_a^{\alpha-2}\big(g(t)\big).\end{aligned}\qquad(2.8)$$

Since $x''(a)=x'(a)=x(a)=0$, it follows that $b_1(\alpha-1)(\alpha-2)+\lambda d_1(\alpha-2)+\frac{\lambda^2-1}{4}w_1=0$.

Multiplying equation (2.8) by t, we obtain

$$t^2x''(t)+(1+\lambda)tx'(t)+\left(\frac{\lambda^2-1}{4}\right)x(t)=J_a^{\alpha-2}\big(g(t)\big).\qquad(2.9)$$

Now, we substitute the transformation

$$\begin{cases}x(t)=t^{\frac{-(1+\lambda)}{2}}u(t),\\ x'(t)=t^{\frac{-(1+\lambda)}{2}}u'(t)-\dfrac{(1+\lambda)}{2}t^{\frac{-(3+\lambda)}{2}}u(t),\\ x''(t)=t^{\frac{-(1+\lambda)}{2}}u''(t)-(1+\lambda)t^{\frac{-(3+\lambda)}{2}}u'(t)\\ \qquad+\dfrac{(1+\lambda)}{2}\dfrac{(3+\lambda)}{2}t^{\frac{-(5+\lambda)}{2}}u(t),\end{cases}$$

into equation (2.9), and then simplify, we obtain

$$t^{\frac{-(1+\lambda)}{2}+2}u''(t)=J_a^{\alpha-2}\big(g(t)\big).$$

Therefore,

$$u''(t)=t^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\big(g(t)\big).$$

Twice integration and then changing the variables in the double integral will lead to

$$u(t)-u'(a)(t-a)-u(a)=\int_a^t(t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\big(g(s)\big)\,ds.$$

The conditions $x(a)=x'(a)=0$ imply that $u(a)=u'(a)=0$. Hence, after reversing the transformation, we obtain

$$x(t)=t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\big(g(s)\big)\,ds.$$

This finishes the proof. ■

3. Existence Theorems:

The fixed point theorems are the basic tools for dealing with the nonlinear differential equations. The idea is to convert the corresponding integral equation into operator equation and then proving this equation has fixed point, which is then the required solution. We shall focus on two fixed point theorems, the Banach and Schauder's fixed point theorems [14].

In view of Theorem 2.4, and Theorem 2.5, we define the operators Φ and Ψ on $C(J,\mathbb{R})$, as

$$\Phi x(t)=t^{-\gamma}\int_a^t s^{\gamma-1}J_a^{\alpha-1}f(s,x(s))ds, \tag{3.1}$$

$$\Psi x(t)=t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\big(g(s,x(s))\big)\,ds. \tag{3.2}$$

Theorem 3.1. The operators Φ and Ψ are completely continuous.

Proof. The continuity of the operators Φ and Ψ follows respectively by the continuity of the functions f and g. Let $\mathcal{B}$ be a bounded proper subset of $C(J,\mathbb{R})$,then, there exist positive real numbers A_f and $A_g$such that $|f(t,x)|\le A_f$, and $|g(t,x)|\le A_g$for any order pair $(t,x)\in J\times\mathcal{B}$. Therefore

$$|\Phi x(t)|\le\frac{A_f\left(\ln\frac{t}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1-\left(\frac{a}{t}\right)^{\gamma}}{\gamma}\right|,\gamma\neq0,$$

and

$$|\Psi x(t)|\le\frac{A_g\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\frac{\left(1-\frac{a}{t}\right)\left(1-\left(\frac{a}{t}\right)^{\frac{(1+\lambda)}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right|,\lambda\neq1.$$

