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# On Fredholm Theory In a Banach Algebra of Operators

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## Introduction

Let $B(X)$ be the Banach algebra of all bounded linear operators on the infinite dimensional Banach space $X$.

We say that two infinite dimensional Banach spaces $X$ and $Y$ form a dual system, denoted $(X,Y)$, if there is defined on $X \times Y$ a nondegenerated bounded bilinear form < , >, [6, P. 6].

Let $T \in B(x)$, if there exists $T^{+} \in B(Y)$ such that

$$<Tx,y>=<x,T^{+}y> \quad \text{for all } x \in X \text{ and } y \in Y,$$

then $T^{+}$ is called a conjugate operator to $T$ relative to the dual system $(X,Y)$. [4, p. 44]

Let $A(X,Y)$ be the algebra of all $T \in B(X)$ that have, with respect to the dual system $(X,Y)$, a conjugate $T^{+} \in B(Y)$. $A(X,Y)$ is a Banach algebra with respect to the normal given by

$$\|T\|_{A} = max\{\|T\|,\|T^{+}\|\}, \quad [4, P. 45]$$

An operator T in $B(X)$ is called a Fredholm operator if both $\alpha(T)$ and $\beta(T)$ are finite, Where $\alpha(T)$ denotes the dimension of the null space of $T$, dim $N(T)$, and $\beta(T)$ denotes the codimension of the image of $T$, $co\dim R(T)$ [2, P. 3]

In this paper we study the connection between some properties of operators $T \in A(X,Y)$ and their conjugates $T^{+}$. For example we show that if $T$ is an operator in $A(X,Y)$, such that the dimension of the null space of $T$ is finite or the codimension of the image of $T^{+}$ is finite, then $R(T^{+}) = N(T)^{\perp}$ and $\beta(T^{+}) = \alpha(T)$ are equivalent. We also prove that if the codimension of the image of $T$ is finite, then the dimension of the null space of the conjugate operator $T^{+}$ is also finite and in fact, is less or equal it.

Now let $\Phi(X)$ be the set of all Fredholm operators in $B(X)$, and for any $T$ in $\Phi(X)$, the index $ind(T)$, is defined by the formula

$$ind(T) = \alpha(T) - \beta(T)$$

let $\Phi_0(X)$ be the set of all Fredholm operators of index zero and $F(X)$ be the set of all finite rank operators.
Also let $[M]$ be the subspace generated or spanned by a non - empty subset $M$ of $X$.

§ **1. Fredholm Operators in *B(X)*.**

In this section we study the connection between the properties of operators $T$ and their conjugates $T^+$.
First we give the following general proposition which appeared in [7, P. 99] for the particular case of the dual system $(X, X')$ where $X'$ is the dual space of $X$ and $(X, X')$ the dual system with bilinear form defined by $< x, x' >= x'(x)$.

**Proposition (1. 1)**

Let $T \in A(X,Y)$, then we have

(1) $R(T)^\perp = N(T^+)$ (2) $N(T^+) = N(T^+)^{\perp\perp}$

(3) $R(T^+)^\perp = N(T)$ (4) $N(T) = N(T)^{\perp\perp}$

**Proof:**

(1) $y \in N(T^+) \Leftrightarrow T^+ y = 0 \Leftrightarrow < x, T^+ y >= 0$ for all $x \in X$

$\Leftrightarrow < Tx, y >= 0$ for all $x \in X$

$\Leftrightarrow y \in R(T)^\perp$

(2) From part (1), it follows that

$$N(T^+)^{\perp\perp} = (R(T)^\perp)^{\perp\perp} = R(T)^\perp = N(T^+)$$

In similar way, we can prove (3) and (4). □
Form the previous proposition, we Conclude the following:

**Corollary (1. 2)**

Let $T \in A(X,Y)$, if $T$ (or $T^+$) is surjective, then $T^+$ (or $T$) is injective.
Also we need the following two lemmas which are known.

**Lemma (1. 3)**

Assume $T \in A(X,Y)$

(1) If $x_1, x_2, ..., x_n$ are elements in $N(T)$ and $y_1, y_2, ..., y$ are elements in $Y$ such that $< x_i, y_k >= \delta_{ik}$ for i, k = 1,2,…,n then $[y_1, y_2, ..., y_n] \cap R(T^+) = \{0\}$

(2) if $w_1, ..., w_n$ are elements in $N(T^+)$ and if $z_1, ..., z$ are elements in $X$ such that $< z_i, w_k >= \delta_{ik}$ for i, k = 1,…, n then $[z_1, ..., z\ ] \cap R(T) = \{0\}$

**lemma (1. 4)**

If E and F are subspaces of a vector space V whose intersection is trivial, then E has a complementary space that contains F.

