Bessaga's conjecture in finite direct sum of d_1 unstable Köthe spaces

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Abstract: In this paper the following result is proved: Let E_1, E_2 be d_1 unstable Köthe spaces, then $E = E_1 \bigoplus E_2$ satisfies Bessaga's conjecture under certain conditions.

Introduction and preliminaries

In [2], Bessaga proved that if E is nuclear Köthe space with basis (e_n) and F is a complemented subspace of E with basis (f_n) , then there exists a non-decreasing sequence (k_n) such that $\lim_{n\to\infty} k_n = \infty$ and (f_n) is quasi-equivalent "QE" to (e_{k_n}), and conjectured that (k_n) can be chosen as a subsequence. Bessaga's conjecture could be shown to hold in power series spaces [8,9] , Dragilev space [1,6], regular d_1 spaces [3], stable regular spaces [3], and unstable Köthe spaces [10]. M.M Dragilev has shown in [5], that an unstable Frèchet space with basis is regular and nuclear. Thus any unstable Köthe space satisfies quasi-equivalent property "QEP". T.Terzioğlu has shown in [13] that for any unstable Köthe space of type d_1 or d_2 the Pelczynski problem "Does every complemented subspace of a nuclear Köthe space have a basis?" has an affirmative solution. Therefore using Prada result in [12], we can show that if E_1 is unstable d_1 Köthe space and E_2 is unstable d_2 Köthe space then $E = E_1 \bigoplus E_2$ satisfies Bessage's conjecture. Z. Nurlu and J. Sarsour in [9] have shown that if all linear continuous maps from E_1 to E_2 are compact and $E = E_1 \bigoplus E_2$ is not isomorphic to any proper subspace of itself then E satisfies Bessaga's conjecture.The author in [11] has shown that if E_1, E_2 are d_1 unstable Köthe spaces without common step space then $E_1 \bigoplus E_2$ satisfies Bessaga's conjecture

The aim of this paper is to prove that if $E_1 and E_2$ are d_1 unstable Köthe spaces with possible common step space and $E = E_1 \bigoplus E_2$ is not isomorphic to any proper subspace of itself, then E satisfies Bessaga's conjecture

Throughout the sequel, all spaces considered will be nuclear Köthe spaces. The cardinality of a set A is $|A|$.

If (a_n^r) is a matrix of non-negative real numbers satisfying

i)$\forall n \; \exists k$ such that $0 < a_n^k$ ii)$a_n^k \leq a_n^{k+1}$ $\forall n, k$
then the sequence space

$$K(a_n^r) = \{(\xi_n) \in C^N : \|(\xi_n)\|_k = \sum_n |\xi_n| a_n^k < \infty \forall k\}$$

is called a Köthe space and the matrix (a_n^k) is Köthe matrix.

The space $K(a_n^r)$ is

i) Nuclear if and only if $\forall k \exists s \left(\frac{a_n^k}{a_n^s}\right) \in \ell_1$
ii) regular (d_0) if for some matrix (b_n^r) generating $K(a_n^r)$ one has

$$\frac{b_{n+1}^k}{b_{n+1}^s} \leq \frac{b_n^k}{b_n^s} \qquad \forall n \in \mathbb{N} \text{ and } s \geq k$$

iii) of type d_1 if

$$\exists k \forall j \exists s \qquad \sup_n \frac{(a_n^j)^2}{a_n^k a_n^s} < \infty$$

iv) of type d_2 if

$$\forall k \exists j \forall s \qquad \sup_n \frac{a_n^k a_n^s}{(a_n^j)^2} < \infty$$

v) of type d_3 if

$$\frac{a_n^r}{a_n^s} \geq \frac{a_n^{r+1}}{a_n^{s+1}} \qquad \forall s \geq r$$

vi)unstable if

$$\exists s \forall p \exists q \forall r \qquad \lim_n \frac{a_{n+1}^p a_n^r}{a_{n+1}^q a_n^s} = 0........(1)$$

A Nuclear Köthe space is of type d_1 if and only if it is of type d_3.