Taking the maximum over J, we deduce that the operators Φ and Ψ are bounded on $C(J,\mathbb{R})$. Next, we show the equicontinuity of Φ and Ψ. For this, let $a\le t_1<t_2\le T$, then

$$|\Phi x(t_2)-\Phi x(t_1)|$$

$$\le\frac{A_f\left(\ln\frac{t_2}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}|(t_2)^{-\gamma}-(t_1)^{-\gamma}|\left|\frac{(t_1)^{\gamma}-(a)^{\gamma}}{\gamma}\right| +\frac{A_f\left(\ln\frac{t_2}{t_1}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1-\left(\frac{t_1}{t_2}\right)^{\gamma}}{\gamma}\right|,$$

and

$$\begin{aligned}&|\Psi x(t_2)-\Psi x(t_1)|\\ &\le t_2^{\frac{-(1+\lambda)}{2}}|t_2-t_1|\int_a^{t_2}s^{\frac{(1+\lambda)}{2}-2}\left|J_a^{\alpha-2}\left(g(s,x(s))\right)\right|ds\\ &+\left|t_2^{\frac{-(1+\lambda)}{2}}-t_1^{\frac{-(1+\lambda)}{2}}\right|\left|\int_a^{t_1}(t_1-s)s^{\frac{(1+\lambda)}{2}-2}\left|J_a^{\alpha-2}\left(g(s,x(s))\right)\right|ds\right.\\ &+t_2^{\frac{-(1+\lambda)}{2}}\int_{t_1}^{t_2}(t_1-s)s^{\frac{(1+\lambda)}{2}-2}\left|J_a^{\alpha-2}\left(g(s,x(s))\right)\right|ds\\ &\le\frac{A_g\left(\ln\frac{t_2}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left|1-\frac{t_1}{t_2}\right|\left|\frac{1-\left(\frac{a}{t_2}\right)^{\frac{(1+\lambda)}{2}-1}}{\frac{(\lambda-1)}{2}}\right|\\ &+\frac{A_g\left(\ln\frac{t_1}{a}\right)^{\alpha-2}(t_1-a)}{\Gamma(\alpha-1)}\left|t_2^{\frac{-(1+\lambda)}{2}}-t_1^{\frac{-(1+\lambda)}{2}}\right|\left|\frac{\left(t_1^{\frac{(1+\lambda)}{2}-1}-a^{\frac{(1+\lambda)}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right|\\ &+\frac{A_g\left(\ln\frac{t_2}{t_1}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left|1-\frac{t_1}{t_2}\right|\left|\frac{\left(1-\left(\frac{t_1}{t_2}\right)^{\frac{(1+\lambda)}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right|.\end{aligned}$$

As $|t_2-t_1|\to0$, then $|\Phi x(t_2)-\Phi x(t_1)|\to0$, and $|\Psi x(t_2)-\Psi x(t_1)|\to0$. These imply that Φ and Ψ are equicontinuous on J. In consequence, it follows by the Arzela-Ascoli theorem that the operators Φ and Ψ are completely continuous. This finishes the proof. ■

We state next the so-called Schauder's fixed point theorem.

Theorem 3.2.[14]If F is a closed , bounded, convex subset of a Banach space X and the mapping $\Delta: U\to U$ is completely continuous, then Δ has a fixed point in F.

Accordingly, if we define a closed , bounded, convex subset F of $C(J,\mathbb{R})$ on which the operators Φ and Ψ, as defined by (3.1)-(3.2), are completely continuous, then the problems (1.1)-(1.2) have the respective solution.

Theorem 3.3. Let B_f and B_g be positive constants such that

$$\lim_{x\to0}\frac{f(t,x)}{x}\le B_f<\infty,\qquad \text{and}\ \lim_{x\to0}\frac{g(t,x)}{x}\le B_g<\infty,$$

then, each problem of (1.1) and (1.2) has a solution.

Proof. The given conditions imply that there exist positive constants ρ_f and ρ_g such that $|f(t,x(t))|\le(1+B_f)\rho_f$ and $|g(t,x(t))|\le(1+B_g)\,\rho_g$. Therefore, define the subsets F_f and F_g of $C(J,\mathbb{R})$ as

$$F_f = \{x \in C(J,\mathbb{R}) : |x(t)| \le \rho_f, t \in J\},$$

and

$$F_g = \{x \in C(J,\mathbb{R}) : |x(t)| \le \rho_g, t \in J\},$$

Hence, F_f and F_g are closed, bounded, and convex subsets of $C(J,\mathbb{R})$. By Theorem 3.1, the operators Φ and Ψ are completely continuous, then by Schauder's fixed point Theorem 3.2, each problem of (1.1) and (1.2) has a solution. This finishes the proof.■

Next result, we show the existence and uniqueness of solution for each problem of (1.1) and (1.2) by using the contraction principle and the so-called Banach fixed point Theorem.

