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Neighborhoods and Partial Sums of Certain Class of Analytic and Multivalent Functions Involving the Cho–Kwon-Srivastava Operator Defined by Convolution

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Abstract
In this present paper, we derive several subordination results for analytic function defined by convolution, and we prove several inclusion relations associated with the δ-neighborhood of certain subclasses of analytic functions with negative coefficients defined by the Cho-Kwon-Srivastava operator by making use of familiar concept of neighborhood of analytic function. The results of partial sums and subordinating results and result of integral means inequalities and also obtained.

Keywords:
Cho-Kwon-Srivastava Operator
Subordination
δ-neighborhood
Convolution

1. Introduction:

Let S_p bethe class of function $f(z)$ of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p},\ (p\in\mathbb{N}),\ (1.1)$$

which are analytic and P-valent in theunit disk $U=\{z:z\in\mathbb{C},\ |z|<1\}$.Also let T_p denote the subclass of S_p consisting of analytic and P-valent functions which can be expressed in the form

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}z^{k+p},\ (a_{k+p}\geq 0)\ (p\in\mathbb{N}),\ (1.2)$$

Let

$$g(z)=z^p-\sum_{k=1}^{\infty}b_{k+p}z^{k+p},(b_{k+p}\geq 0)\ (1.3)$$

Then the Hadamard product (or convolution) $f*g$ of the analytic functions f and g isdefined by

$$(f*g)(z)=z^p-\sum_{k=1}^{\infty}a_{k+p}b_{k+p}z^{k+p},(a_{k+p}\geq 0,b_{k+p}\geq 0)\ (1.4)$$

Definition 1.1 ([3] see also [7,9]), we define the δ-neighborhood of function $f(z)\in T_p$ is definedby

$$N_\delta(f)=\left\{k:k\in T_p,k(z)=z^p-\sum_{k=1}^{\infty}c_{k+p}z^{p+k}\ \text{and}\ \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}-c_{k+p}\right|\leq\delta\right\}.(1.5)$$

In particular, for the identity function $h(z)=z^p$, we immediately have

$$N_\delta(h)=\left\{k:k\in T_p,k(z)=z^p-\sum_{k=1}^{\infty}c_{k+p}z^{p+k}\ \text{and}\ \sum_{k=1}^{\infty}(k+p)\left|c_{k+p}\right|\leq\delta\right\}(1.6)$$

Definition 1.2 [10]
The linear operator $L_p(a,c):S_p\to S_p$, is defined by

$$L_p(a,c)\,f(z)=\phi_p(a,c;z)*f(z)\ (z\in U)\ ,(1.7)$$

where $\phi_p(a,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k\ z^{k+p}}{(c)_k}$, (1.8)

$(a)_k$ is the Pochammer symbol.

Definition 1.3 [2] introduced the linear operator $L_p^\lambda(a,c): S_p \to S_p$ analogons to $L_p(a,c)$, defined by

$L_p^\lambda(a,c)f(z) = \phi_p^\lambda(a,c;z) * f(z)\ (p \in N; a,c \in R/Z_0^-; \lambda > -p)$, (1.9)

where $\phi_p^\lambda(a,c;z)$ is the function defined in termsof the Hadmard product (or Convolution) by the followingcondition

$$\phi_p^\lambda(a,c;z) * \phi_p(a,c;z) = \frac{z^p}{(1-z)^{p+\lambda}}. \quad (1.10)$$

We can easily find from (1.7), (1.8) and (1.9) and for the function $f(z) \in T_p$ that

$$L_p^\lambda(a,c) = z^p - \sum_{k=1}^{\infty} \frac{(p+\lambda)_k (c)_k}{k!(a)_k} a_{k+p}\, z^{p+k} \quad (a_{k+p} \ge 0) \quad .(1.11)$$

Itis easily verified from (1.11) that

$$z\left(L_p^\lambda(a+1,c)f\right)'(z) = aL_p^\lambda(a,c)f(z) - (a-p)L_p^\lambda(a+1,c)f(z), \quad (1.12)$$

And

$$z\left(L_p^\lambda(a,c)f\right)'(z) = (\lambda+p)L_p^{\lambda+1}(a,c)f(z) - \lambda L_p^\lambda(a,c)f(z). \quad (1.13)$$

Also by specializing the parameters a, c, λ we obtain

Remark 1.1 If we put $a = p+1, c = \lambda = 1$, in (1.11)then we get

$$L_p^1(p+1,1)f(z) = f(z). (1.14)$$

Remark 1.2If we put $a = p, c = \lambda = 1$, in (1.11)then we get

$$L_p^1(p,1)f(z) = \frac{zf'(z)}{p}. (1.15)$$

Remark 1.3 If we put $c = a, \lambda = n$, in (1.12)then we get

$$L_p^n(a,a)f(z) = D^{n+p-1}f(z) \qquad (n > -p), (1.16)$$

where $D^{n+p+1}f(z)$ is the well-Known Ruchewehy derivative of order $n+p-1$.