Let X, Y be Fréchet spaces. The basic sequences (x_n) in X and (y_n) in Y are called "QE", if there exists a permutation π of $\mathbb{N}$ and scalars (t_n) such that $\sum \xi_n x_n$ converges in X if and only if $\sum \xi_n t_n y_{\pi(n)}$ converges in Y. If any two unconditional bases in a fréchet space E are QE then E is said to have "QEP".

Two system of norms $(\ \ _r)$, $(\|\ \ \|_r)$ defined on X are equivalent if for all p there exists c,q such that $x|_p \leq c\|x\|_q$ and for all r there exist k,s such that $\|x\|_r \leq k|x|_s$ for all x in X

Combining a lemma of V. P. Kondakov [7] and C. Bessaga [2] one can find a sequence of norms so that both conditions in lemma (1) are simultaneously satisfied (see also [9,10,11])

lemma 1. Let E be a d_1 nuclear Köthe space with basis (e_n). let F be a complemented subspace with basis (f_n) of E, then for all m in $\mathbb{N}$ there exist a sequence of norms $(\ \ _r)$resp. $(\|\ \ \|_r)$ defining the topologies of E resp. F, a continuous projection $Q : E \Rightarrow F$, and a non-decreasing sequence (k_n), with $\lim_{n\to\infty} k_n = \infty$ such that

a) $|Qe|_r \leq (2^m)^{-r}\|Qe\|_{r+1} \leq (2^m)^{-2(r+1)}|e|_{r+2} \qquad \forall e \in E\,\forall r \in \mathbb{N}$.........(2)

b) $\frac{|e_n|_r}{|e_n|_s} \geq \frac{|e_n|_{r+1}}{|e_n|_{s+1}} \qquad \forall r, s \in \mathbb{N}$ and $r \leq s$

c) $\frac{\|f_n\|_r}{\|f_n\|_s} = \frac{|e_{k_n}|_r}{|e_{k_n}|_s} \qquad \forall n, r, s$

Note. If E is regular then we can find a sequence of norms satisfying the condition of lemma (1) and $(|e_n|_r)$ is regular

lemma 2. Let $(\ \ _n)$ be any regular system of norms defining the topology of the unstable Köthe space E with basis (e_n). Then there exists s in $\mathbb{N}$ such that for all $k \geq s$ there exist $q \geq k$, for all r in $\mathbb{N}$ there exist M in $\mathbb{N}$ such that

$$\frac{|e_n|_k}{|e_n|_q} \leq \frac{|e_m|_k}{|e_m|_r} \qquad \forall n \geq m \geq M$$

Proof. Since E is unstable Köthe space then

$$\exists s\,\forall p\,\exists q\,\forall r\ \lim_{n\to\infty} \frac{|e_{n+1}|_p\,|e_n|_r}{|e_{n+1}|_q|e_n|_s} = 0$$

Let $k \geq s$, p=k then $\lim_{n\to\infty} \frac{|e_{n+1}|_k\, |e_n|_r}{|e_{n+1}|_q\, |e_n|_k} = 0$.

Therefore there exists M such that for all $n \geq M$ $\quad \frac{|e_{n+1}|_k\, |e_n|_r}{|e_{n+1}|_q\, |e_n|_k} \leq 1$.

Since E is regular then for all $m \geq n \geq M$ we have $\frac{|e_m|_k}{|e_m|_q} \leq \frac{|e_n|_k}{|e_n|_r}$.