Now we can state one of our main results.

**Theorem (1. 5)**

Let $T \in A(X,Y)$

(1) If $\alpha(T) < \infty$ then $R(T^+) = N(T)^{\perp}$ iff $\beta(T^+) = \alpha(T)$

(2) If $\alpha(T^+) < \infty$ then $R(T) = N(T^+)$ iff $\beta(T) = \alpha(T^+)$

**proof**

(1) $\Rightarrow$) If $\alpha(T) = 0$, then it is clear that $\beta(T^+) = \alpha(T)$.

Now , assume that $\alpha(T) = n > 0$ and $\{x_1,...,x_n\}$ is a basis of $N(T)$, there exist n linearly independent elements $y_1, y_2,..., y_n$ in $Y$ such that $< x_i, y_k >= \delta_{ik}$ for i, k = 1, …, n. [5, P. 18], [6, P. 63]. So $[y_1,..., y_n] \cap R(T^+) = \{0\}$ [lemma 1. 3].

For every $y$ in $Y$, we define

$$w_y = y - \sum_{i=1}^{n} < x_i, y > y_i \text{ ; so}$$

$$< x_k, w_y >=< x_k, y > - \sum_{i=1}^{n} < x_i, y >< x_k, y_i >= 0 \quad \text{for all k, } 1 \le k \le n$$

this implies that $w_y \in N(T)^{\perp}$ so $w_y \in R(T^+)$

Hence, every $y \in Y$ can be represented in the form

$$y = \alpha_1 y_1 + ... + \alpha_n y_n + w_y \quad , \quad w_y \in R(T^+) \text{ and } \alpha_i =< x_i, y >$$

Therefor, $[y_1,..., y_n] \oplus R(T^+) = Y$ and so

$$\operatorname{codim} R(T^+) = n = \dim N(T) \text{ that is } \beta(T^+) = \alpha(T)$$

$\Leftarrow$) Clearly, $R(T^+) \subset N(T)^{\perp}$ [3, P. 90, P. 134]. To prove that $N(T)^{\perp} \subset R(T^+)$, if $\alpha(T) = 0$, then $N(T) = 0$ implies that $N(T)^{\perp} = Y$ and hence $R(T^+) = N(T)^{\perp}$. Assume that $\alpha(T) = n > 0$ and $\{x_1,...,x_n\}$ is a basis for $N(T)$ [5, P. 18] [6, P. 63].

There exist n linearly independent elements $y_1,..., y_n$ in $Y$ such that

$< x_i, y_k >= \delta_{ik}$ for i, k = 1,…,n

so $[y_1,..., y_n] \cap R(T^+) = \{0\}$ [lemma 1. 3] and $R(T^+)$ has a complementary space that contains $[y_1,..., y_n]$ since

$dim[y_1,...,y_n] = n = \alpha(T) = \beta(T^+)$ hence $Y = [y_1,...,y\ ] \oplus R(T^+)$.

Now, if $y \in N(T)^{\perp} \subseteq Y$ , then

$y = \alpha_1 y_1 + ... + \alpha_n y_n + T^+ w$ $\alpha_i \in C, w \in Y$ since for all k, $1 \le k \le n$, we have

$$0 = < x_k, y > = \sum \alpha_i < x_k, y_i > + < x_k, T^+ w > = \alpha_k + < Tx_k, w > = \alpha_k$$

hence $y = T^+ w$ , i. e. $y \in R(T^+)$ , thus $N(T)^{\perp} \subseteq R(T^+)$ therefore $R(T^+) = N(T)^{\perp}$.

(2) Assume $T \in A(X,Y)$ and $\alpha(T^+) < \infty$ , then $T^+ \in A(Y,X)$ and T is the conjugate operator to $T^+$ and since $\alpha(T^+) < \infty$ , then by part one $R(T) = N(T^+)$ and $\beta(T) = \alpha(T^+)$ are equivalent. □

Now, suppose $T \in A(X,Y)$. In the following theorem we show that if the codimension of the image of *T* is finite then the dimension of the null space of the conjugate operator $T^+$ to *T* is also finite.

**Theorem (1. 6)**

Let $T \in A(X,Y)$

(1) If $\beta(T) < \infty$ , then $\alpha(T^+) < \infty$ . In particular $\alpha(T^+) \le \beta(T)$.