Now making use of Cho-Kwon-Srivastava operator $L_p^\lambda(a,c)$ defined by (1.11), we introduced the following subclass $S_p(f,g,a,b,c,\lambda,\beta)$ of p-valentanalytic function .

Definition 1.4for $p \in N; a,c \in R/Z_0^-; b \in C^*; \lambda > -p; 0 < \beta \le 1$ and for all $z \in U$, let $S_p(f,g,a,b,c,\lambda,\beta)$ denote the subclass of S_p consisting of p-valent analytic function $f(z)$ and $g(z)$ defined by (1.2) and (1.3), respectively, and satisfying

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)} - p\right)\right\} \ge \beta\left|\frac{1}{b}\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)} - p\right)\right|, \quad (1.17)$$

where, $(f*g)(z)$ defined by (1.4).

Remark 1.4 Setting $a = p+1$ and $c = \lambda = 1$ in definition 1.4 then we obtain the class $S_p(f,g,p+1,b,1,1,\beta)$ satisfying

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{z(f*g)'(z)}{(f*g)(z)} - p\right)\right\} \ge \beta\left|\frac{1}{b}\left(\frac{z(f*g)'(z)}{(f*g)(z)} - p\right)\right|,$$

$(z \in U; p \in N; b \in C^*; 0 < \beta \le 1)$.

Remark 1.5 If we put $p = c = \lambda = 1,\ b = 1-\gamma\ (0 \le \gamma < 1)$ and $g(z) = \frac{z}{1-z}$ in Remark 1.4, the class $S_1\left(f, \frac{z}{1-z}, 2, 1-\gamma, 1, 1, \beta\right)$ reduces to the class $UST(\gamma,\beta)$ (see[4], [5] and [8]).

Remark 1.6 If we put $\beta = 1$ in Remark 1.5 , the class $S_1\left(f, \frac{z}{1-z}, 2, 1-\gamma, 1, 1, 1\right)$ is reduced to the class $S^*(\gamma)$ (class of starlike function of order γ) (see[1]).

2. Coefficient estimates:

Lemma 2.1let f and g definedby(1.2) and (1.3), respectively,satisfy the following condition:

$$\sum_{k=1}^{\infty}\left[k(\beta+1) + |b|\right]\phi_k^\lambda(a,c)\left|a_{p+k}\right|b_{p+k} \le |b| \quad ,(2.1)$$

$(p \in N; a,c \in R/Z_0^-; b \in C^*; \lambda > -p; 0 < \beta \le 1;\ z \in U)$,

Where $\phi_k^\lambda(a,c) = \frac{(p+\lambda)_k (c)_k}{k!(a)_k}$,(2.2) then $f(z)\in S_p\left(f,g,a,b,c,\lambda,\beta\right)$.

Proof. Itsuffices to show that .

$$\beta\left|\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right)\right|-\mathrm{Re}\left\{\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right)\right\}\le|b|.$$

Let (2.2) hold . Then we have

$$\beta\left|\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right)\right|-\mathrm{Re}\left\{\left(\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right)\right\}\le\frac{(1+\beta)\sum_{k=1}^{\infty}k\phi_k^\lambda(a,c)b_{p+k}\left|a_{p+k}\right|}{1-\sum_{k=1}^{\infty}\phi_k^\lambda(a,c)b_{p+k}\left|a_{p+k}\right|}\le|b|.$$

This complete the proof of lemma 1 .

3. Inclusion relationships involving $N_\delta(h)$ of the class $S_p\left(f,g,a,b,c,\lambda,\beta\right)$:

In our investigation of the inclusion relation involving $N_\delta(h)$, we shall require the following Theorem .