Corollary 1. Let K be unstable nuclear space with basis (e_n). If (e_{j_n}) and (e_{i_n}) are two subsequences of (e_n) such that (e_{i_n}) QE to (e_{j_n}), then there exists, M such that $i_n = j_n$ for all $n \geq M$.

lemma 3. let E be unstable Köthe space with basis (e_n) and topology defined by increasing sequence of norms $(|\cdot|_r)$, then for any natural number r and

$$A_m = \left\{n : \frac{|e_m|_r}{|e_m|_{s+4}} \leq \frac{|e_n|_r}{|e_n|_{s+2}} \leq \frac{|e_m|_r}{|e_m|_s}, s \geq r\right\}$$

there exists M in $\mathbb{N}$ such that for all $m \geq M$, $\{A_m\} = \{m\}$

Proof. Let $(|\cdot|_n)$ be an increasing sequence of norms defining the topology of E, then $|x|_s \leq |x|_{s+1}$ $\forall s \in \mathbb{N}, x \in E$ and let $r \in N$ be arbitrary and fix it then $m \in A_m$ $\forall m \in \mathbb{N}$. Assume that there exists a subsequence (A_{k_n}) of (A_n) such that $|A_{k_n}| \geq 1$ for all n$\in \mathbb{N}$. Since E is nuclear space then we can say that $\forall n \neq m$ $A_{k_n} \cap A_{k_m} = \phi$. Therefore we can find $(e_{k_n}), (e_{j_m})$ subsequences of (e_n) such that $e_{k_n} \notin (e_{j_n})$ $\quad \forall n \in N$, also we have

$$\frac{|e_{k_m}|_r}{|e_{k_m}|_{s+4}} \leq \frac{|e_{j_n}|_r}{|e_{j_n}|_{s+2}} \leq \frac{|e_{k_m}|_r}{|e_{k_m}|_s}, \forall s \geq r$$

Let $\lambda_n = \frac{|e_{j_n}|_r}{|e_{k_n}|_r}$ $\forall n \in \mathbb{N}$, then we have $|e_{j_n}|_s \leq \lambda_n |e_{k_n}|_{s+2} \leq |e_{j_n}|_{s+4}$ $\forall n \in \mathbb{N}, s \geq r$.

Let $\||e_{j_n}\||_s = \lambda_n |e_{k_n}|_s$ $\forall n \in \mathbb{N}, s \geq r$, $J = (j_n), K = (k_n)$ and

$$\||e\||_t = \sum_{n \notin J} |e'_n(e)|\, |e_n|_t + \sum_{n \in J} |e'_n(e)|\, \||e_n\||_t,$$

then we have $(\||\cdot\||_t)$ equivalent to $(|\cdot|_t)$ and

$$\frac{\||e_{j_n}\||_s}{\||e_{j_n}\||_t} = \frac{|e_{k_n}|_s}{|e_{k_n}|_t} = \frac{\||e_{k_n}\||_s}{\||e_{k_n}\||_t}.$$

But this contradicts the result in lemma (2) . Therefore there exists, M such that for all $m > M$ we have $\{A_m\} - \{m\}$.

Remark.

In Frèchet space E, if a basic sequence (x_n) is QE to a basic sequence (y_n) and to a basic sequence (e_n) then (y_n) is QE to (e_n).Also we can find a system of norms $(\ \ _r)$ defining the topology of E such that $e_n|_r - y_n|_r$.

Main result

proposition 1. Let $E_1 - K(a^r_{n,1}), E_2 - K(a^r_{n,2})$ be regular d_1 spaces with basis (e^1_n) resp.(e^2_n) , let F be a complemented subspace with basis (f_n) of $E - E_1 \oplus E_2$, let $e_{2n} - e^1_n$ and $e_{2n-1} - e^2_n$. Then, for any increasing sequence of norms $(\ \ _r)$ resp. $(\|\ \ \|_r)$ defining the topologies of E resp. F, and satisfying the conditions of lemma (1) we have

$$\{n : \frac{|e_m|_r}{e_m|_{s+8}} < \frac{|e_n|_r}{e_n|_{s+4}} < \frac{e_m|_r}{|e_m|_s}, s > r\} > \{n : \frac{e_m|_r, \quad s > r}{|e_m|_s} - \frac{\|f_n\|_r}{\|f_n\|_s}\}$$