Theorem 3.4. Let f and g be Lipschitzian functions that satisfying the conditions

$$\begin{cases} |f(t,x) - f(t,y)| \le C_f|x-y|, \\ |g(t,x) - g(t,y)| \le C_g|x-y|, \end{cases}$$

where $t \in J, x, y \in \mathbb{R}$, and $C_f, C_g > 0$. Then, each problem of (1.1) and (1.2) has a unique solution whenever

$$\vartheta_f = \frac{C_f\left(\ln\frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\gamma}\right| < 1,$$

and

$$\vartheta_g = \frac{2C_g\left(\ln\frac{T}{a}\right)^{\alpha-2}}{|\lambda-1|\Gamma(\alpha-1)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\frac{(1+\lambda)}{2}-1}\right| < 1.$$

Proof. The continuity of f and g implies that there exist positive constants D_f and D_g such that $\max\{|f(t,0)| : t \in J\} \le D_f$ and $\max\{|g(t,0)| : t \in J\} \le D_g$. We show firstly that $\Phi\mathfrak{B}_{r_f} \subset \mathfrak{B}_{r_f}$, and $\Psi\mathfrak{B}_{r_g} \subset \mathfrak{B}_{r_g}$, where $\mathfrak{B}_{r_f}$ and $\mathfrak{B}_{r_g}$ are defined by $\mathfrak{B}_{r_f} = \{x \in C(J,\mathbb{R}) : \|x\| \le r_f\}$, and $\mathfrak{B}_{r_g} = \{x \in C(J,\mathbb{R}) : \|x\| \le r_g\}$, such that r_f and r_g are given by

$$r_f \ge \frac{D_f\left(\ln\frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\gamma}\right|\left(1-\vartheta_f\right)^{-1},$$

$$r_g \ge \frac{2D_g\left(\ln\frac{T}{a}\right)^{\alpha-2}}{|\lambda-1|\Gamma(\alpha-1)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\frac{(1+\lambda)}{2}-1}\right|\left(1-\vartheta_g\right)^{-1}.$$

For doing this, let $x \in \mathfrak{B}_{r_f}$, then

$$\begin{aligned}|\Phi x(t)| \le\ & t^{-\gamma}\int_a^t s^{\gamma-1}J_a^{\alpha-1}\left(|f(s,x(s)) - f(s,0)| + |f(s,0)|\right)ds \\ &\le \frac{\left(C_f\|x\| + D_f\right)\left(\ln\frac{T}{a}\right)^{\alpha-1}}{\Gamma(\alpha)}\left|\frac{1-\left(\frac{a}{t}\right)^{\gamma}}{\gamma}\right| \\ &\le \frac{D_f\left(\ln\frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\gamma}\right| + r\vartheta_f \\ &\le r.\end{aligned}$$

For $x \in \mathfrak{B}_{r_g}$, we have

$$\begin{aligned}|\Psi x(t)| \le\ & t^{\frac{-(1+\lambda)}{2}}\int_a^t (t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\left(\left|g(s,x(s)) - g(s,0)\right| + |g(s,0)|\right)ds \\ &\le \frac{\left(C_g\|x\| + D_g\right)\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\frac{\left(1-\frac{a}{t}\right)\left(1-\left(\frac{a}{t}\right)^{\frac{(1+\lambda)}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right| \\ &\le \left(1-\vartheta_g\right)r + \vartheta_g r = r.\end{aligned}$$

Next step, is showing the contraction principle. For doing this, let $x, y \in C(J,\mathbb{R})$, then

$$\begin{aligned}|\Phi x(t) - \Phi y(t)| &\le t^{-\gamma}\int_a^t s^{\gamma-1}J_a^{\alpha-1}\left(|f(s,x(s)) - f(s,y(s))|\right)ds \\ &\le \frac{C_f\left(\ln\frac{T}{a}\right)^{\alpha-1}}{|\gamma|\Gamma(\alpha)}\max_{t\in J}\left|1-\left(\frac{a}{t}\right)^{\gamma}\right|\|x-y\| \\ &\le \vartheta_f\|x-y\|,\end{aligned}$$