Theorem 3.1A function $f(z)\in T_p$ defined by (1.2) belongs to the class $S_p\left(f,g,a,b,c,\lambda,\beta\right)$ if and only if

$$\sum_{k=1}^{\infty}\left[k(\beta+1)+|b|\right]\phi_k^\lambda(a,c)a_{p+k}b_{p+k}\le|b|\quad ,(3.1)$$

($p\in N; a,c\in R/Z_0^-; b\in C^*; \lambda>-p; 0<\beta\le1$; $z\in U$),

where $\phi_k^\lambda(a,c)$ defined by(2.2) . The result is sharp .

Proof . we only need to prove the " only if"part of Theorem 3.1.

Let $f(z)$ defined by(1.1) belongs to the class $S_p\left(f,g,a,b,c,\lambda,\beta\right)$,we can write

$$\beta\left|\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right|-\mathrm{Re}\left\{\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right\}$$

$$\le(1+\beta)\left|\frac{z\left[L_p^\lambda(a,c)(f*g)(z)\right]'}{L_p^\lambda(a,c)(f*g)(z)}-p\right|=\frac{(1+\beta)\left|\sum_{k=1}^{\infty}k\phi_k^\lambda(a,c)b_{p+k}a_{p+k}z^k\right|}{\left|1-\sum_{k=1}^{\infty}\phi_k^\lambda(a,c)b_{p+k}a_{p+k}z^k\right|}$$

or equivalently,

$$\mathrm{Re}\left\{\frac{\sum_{k=1}^{\infty}k\phi_k^\lambda(a,c)b_{p+k}a_{p+k}z^k}{1-\sum_{k=1}^{\infty}\phi_k^\lambda(a,c)b_{p+k}a_{p+k}z^k}\right\}\le\frac{|b|}{(1+\beta)}\quad .(3.2)$$

We now chose values of z on the real axis and let $z\to1$ through the real values, then the inequality (3.2) immediately yields the desired condition (3.1) .

Theorem 2 If $\delta=\dfrac{a(p+1)|b|}{c\left[(1+\beta)+|b|\right](p+\lambda)b_{p+1}}\quad (p>|b|)$,(3.3)

andassume that $k\ge\dfrac{2a+1-c(p+\lambda)}{(p+c+\lambda)}$

then $S_p\left(f,g,a,b,c,\lambda,\beta\right)\subset N_\delta(h)$.(3.4)

Proof. Let $f(z)\in S_p\left(f,g,a,b,c,\lambda,\beta\right)$, then in view of Theorem 1 and under the condition we have

$$\left[(\beta+1)+|b|\right]\phi_1^\lambda(a,c)\sum_{k=1}^{\infty}a_{p+k}b_{p+k}\le\sum_{k=1}^{\infty}\left[k(\beta+1)+|b|\right]\phi_k^\lambda(a,c)a_{p+k}b_{p+k}\le|b|$$

,

Whichyields $\displaystyle\sum_{k=1}^{\infty}a_{p+k}\le\frac{a|b|}{\left[(\beta+1)+|b|\right]c(p+\lambda)b_{p+1}}$.(3.5)

On the other hand , we also find from (3.1) and (3.5)

$$\sum_{k=1}^{\infty}(\beta+1)(p+k)\phi_k^\lambda(a,c)a_{p+k}b_{p+k}$$

$$+\sum_{k=1}^{\infty}\left[|b|-p(\beta+1)\right]\phi_k^\lambda(a,c)a_{p+k}b_{p+k}\le|b|$$

,

Or,

$$\frac{c(p+\lambda)b_{p+1}}{a}\sum_{k=1}^{\infty}(\beta+1)(p+k)a_{p+k}$$

$$\le|b|+\sum_{k=1}^{\infty}\left[p(\beta+1)-|b|\right]\phi_k^\lambda(a,c)a_{p+k}b_{p+k}$$

$$\le\frac{(p+1)|b|(\beta+1)}{(\beta+1)+|b|}$$

Hence

$$\sum_{k=1}^{\infty}(p+k)a_{p+k} \le \frac{a(p+1)|b|}{c[(\beta+1)+|b|](p+\lambda)b_{p+1}} = \delta \qquad (p>|b|), \quad (3.6)$$

then $f(z)\in N_\delta(h)$. This completes the proof of Theorem 3.2

Corollary 3.2 If $\frac{(p+1)|b|}{[(\beta+1)+|b|]b_{p+1}} = \delta \qquad (p>|b|)$, (3. 7)

and assume that $k \ge 1$, then $S_p(f,g,p,b,\beta)\subset N_\delta(h)$ (3. 8)

Proof. Putting $a = p+1$, $c=\lambda=1$ in the Theorem 3.2.