Proof. By lemma (1) we can find increasing sequences of norms $(\ \ _r)(\|\ \ \|_r)$ defining the topologies of E resp. F, such that $(|e_{2n-i+1}|_r)$ i=1,2 is regular, a continuous projection $Q : E \Rightarrow F$, and a non-decreasing sequence (k_n), with $\lim_{n\to\infty} k_n - \infty$ such that

a) $Qe|_r < (2^4)^{-r}\|Qe\|_{r+1} < (2^4)^{-2(r+1)}|e|_{r+2} \quad \forall e \in E\ \forall r..........(2)$

b) $\frac{e_n|_r}{e_n|_s} > \frac{e_n|_{r+1}}{e_n|_{s+1}} \quad \forall n, r, s \in \mathbb{N}$ *and* $s > r$........................⋆

c) $\frac{\|f_n\|_r}{\|f_n\|_s} - \frac{e_{k_n}|_r}{e_{k_n}|_s} \quad \forall n, r, s$

Choose $r > 1$ and fix it. Let

$$A_m - \{n : \frac{|e_m|_r}{e_m|_{s+8}} < \frac{|e_n|_r}{e_n|_{s+4}} < \frac{e_m|_r}{e_m|_s}, s > r\} \text{ and } K - \{n : \frac{e_m|_r}{e_m|_s} - \frac{\|f_n\|_r}{\|f_n\|_s}\}$$

Assume that $A_m < K$ then

$$\exists g \in E, g \neq 0, \text{ and } g - \sum_{n\notin A} e'_n(g)e_n - \sum_{n\in K} f'_n(g)f_n$$

Let

$$V_1 = \left\{2n \notin A : e'_{2n}(g) \neq 0 \text{ and } \exists \xi_{2n} \ni \frac{|e_{2n}|_r}{|e_{2n}|_{\xi_{2n}+4}} < \frac{|e_m|_r}{|e_m|_{\xi_{2n}+8}}\right\}$$

$$V_2 = \left\{2n-1 \notin A : e'_{2n-1}(g) \neq 0 \text{ and } \exists \xi_{2n-1} \ni \frac{|e_{2n-1}|_r}{|e_{2n-1}|_{\xi_{2n-1}+4}} < \frac{|e_m|_r}{|e_m|_{\xi_{2n-1}+8}}\right\}$$

$$P_1 = \left\{2n \notin A \cup V_1 : e'_{2n}(g) \neq 0 \text{ and } \exists \eta_{2n} \ni \frac{|e_{2n}|_r}{|e_{2n}|_{\eta_{2n}+4}} > \frac{|e_m|_r}{|e_m|_{\eta_{2n}}}\right\}$$

$$P_2 = \left\{2n-1 \notin A \cup V_2 : e'_{2n-1}(g) \neq 0 \text{ and } \exists \eta_{2n-1} \ni \frac{|e_{2n-1}|_r}{|e_{2n-1}|_{\eta_{2n-1}+4}} > \frac{|e_m|_r}{|e_m|_{\eta_{2n-1}}}\right\}$$

Let $a_1 = \min_{2n \in V_1} \xi_{2n}$, $a_2 = \min_{2n-1 \in V_2} \xi_{2n-1}$
$b_1 = \max_{2n \in P_1} \eta_{2n}$, $b_2 = \max_{2n-1 \in P_2} \eta_{2n-1}$.

Then by regularity of $(|e_{2n}|_r)$, $(e_{2n-1}|_r)$ and $(\star)$ we have

$$\frac{|e_{2n}|_{r+1}}{|e_{2n}|_{a_1+5}} < \frac{|e_m|_r}{|e_m|_{a_1+8}} \quad \forall 2n \in V_1$$

$$\frac{|e_{2n-1}|_{r+1}}{|e_{2n-1}|_{a_1+5}} < \frac{|e_m|_r}{|e_m|_{a_2+8}} \quad \forall 2n-1 \in V_2$$

$$\frac{|e_m|_{r+1}}{|e_m|_{b_1+1}} < \frac{|e_{2n}|_r}{|e_{2n}|_{b_1+4}} \quad \forall 2n \in P_1$$

$$\frac{|e_m|_{r+1}}{|e_m|_{b_2+1}} < \frac{|e_{2n-1}|_r}{|e_{2n-1}|_{b_2+4}} \quad \forall 2n-1 \in P_2 .$$