and

$$\begin{aligned}|\Psi x(t) - \Psi y(t)| \le\ & t^{\frac{-(1+\lambda)}{2}} \times \int_a^t (t-s)s^{\frac{(1+\lambda)}{2}-2}J_a^{\alpha-2}\left(|g(s,x(s)) - g(s,y(s))|\right)ds \\ &\le \frac{C_g\left(\ln\frac{t}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left|\frac{\left(1-\frac{a}{t}\right)\left(1-\left(\frac{a}{t}\right)^{\frac{(1+\lambda)}{2}-1}\right)}{\frac{(\lambda-1)}{2}}\right| \times \|x-y\| \le \vartheta_g\|x-y\|.\end{aligned}$$

As $\vartheta_f, \vartheta_g < 1$, the contraction principles are satisfied. By Banach fixed point Theorem, there exists a unique solution for each problem of (1.1) and (1.2). The proof is completed.■

Remark 3.5 If $\gamma = 0$ in equation (1.1), then, instead of (2.2), the solution will be

$$x(t) = J_a^{\alpha}f(t,x(t)).$$

Hence, all the above result will be simpler. The other particular value is $\lambda = 1$, the problem (1.2) has the following solution

$$x(t) = \frac{1}{t}\int_a^t \left(\frac{t}{s} - 1\right) J_a^{\alpha-2}\big(g(s, x(s))\big)\, ds,$$

which has a maximum $\frac{A_g\left(\ln\frac{T}{a}\right)^{\alpha-2}}{\Gamma(\alpha-1)}\left(\frac{T-a}{a^2}\right)$, and is again simple to consider. Therefore, we omit these particular cases.

We close this article by the following examples.

Example 3.6 Consider the following fractional Hadamard differential equation

$$\begin{cases} \left({}^H D_a^{1.5} - {}^H D_a^{0.5}\right)x(t) = Ct\, sinx(t), t \in [1, e], \\ x(1) = x'(1) = 0\,. \end{cases} \quad (3.3)$$

Here $\alpha = 1.5$, $\gamma = -1$, and $f\big(t, x(t)\big) = Ct\, sinx(t)$. We notice that

$$\lim_{x \to 0} \frac{Ct \sin x}{x} = Ct \le Ce,$$

and

$$\vartheta_f = \frac{C_f}{\Gamma(1.5)} \max_{t \in J} |1 - t| = 1.9526C.$$

Therefore, choosing any real number $0 < C \le 0.5$, the Theorems 3.3 and 3.4 can be applied, hence the problem (3.3) has a unique solution in $C([1, e], \mathbb{R})$.

Example 3.6 Consider the following fractional Hadamard differential equation

$$\begin{cases} \left({}^H D_a^{2.8} + 3\,{}^H D_a^{1.8} + 2\,{}^H D_a^{0.8}\right)x(t) = \frac{Bt|x(t)|}{1 + |x(t)|}, 1 \le t \le e, \\ x(1) = x'(1) = x''(1) = 0\,, \end{cases} \quad (3.4)$$

Here $\alpha = 2.8$, $\lambda = 3$, and $g\big(t, x(t)\big) = \frac{Bt|x(t)|}{1+|x(t)|}$.

$$\lim_{|x| \to 0} \frac{Bt}{1 + |x|} = Bt \le Be,$$

and

$$\vartheta_g = \frac{C_g}{\Gamma(1.8)} \max_{t \in J} \left|1 - \frac{1}{t}\right| \le 1.85B.$$

Therefore, choosing any real number $0 < B \le 0.54$, the Theorems 3.3 and 3.4 can be applied, hence the problem (3.4) has a unique solution in $C([1, e], \mathbb{R})$.

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وجود حل وحيد لمعادلات هادامارد التفاضلية التتابعية الكسرية

في هذه المقالة، لقد حصلنا على الشروط الكافية لمسالة وجود حل وحيد للمعادلات التفاضلية الغير الخطية التتابعية ذوات الرتب الكسرية $1<\alpha<2$ و $2<\alpha<3$ وتستند النتائج التي تم الحصول عليها نظريات النقطة الثابتة لبناخ وشوادر. واستعرضنا بعض الأمثلة لشرح تطبيق النظريات.

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