Corollary 3.3 If $\delta = \frac{1-\gamma}{2+\beta-\gamma}, \qquad 0\le\gamma<1$, (3. 9)

and assume that $k \ge 1$. Then $UST(\gamma,\beta)\subset N_\delta(h)$ (3. 10)

Proof. Putting $a=p+1, p=c=\lambda=1, \; b=1-\gamma \; (0\le\gamma<1)$, and $g(z)=\frac{z}{1-z}$ in the Theorem 3.2 .

Corollary 3.4 If $\delta = \frac{1-\gamma}{3-\gamma}, \qquad 0\le\gamma<1$, (3. 11)

and assume that $k \ge 1$. Then $S^*(\gamma)\subset N_\delta(h)$ (3. 12)

Where $S^*(\gamma)$ the class of starlike functions of order γ.

Proof. Putting $a=p+1, p=c=\lambda=\beta=1, \; b=1-\gamma \; (0\le\gamma<1)$, and $g(z)=\frac{z}{1-z}$ in the Theorem 3.2.

4. $\delta-$neighborhood for the class $S_p^{(\alpha)}(f,g,a,b,c,\lambda,\beta)$.

In this section, we determine the neighborhoodfor the class $S_p^{(\alpha)}(f,g,a,b,c,\lambda,\beta)$, which is define as follows.

Definition4.1 A function $f(z)\in T_p$ is said to be in the class $S_p^{(\alpha)}(f,g,a,b,c,\lambda,\beta)$ if there exist a function $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ such that,

$$\left|\frac{f(z)}{k(z)}-1\right| < p-\alpha, (z\in U, 0\le\alpha<p) . \quad (4.1)$$

Theorem 4.1 let $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ and

$$\alpha = p - \frac{\delta b_{p+1}\phi_1^\lambda(a,c)[\beta+1+|b|]}{(p+1)[b_{p+1}\phi_1^\lambda(a,c)(\beta+1+|b|)-|b|]} \quad ,(4.2)$$

then $N_\delta(k)\subset S_p^{(\alpha)}(f,g,a,b,c,\lambda,\beta)$. (4.3)

Proof. Suppose that $f(z)\in N_\delta(k)$. we find that from (1.4)

$$\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}-c_{k+p}\right| \le \delta, \quad (4.4)$$

which immediately have

$$\sum_{k=1}^{\infty}\left|a_{k+p}-c_{k+p}\right| \le \frac{\delta}{(1+p)} \qquad (p\in N), \quad (4.5)$$

as $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$, then we have from Theorem 3.1

$$\sum_{k=1}^{\infty}c_{p+k} \le \frac{|b|}{\phi_1^\lambda(a,c)b_{p+1}[|b|+(\beta+1)]} \quad ,(4.6)$$

so that

$$\left|\frac{f(z)}{k(z)}-1\right| \le \frac{\delta b_{p+1}\phi_1^\lambda(a,c)[\beta+1+|b|]}{(p+1)[b_{p+1}\phi_1^\lambda(a,c)(\beta+1+|b|)-|b|]} = p-\alpha$$

provided that α is given by (3.4.2). thus $f(z)\in S_p^{(\alpha)}(f,g,a,b,c,\lambda,\beta)$. This completes the proof of Theorem 4.1 .

Corollary 4.1 let $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ and

$$\alpha = p - \frac{\delta b_{p+1}[\beta+1+|b|]}{(p+1)[b_{p+1}(\beta+1+|b|)-|b|]}, \quad (4.7)$$

then $N_\delta(k)\subset S_p^{(\alpha)}(f,g,p+1,b,1,1,\beta)$. (4.8)

Proof. Putting $a = p+1$, $c=\lambda=1$ in the Theorem 3.2.

Corollary 4.2 let $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ and

$$\alpha = 1 - \frac{\delta(2+\beta-\gamma)}{2(2+\beta)(1-\gamma)+2\gamma^2}, \quad (4.9)$$

then $N_\delta(k)\subset UST^{(\alpha)}(\gamma,\beta)$. (4.10)

Proof. Putting $a=2, p=c=\lambda=1, b=1-\gamma \; (0\le\gamma<1)$, and $g(z)=\frac{z}{1-z}$ in the Theorem 4.1 .