Let

$$g_1 = \sum_{n \in V_1} e'_n(g) \sum_{m \in K} f'_m(e_n) f_m ,$$

$$g_2 = \sum_{n \in V_2} e'_n(g) \sum_{m \in K} f'_m(e_n) f_m ,$$

$$h_1 = \sum_{n\in P_1} e'_n(g) \sum_{m\in K} f'_m(e_n) f_m ,$$

$$h_2 = \sum_{n\in P_2} e'_n(g) \sum_{m\in K} f'_m(e_n) f_m .$$

Then $g = g_1 + g_2 + h_1 + h_2$.

$$
\begin{aligned}
\|g_i\|_{r+1} &= \sum_{n\in V_i} |e'_n(g)| \sum_{t\in K} |f'_t(e_n)| \, \|f_t\|_{r+1} \le \sum_{n\in V_i} |e'_n(g)| \, \|Qe_n\|_{r+1} \\
&\le (2^4)^{-2} \sum_{n\in V_i} |e'_n(g)| \, |e_n|_{r+2} \\
&\le 2^{-8} \sum_{n\in V_i} |e'_n(g)| \frac{|e_m|_r}{|e_m|_{a_i+8}} |e_n|_{a_i+6} \\
&\le 2^{-8} \frac{|e_m|_r}{|e_m|_{a_i+8}} |g|_{a_i+6} \le 2^{-8} \frac{|e_m|_r}{|e_m|_{a_i+8}} \|g\|_{a_i+8} \\
&\le 2^{-8} \sum_{t\in K} |f'_t(g)| \frac{|e_m|_r}{|e_m|_{a_i+8}} \|f_t\|_{a_i+8} \\
&\le 2^{-8} \sum_{t\in K} |f'_t(g)| \, \|f_t\|_r = 2^{-8} \|g\|_r \\
\|h_i\|_{r+1} &= \sum_{n\in P_i} |e'_n(g)| \sum_{t\in K} |f'_t(e_n)| \, \|f_t\|_{r+1} \\
&\le \sum_{n\in P_i} |e'_n(g)| \sum_{t\in K} |f'_t(e_n)| \frac{|e_m|_{r+1}}{|e_m|_{b_i+1}} \|f_t\|_{b_i+1} \\
&\le \sum_{n\in P_i} |e'_n(g)| \, \|Qe_n\|_{b_i+1} \frac{|e_m|_{r+1}}{|e_m|_{b_i+1}} \\
&\le 2^{-8} \sum_{n\in P_i} |e'_n(g)| \, |e_n|_{b_i+2} \frac{|e_m|_{r+1}}{|e_m|_{b_i+1}} \\
&\le 2^{-8} \sum_{m} |e'_n(g)| \, |e_n|_r = 2^{-8} |g|_r
\end{aligned}
$$

Thus we get the contradiction that

$$\|g\|_{r+1} \leq \|g_1\|_{r+1} + \|g_2\|_{r+1} + \|h_1\|_{r+1} + \|h_2\|_{r+1}$$

$$\leq 2^{-8}(\|g\|_r + \|g\|_r + \|g\|_r + \|g\|_r \leq \|g\|_{r+1}$$

Therefore $|A_m| \geq |K|$.

lemma4. Let E_1, E_2 be unstable Köthe spaces with bases $(e_{n,1})$, $(e_{n,2})$ of E_1, E_2 resp. and let $(|\ |_r)$, $(\|\ \|_r)$ be sequences of norms defining the topologies of E_1, E_2 resp. Let

$$W_{m,1} = \{n : \frac{|e_{m,1}|_r}{|e_{m,1}|_{s+4}} \leq \frac{\|e_{n,2}\|_r}{\|e_{n,2}\|_{s+2}} \leq \frac{|e_{m,1}|_r}{|e_{m,1}|_s}, s \geq r\}$$

and

$$W_{m,2} = \{n : \frac{\|e_{m,2}\|_r}{\|e_{m,2}\|_{s+4}} \leq \frac{|e_{n,1}|_r}{|e_{n,1}|_{s+2}} \leq \frac{\|e_{m,2}\|_r}{\|e_{m,2}\|_s}, s \geq r\}.$$

Then for all $i = 1, 2$ $\exists M$ $\forall n \geq M$ $|W_{n,i}| \leq 1$.