(2) If $\beta(T^+) < \infty$ , then $\alpha(T) < \infty$ . In particular $\alpha(T) \le \beta(T^+)$.

**Proof:**

(1) If $\beta(T) = 0$ , then $R(T) = X$ this implies that the conjugate operator $T^+$ to $T$ is injective [ Corollary 1. 2] so $\alpha(T^+) = 0$.

Now we may assume that $\beta(T) = m > 0$ , and assume the contrary, that $N(T^+)$ is infinite dimensional, therefore one can choose n linearly independent elements $y_1,...,y_n$ in $N(T^+)$ such that $n > m$.

There exist n linearly independent elements $x_1,...,x_n$ in $X$ such that

$< x_i, y_k > = \delta_{ik}$ i, k = 1, 2, …, n

then $[x_1,...,x_n] \cap R(T) = \{0\}$ (lemma 1. 3).

Since $R(T)$has a complementary space which contains $[x_1, x_2,...,x_n]$, then

$$n = dim[x_1,...,x_n] \le co\,dim\,R(T) = \beta(T) = m$$

and this is contradiction.

Thus $\alpha(T^+)$ must be finite and also $\alpha(T^+) \leq \beta(T)$.

(2) Since $T \in A(X,Y)$ then $T^+ \in A(Y,X)$ and since $\beta(T^+) < \infty$, therefore by part (1) $\alpha(T) \leq \beta(T^+)$.□

Using this theorem we have the following Corollary

**Corollary (1. 7)**

Let $T \in A(X,Y)$, if $\beta(T)$ and $\beta(T^+)$ are finite, then $T \in \Phi(X)$ and $T^+ \in \Phi(Y)$.

Form our preceding theorems (1. 5) and (1. 6), we obtain directly the following result.

**Corollary (1. 8)**

Let $T \in A(X,Y)$, then

(1) If $\beta(T) < \infty$, then $R(T) = N(T^+)^{\perp}$ if and only if $\beta(T) = \alpha(T^+)$.

(2) If $\beta(T^+) < \infty$, then $R(T^+) = N(T)$ if and only if $\beta(T^+) = \alpha(T)$.

It was proved in [4, P. 111] that if $T \in \Phi(X), T^+ \in \Phi(Y)$ and $ind(T) = -ind(T^+)$ then $R(T) = N(T^+)$, $R(T^+) = N(T)$

$$\beta(T) = \alpha(T^+) \quad , \quad \beta(T^+) = \alpha(T)$$

We prove a stronger result as follows

**Theorem (1. 9)**

Let $T \in A(X,Y), T^+ \in \Phi(Y)$, then the following statements are equivalent

(1) $R(T) = N(T^+)$ and $R(T^+) = N(T)$

(2) $\beta(T) = \alpha(T^+)$ and $\beta(T^+) = \alpha(T)$

(3) $ind(T) = -ind(T^+)$

**Proof:**

$(1) \Leftrightarrow (2)$ follows from Theorem (1. 4)

$(2) \Rightarrow (3)$ trivial.

$(3) \Rightarrow (2)$ since $\beta(T)$ and $\beta(T^+)$ are finite, by Theorem (1. 6), $\alpha(T^+) \leq \beta(T)$ and $\alpha(T) \leq \beta(T^+)$, then

$$ind(T) = \alpha(T) - \beta(T) \leq \beta(T^+) - \alpha(T^+) = -ind(T^+) = ind(T)$$

Therefore, $\alpha(T) - \beta(T) = \beta(T^+) - \alpha(T^+)$.

Thus $\alpha(T) = \beta(T^+)$, and $\alpha(T^+) = \beta(T)$

From the previous theorem, we have the following corollary.

**Corollary (1. 10)**

Assume $T \in A(X,Y), T \in \Phi^0(X), T^+ \in \Phi^0(Y)$ , then $T$ and $T^+$ are bijective if and only if one of them is injective or surjective.
Let $T \in A(X,Y)$ , Jorgens in [4, P. 111] proved that if $T \in \Phi(X), T^+ \in \Phi(Y)$ , and $ind(T) = -ind(T^+)$ , then there exist projections $P$ and $Q$ in $F(A) = A(X,Y) \cap F(X)$ such that

$$R(P) = N(T) \qquad N(P^+) = R(T^+)$$
$$N(Q) = R(T) \qquad R(Q^+) = N(T^+)$$

In fact, we can say more than this

**Theorem (1. 11)**

Let $T \in A(X,Y)$. Then the following are equivalent:

(1) $T \in \Phi(X), T^+ \in \Phi(Y)$ , and $ind(T) = -ind(T^+)$

(2) There exist projections $P$ and $Q$ in $F(A)$ such that

$$R(P) = N(T) \qquad N(P^+) = R(T^+)$$
$$N(Q) = R(T) \qquad R(Q^+) = N(T^+)$$

**Proof**

$(1) \Rightarrow (2)$

Let $\{x_1,...,x_n\}$ be a basis of $N(T)$, then there exist n linearly independent elements $y_1,...,y_n$ in y such that $< x_i, y_k >= \delta_{ik}$ i, k=1, …,n [5, P. 18] [6, P. 63]

Define $P$ by

$$Px = \sum_{i=1}^{n} < x, y_i > x_i \quad . \quad [4, p. 47]$$

So $P \in F(A)$, and since for all $k$, $1 \le k \le n, Px_k = x_k$ , $P$ is a projection and $R(P) = [x_1,...,x_n] = N(T)$. Clearly, the conjugate operator $P^+$ to $P$ is given by

$$P^+ y = \sum_{i=1}^{n} < x_i, y > y_i \text{ , thus}$$

$$N(P^+) = \{y \in Y :< x_i, y >= 0 \quad \text{for all i, } 1 \le i \le n\}$$
$$= \{ y \in Y :< x, y >= 0 \quad \text{for all } x \in N(T)\} = N(T)^\perp$$

but $R(T^+) = N(T^+)^\perp$ [Theorem 1.9] , hence $N(P^+) = R(T^+)$.

We can define the projection $Q$ in $F(A)$ by $Qx = \sum_{i=1}^{n} < x, w_i > z_i$

With $w_1,...,w_n$ form a basis of $N(T^+)$ and $z_1,...,z_n$ linearly independent elements in $X$ such that $<z_i, w_k>=\delta_{ik}$

In a similar way and with the aid of $R(T)=N(T^+)^{\perp}$ we can show that

$$N(Q)=R(T) \ , \ R(Q^+)=N(T^+)$$

$(2)\Rightarrow(1)$

To prove that $T\in\Phi(X)$ , it follows from $P\in F(X)$ that $dim\,R(P)<\infty$ and since $R(P)=N(T)$ , so $\alpha(T)<\infty$. Since $Q$ is a projection and $N(Q)=R(T)$. , this implies that $R(Q)$ is a Complementary space to $R(T)$ and since $Q\in F(X)$, then $dim\,R(Q)<\infty$ and so $\beta(T)<\infty$. Now, we show that $T^+\in\Phi(Y)$.

Since $P$ and $Q$ are projections of finite rank, we can easily show that $P^+$ and $Q^+$ are also projections of finite rank, therefore,

$\alpha(T^+)=dim\,N(T^+)=dim\,R(Q^+)<\infty$ and by $N(P^+)=R(T^+)$

$\beta(T^+)=co\,dim\,R(T^+)=dim\,R(P^+)<\infty$.

It remains to prove that. $ind(T)=-ind(T^+)$ For this, we show that $\alpha(T^+)=\beta(T)$. By hypothesis, there exists a projection $Q$ in $F(A)$ such that $N(Q)=R(T)$ and $R(Q^+)=N(T^+)$ , by [4, P.47] $Q$ can be represented in the form $Qx=\sum_{i=1}^{n}<x,y_i>x_i$ ,

where $x_1,...,x_n$ are linearly independent in $X$ and $y_1,y_2,...,y_n$ are linearly independent in $Y$. By [3, P. 125] $R(Q)=[x_1,...,x_n]$ . and since $N(Q)=R(T)$ , therefore $X=[x_1,...,x_n]\oplus R(T)$ and so $\beta(T)=n$. Now , since $R(T)=N(Q)=\{z\in X:<z,y_k>=0,k=1,...,n\}$,

then for all $k,1\le k\le n,0=<T_x,y_k>=<x,T^+y_k>$ for all $x\in X$ and since $(X,Y)$ is a dual system, then $T^+y_k=0$ for all $k,1\le k\le n$, that is $y_k\in N(T^+)$ which are linearly independent in $N(T^+)$ , therefore $n\le dim(N^+)$ and so $\beta(T)\le\alpha(T^+)$ . Since $\beta(T)\le\infty$ then $\alpha(T^+)\le\beta(T)$ [Theorem 1. 6] . Hence $\alpha(T^+)=\beta(T)$. Similarly, with the aid of the projection $P,\alpha(T)\le\beta(T^+)$. Thus

$$ind(T)=\alpha(T)-\beta(T)=\beta(T^+)-\alpha(T^+)=-ind(T^+)$$

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