Corollary 4.3 let $k(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ and

$$\alpha = 1 - \frac{\delta(3-\gamma)}{6(1-\gamma)+2\gamma^2} \quad ,(4.11)$$

then $N_\delta(k)\subset ST^{(\alpha)}(\gamma)$. (4.12)

Proof. Putting $a=2, p=c=\lambda=\beta=1, b=1-\gamma\ (0\le\gamma<1)$, and $g(z)=\dfrac{z}{1-z}$ in the Theorem 4.1.

5. Subordination Results:

Definition 5.1 A sequence $\{b_{p+k}\}_{k=0}^{\infty}$ of complex is called subordination factor sequence if for any regular and convex function

$$k(z)=\sum_{k=0}^{\infty}c_{p+k}z^{p+k}\ ,\ \text{with}\ c_p=1, z\in U,$$

$$\sum_{k=0}^{\infty}b_{p+k}c_{p+k}z^{p+k}\prec g(z)\quad(z\in U).\ (5.1)$$

Lemma 5.1 [11] the sequence $\{b_{p+k}\}_{k=0}^{\infty}$ is subordination factor if and only if

$$\operatorname{Re}\left\{1+2\sum_{k=0}^{\infty}b_{p+k}z^{p+k}\right\}>0\qquad(z\in U)\ .(5.2)$$

Now, we obtain the subordination result for the class $S_p(f,g,a,b,c,\lambda,\beta)$.

Theorem 5.1 let $f(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ of the form (1.2) and

$m(z)=\sum_{k=0}^{\infty}c_{p+k}z^{p+k}$ $\quad c_p=1$, be regular and convex function in U, then

$$\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}\left(f*m\right)(z)\prec m(z)\quad ,(5.3)$$

moreover,

$$\operatorname{Re}\{f(z)\}>-\frac{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}\quad ,(5.4)$$

and the subordination result (5.3)is sharp for the maximum factor

$$\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}\ .(5.5)$$

Proof. Let $f(z)\in S_p(f,g,a,b,c,\lambda,\beta)$ of the form (1.1) and $m(z)=\sum_{k=0}^{\infty}c_{p+k}z^{p+k}$ $\quad c_p=1$, be regular and convex function in U .to show subordinating result (5.3), we need to show that

$$\left\{\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}a_{p+k}\right\}_{k=0}^{\infty}\ ,(5.6)$$

isa subordinating factor sequence with $a_p=1$ which in view of lemma 5.1is true if

$$\operatorname{Re}\left\{1+\sum_{k=0}^{\infty}\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left[(\beta+1)+|b|\right]}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left[(\beta+1)+|b|\right]+|b|\right]}a_{p+k}z^{p+k}\right\}>0\ ,\ (z\in U).(5.7)$$

Since

$$k\left[(\beta+1)+|b|\right]b_{p+k}\phi_k^{\lambda}(a,c)\ge\left[(\beta+1)+|b|\right]b_{p+1}\phi_1^{\lambda}(a,c)>0\ (k\ge1),$$

on using Theorem 3.1 , we have for $|z|=r<1$,

$$\operatorname{Re}\left\{1+\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left[(\beta+1)+|b|\right]}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left[(\beta+1)+|b|\right]+|b|\right]}\sum_{k=0}^{\infty}a_{p+k}z^{p+k}\right\}=$$

$$\operatorname{Re}\left\{1+\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}z^{p}+\right.$$

$$\left.\frac{1}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}\sum_{k=1}^{\infty}b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)a_{p+k}z^{p+k}\right\}$$

$$\ge1-\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}r^{p}-\frac{|b|}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}r^{p+k}$$

$$\ge1-\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}-\frac{|b|}{\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}=0\ .$$

Which evidently prove of (5.7), and hence the subordinationresult (5.3), we easily get the result (5.4), and for the function

$$f(z)=z^{p}-\frac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}z^{p+1}\in S_p\left(f,g,a,b,c,\lambda,\beta\right),$$

it can be verified $\dfrac{b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)}{2\left[b_{p+1}\phi_1^{\lambda}(a,c)\left(\beta+1+|b|\right)+|b|\right]}$ isthe maximum factorfor the subordinationresult in (5.3).