Proof. Assume that there exists M in $\mathbb{N}$ such that for all $n \geq M$ $\exists k_n$ such that $|W_{k_n,i}| \geq 1$. Since $E_i, i = 1, 2$ is nuclear, then we can assume that $\exists (j_n), (i_n) \ni i_n \neq j_m \forall n, m$ and $i_n, j_n \in W_{k_n,i}$.
W.L.O.G. Let $i = 1$, then we have

$$\frac{|e_{k_n,1}|_r}{|e_{k_n,1}|_{s+4}} \leq \frac{\|e_{i_n,2}\|_r}{\|e_{i_n,2}\|_{s+2}} \leq \frac{|e_{k_n,1}|_r}{|e_{k_n,1}|_s}$$

$$\frac{|e_{k_n,1}|_r}{|e_{k_n,1}|_{s+4}} \leq \frac{\|e_{j_n,2}\|_r}{\|e_{j_n,2}\|_{s+2}} \leq \frac{|e_{k_n,1}|_r}{|e_{k_n,1}|_s} \forall s \geq r, \text{ and } n \in \mathbb{N}$$

Let $t_{j_n} = \frac{\|e_{j_n,2}\|_r}{|e_{k_n,1}|_r}$ and $t_{i_n} = \frac{\|e_{i_n,2}\|_r}{|e_{k_n,1}|_r}$. Then we have

$$\|e_{i_n,2}\|_s \leq t_{i_n} |e_{k_n,1}|_{s+2} \leq \|e_{i_n,2}\|_{s+4}$$

$$\|e_{j_n,2}\|_s \leq t_{j_n} |e_{k_n,1}|_{s+2} \leq \|e_{j_n,2}\|_{s+4}$$

let $|\|e_{j_n,2}\||_s = t_{j_n} |e_{k_n,1}|_s$, and $|\|e_{i_n,2}\||_s = t_{i_n} |e_{k_n,1}|_s$. Also let

$$\overline{|e_{n,2}|}_r = \begin{cases} \|e_{n,2}\|_r & \text{if } n \notin (j_n) \cup (i_n) \\ |\|e_{n,2}\||_r & \text{if } n \in (j_n) \cup (i_n) \end{cases}$$

Then we have

$$(\bar{\ }\ _r) = (\|\ \|_r) \text{ and } \frac{\bar{e}_{j_{n,2}}|_r}{|e_{j_{n,2}}|_s} = \frac{\bar{e}_{i_{n,2}}|_r}{|e_{i_{n,2}}|_s}.$$

But this contradicts the result in lemma 2 because E_2 is unstable Köthe space. Therefore for all $i = 1, 2\ \exists M\ \forall n \geq M,\ \ W_{n,i}\ \leq 1$

lemma 5. Let E_i be unstable Köthe space with basis (e^i_n), $i = 1, 2$, $(\ _r)$ be system of norms defining the topology of $E = E_1 \oplus E_2$. Let $e_{2n} = e^1_n$ and $e_{2n-1} = e^2_n$, and

$$A^i_m = \{n : \frac{|e^i_m|_r}{e^i_m|_{s+4}} < \frac{|e_n|_r}{e_n|_{s+2}} < \frac{e^i_m|_r}{|e^i_m|_s}, \quad s > r.\}$$

Then there exists M in N such that

$\forall m, n \geq M |A^i_n\ \leq 2,$ and if $A^i_n \cap A^j_m \neq \phi,$ then $A^i_n \subset A^j_m$ or $A^j_m \subset A^i_n$.

proof. Assume that there exist $(A^i_{i_n}) \ni A^i_{i_n}\ \geq 2\ \forall n \in \mathbb{N}$, i=1,2. Since E_i is unstable Köthe space then by lemma(3) there exists $M \in \mathbb{N}\ \forall n \geq M, A^i_{i_n} \cap \{2n\ \ i + 1 : n \in \mathbb{N}, i = 1 \text{ or } 2\}$ has only one element. Since E_i is nuclear space and $A^i_{i_n}\ \geq 2$ then there exists subsequences $(r_n), (s_n)$ such that $r_n \notin (s_n) \forall n \in \mathbb{N}$ and

$$\frac{|e^i_{j_n}|_r}{e^i_{j_n}|_{s+4}} < \frac{|e^j_{s_n}|_r}{e^j_{s_n}|_{s+2}} < \frac{e^i_{j_n}|_r}{|e^i_{j_n}|_s}, s > r$$

$$\frac{|e^i_{j_n}|_r}{e^i_{j_n}|_{s+4}} < \frac{|e^j_{r_n}|_r}{e^j_{r_n}|_{s+2}} < \frac{e^i_{j_n}|_r}{|e^i_{j_n}|_s}, \quad s > r, i, j = 1, 2 \text{ and } i \neq j.$$

Therefore we have

$$\frac{|e^j_{s_n}|_r}{e^j_{s_n}|_{s+8}} < \frac{|e^j_{r_n}|_r}{e^j_{r_n}|_{s+4}} < \frac{e^j_{s_n}|_r}{|e^j_{s_n}|_s}, \quad s > r.$$

So by the remark $(e^j_{s_n})$ is QE to $(e^j_{r_n})$ and there exists a sequence of norms $(\|\ \|_r)$ such that $\frac{\|e^j_{s_n}\|_r}{\|e^j_{s_n}\|_s} = \frac{\|e^j_{r_n}\|_r}{\|e^j_{r_n}\|_s}$ which contradicts the result in lemma (2) therefore there exist $M \in \mathbb{N}, \forall n \geq M, |A_n| \leq 2$.

Assume that there exists $(A^1_{j_n}), (A^2_{i_n})$ such that $A^1_{i_n} \cap A^2_{j_n} \neq \phi$; $A^1_{j_n} \not\subset A^2_{i_n}$ and $A^2_{i_n} \not\subset A^1_{j_n}$ since there exists $M \in \mathbb{N} \ni |A_n| \leq 2 \, \forall n \geq M$, then by lemmas (3 and 4) there exists a subsequence (l_n) of $\mathbb{N} \ni l_m \notin (j_n) \, \forall m$, $2l_n \in A^2_{i_n}$ and $2i_n - 1 \in A^1_{j_n}$. So we have

$$\frac{|e^1_{j_n}|_r}{|e^1_{j_n}|_{s+4}} \leq \frac{|e^2_{i_n}|_r}{|e^2_{i_n}|_{s+2}} \leq \frac{|e^1_{j_n}|_r}{|e^1_{j_n}|_s}, \ s \geq r$$

$$\frac{|e^2_{i_n}|_r}{|e^2_{i_n}|_{s+4}} \leq \frac{|e^1_{l_n}|_r}{|e^1_{l_n}|_{s+2}} \leq \frac{|e^2_{i_n}|_r}{|e^2_{i_n}|_s}, \ s \geq r \ .$$

Hence

$$\frac{|e^1_{j_n}|_r}{|e^1_{j_n}|_{s+8}} \leq \frac{|e^1_{l_n}|_r}{|e^1_{l_n}|_{s+4}} \leq \frac{|e^1_{j_n}|_r}{|e^1_{j_n}|_s}, \ s \geq r \ .$$

Therefore by the remark $(e^1_{J_n})$ is QE to $(e^1_{l_n})$ and there exists a sequence of norms $(\| \ \|_r)$ such that $\frac{\|e^1_{j_n}\|_r}{\|e^1_{j_n}\|_s} = \frac{\|e^1_{l_n}\|_r}{\|e^1_{l_n}\|_s}$ which contradicts the result in lemma (2). So there exists $M \in \mathbb{N}$ such that, $\forall m, n \geq M$, if $A_n \cap A_m \neq \phi$ then $A_m \subset A_n$ or $A_n \subset A_m$.