6. Partial Sum:

In this section, we determineinequalities involving partial sums of $f(z)\in T_p$ where the Partial sum of $f(z)\in T_p$ defined by (1.2) is defined as follows

$f_0(z)=z^p$ and $f_n(z)=z^p-\sum_{k=1}^{n}a_{k+p}\,z^{p+k}\quad (a_{k+p}\geq 0; n\geq 1)$. (6.1)

Theorem 6.1 Let $f(z)\in T_p$ is defined by (1.2) belong to $S_p\left(f,g,a,b,c,\lambda,\beta\right)$,

The

$$\operatorname{Re}\left\{\frac{(f*g)(z)}{(f*g)_n(z)}\right\}>1-\frac{1}{H_{n+1}(f,g,a,b,c,\lambda,\beta)}\quad (z\in U, n\in N), \quad (6.2)$$

and

$$\operatorname{Re}\left\{\frac{(f*g)_n(z)}{(f*g)(z)}\right\}>\frac{H_{n+1}(f,g,a,b,c,\lambda,\beta)}{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)}\quad ,(6.3)$$

Where

$$H_{n+1}(f,g,a,b,c,\lambda,\beta)=\frac{\left[(n+1)(\beta+1)+|b|\right](c)_{n+1}(p+\lambda)_{n+1}}{(a)_{n+1}(1)_{n+1}|b|}\quad ,(6.4)$$

$(p\in N; a,c\in R/Z_0^-; b\in C^*; \lambda>-p; 0<\beta\leq 1;\ z\in U)$.

Proof. Let $f(z)\in T_p$ is defined by (1.2) belong to $S_p\left(f,g,a,b,c,\lambda,\beta\right)$, then from Theorem 3.1 and using $H_{n+1}(f,g,a,b,c,\lambda,\beta)>H_n(f,g,a,b,c,\lambda,\beta)>1$, (6.5) we get

$$\sum_{k=1}^{n}a_{p+k}b_{p+k}+H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}\leq\sum_{k=1}^{\infty}H_k(f,g,a,b,c,\lambda,\beta)a_{p+k}b_{p+k}\leq 1. \quad (6.6)$$

Set

$$g_1(z)=H_{n+1}(f,g,a,b,c,\lambda,\beta)\left\{\frac{(f*g)(z)}{(f*g)_n(z)}-\left\{1-\frac{1}{H_{n+1}(f,g,a,b,c,\lambda,\beta)}\right\}\right\} \quad (6.7)$$

$$=1-\frac{H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}{1-\sum_{k=1}^{n}a_{p+k}b_{p+k}z^k},$$

which is analytic in U and $g_1(0)=0$. If (6.7) holds we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right|=\left|\frac{-H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}{2\left(1-\sum_{k=1}^{n}a_{p+k}b_{p+k}z^k\right)-H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}\right|$$

$$\leq\frac{H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}}{2-2\sum_{k=1}^{n}a_{p+k}b_{p+k}-H_{n+1}(f,g,a,b,c,\lambda,\beta)\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}}\leq 1,$$

which show the $\operatorname{Re}\{g_1(z)\}>0$,and from (6.7), we obtain the inequality (6.2).

Similarly , if we put

$$g_2(z)=1+H_{n+1}(f,g,a,b,c,\lambda,\beta)\left\{\frac{(f*g)_n(z)}{(f*g)(z)}-\frac{H_{n+1}(f,g,a,b,c,\lambda,\beta)}{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)}\right\}$$

(6.8)

$$=1-\frac{\{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)\}\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}{1-\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^k},$$

and making use (6.5), we find that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right|=\left|\frac{-\{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)\}\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}{2\left(1-\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^k\right)-\{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)\}\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}z^k}\right|$$

$$\leq\frac{\{1+H_{n+1}(f,g,a,b,c,\lambda,\beta)\}\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}}{2-2\sum_{k=1}^{n}a_{p+k}b_{p+k}-\{H_{n+1}(f,g,a,b,c,\lambda,\beta)-1\}\sum_{k=n+1}^{\infty}a_{p+k}b_{p+k}}\leq 1,$$

which proved the inequality (6.3).

7. Integral means inequalities:

Lemma 7.1 If the functions f and g are analytic in U with $g \prec f$,then for
$\eta > 0$ and $0 < r < 1$,

$$\int_0^{2\pi} \left| g\left(re^{i\theta}\right)\right|^{\eta} \, d\theta \le \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^{\eta} \, d\theta \, .$$

Proof. In 1925, Littlewood [6]

Theorem 7.1 let $f(z) \in S_p\left(f, g, a, b, c, \lambda, \beta\right)$, and suppose that

$$\sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \le \frac{|b|}{\left[j(\beta+1)+|b| \right]} \quad (j=1,2,\ldots), (7.1)$$

If there exist analytic function $w(z)$ given by

$$\{w(z)\}^j = \frac{\left[j(\beta+1)+|b| \right]}{|b|} \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^k \; (7.2)$$

thenfor $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| L_p^{\lambda}(a,c)(f*g)(z)\right|^{\sigma} d\theta \le \int_0^{2\pi} \left| L_p^{\lambda}(a,c)(f*g)_{p+j}(z)\right|^{\sigma} d\theta \;\; (\sigma > 0), \; (7.3)$$

where $\left(f*g\right)_{p+j}(z)$ is given by

$$\left(f*g\right)_{p+j}(z) = z^p - \frac{|b|}{\phi_j^{\lambda}(a,c)\left[j(\beta+1)+|b| \right]} z^{p+j} . (7.4)$$

Proof. By virtue of (1.11) and (7.4) we have

$$L_p^{\lambda}(a,c)\left(f*g\right)(z) = z^p - \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^{k+p} \; (7.5)$$

and

$$L_p^{\lambda}(a,c)\left(f*g\right)_{p+j}(z) = z^p - \frac{|b|}{\left[j(\beta+1)+|b| \right]} z^{p+j} , (7.6)$$

then we must show that

$$\int_0^{2\pi} \left| z^p - \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^{k+p} \right|^{\sigma} d\theta \le \int_0^{2\pi} \left| z^p - \frac{|b|}{\left[j(\beta+1)+|b| \right]} z^{p+j} \right|^{\sigma} d\theta \;\; (\sigma > 0),$$

by lemma 7.1,it is sufficient to show that

$$1 - \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^k \prec 1 - \frac{|b|}{\left[j(\beta+1)+|b| \right]} z^j . (7.7)$$

Setting

$$1 - \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^k = 1 - \frac{|b|}{\left[j(\beta+1)+|b| \right]} \{w(z)\}^j ,$$

which readily yields (7.2) and $w(0) = 0$ .Now, we prove that the analytic function $w(z)$ satisfies $|w(z)| < 1$, $z \in U$. Using (7.1) , we obtain

$$\left| \{w(z)\}^j \right| = \left| \frac{\left[j(\beta+1)+|b| \right]}{|b|} \sum_{k=1}^{\infty} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} \, z^k \right|$$

$$< \frac{\left[j(\beta+1)+|b| \right]}{|b|} \sum_{k=1}^{\infty,} \phi_k^{\lambda}(a,c) a_{p+k} b_{p+k} < 1 .$$

This completes the proof of the Theorem .

Corollary 3.8 Let $f(z) \in S_p\left(f, g, a, b, c, \lambda, \beta\right)$ and suppose that

$$\sum_{k=1}^{\infty} a_{p+k} b_{p+k} \le \frac{|b|}{\left[j(\beta+1)+|b| \right]} \quad (j=1,2,\ldots), (7.8)$$

if there exist analytic function $w(z)$ given by

$$\{w(z)\}^j = \frac{\left[j(\beta+1)+|b| \right]}{|b|} \sum_{k=1}^{\infty} a_{p+k} b_{p+k} \, z^k , (7.9)$$

thenfor $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| (f*g)(z)\right|^{\sigma} d\theta \le \int_0^{2\pi} \left| (f*g)_{p+j}(z)\right|^{\sigma} d\theta \;\; (\sigma > 0), (7.10)$$

where $\left(f*g\right)_{p+j}(z)$ is given by

$$\left(f*g\right)_{p+j}(z) = z^p - \frac{|b|}{\left[j(\beta+1)+|b| \right]} z^{p+j} . (7.11)$$

Proof. Putting $a = p+1, c = \lambda = 1$ in Theorem 7.1.

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في هذا البحث تم دراسة مجموعه من الفصول الجزئية المعينة للدوال التحليلية ذات الحدود السالبة باستخدام معامل شو كون ومن ثم درسة خصائص هذه الفصول متمثلة دراسة بعض النظريات الهامة مثل نظريات معامل التقديرات , نظريات الجوار, المجاميع الجزئية ونظريات الالتواء والتبعية ونظريات أخرى

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