Theorem 1. Let G_i be d_1 unstable Köthe space with regular representation $G_i = K(a^r_{n,i})$, and let $(p_i), (q_i), p_1 = q_1 = 1$ be a subsequence of $\mathbb{N}$. Let $b^r_{n,1} = a^r_{i,1}$ if $p_i \leq n \leq p_{i+1}$ and let $b^r_{n,2} = a^r_{i,2}$ if $q_i \leq n \leq q_{i+1}$. Let $E_i = K(b^r_{n,i})$. Then if $E = E_1 \bigoplus E_2$ is not isomorphic to any proper subspace of E, then E satisfies Bessaga's conjecture.

Proof:- Since G_i is regular then E_i is regular . Let F be a complemented subspace with a basis (f_n) of E, let $(e_{n,i})$ be a basis of E_i, and let $(e_n) = (e_{n,1}) \cup (e_{n,2})$ then (e_n) is a basis of E. By lemma (1) and the note we can find increasing sequence of norms $(| \ |_r)$, $(\| \ \|_r)$ defining the topologies of E resp. F, a sequence (k_n) and a continuous projection $Q : E \to F$ such that

1. $\frac{|e_{n,i}|_r}{|e_{n,i}|_s} \geq \frac{|e_{n+1,i}|_r}{|e_{n+1,i}|_s}$ $\forall s \geq r, n \in \mathbb{N}$, and i=1,2

2. $\frac{\|f_n\|_r}{\|f_n\|_s} = \frac{|e_{k_n}|_r}{|e_{k_n}|_s}$

3. $\frac{|e_n|_r}{|e_n|_s} \geq \frac{|e_n|_{r+1}}{|e_n|_{s+1}}$ $s \geq r$

4. $|Qe_n|_r \leq (2^4)^{-r} \|Qe_n\|_{r+1} \leq (2^4)^{-2(r+1)} |e_n|_{r+2}$

Let $r \in N$ be arbitrary and fix it, $A_m = \{n : \frac{\|f_m\|_r}{\|f_m\|_{s+4}} \leq \frac{|e_n|_r}{|e_n|_{s+2}} \leq \frac{\|f_m\|_r}{\|f_m\|_s}, s \geq r\}$. By (2) we can say that

$$A_m = \{n : \frac{|e_{k_m}|_r}{|e_{k_m}|_{s+4}} \leq \frac{|e_n|_r}{|e_n|_{s+2}} \leq \frac{|e_{k_m}|_r}{|e_{k_m}|_s}, s \geq r\}$$

We are going to show that there exists $M \in \mathbb{N}$ such that if $m_1, m_2, ..., m_k$, and $m_i \geq M$, is any distinct indices then $\bigcup_{i=1}^{k} A_{m_i} \geq k$. But $A_{m_i} \cap A_{m_{i+1}} \neq \phi, \forall i$, then by lemma (5) there exists $1 \leq t \leq k$ such that $A_{m_i} \subset A_{m_t}, \forall i = 1, 2, ..., k$. So It is enough to show that $A_m \geq H_m$ where

$$H_m = \{n : \frac{|e_{k_m}|_r}{|e_{k_m}|_{s+4}} \leq \frac{\|f_n\|_r}{\|f_n\|_{s+2}} \leq \frac{|e_{k_m}|_r}{|e_{k_m}|_s}, s \geq r\},$$

because if $A_i \subset A_m$ then $i \in H_m$.

Assume that for each $m \in \mathbb{N}$ there exists $i_m \geq m$ such that

$$A_{i_m} \leq H_{i_m}$$

Let $t_n = \frac{\|f_n\|_r}{|e_{k_{i_m}}|_r}$ then we have

$$\|f_n\|_s \leq t_n |e_{k_{i_m}}|_{s+2} \leq \|f_n\|_{s+4}$$

Let $\overline{\|f_n\|} = t_n |e_{k_{i_m}}|$ $\forall n \in H_{i_m}$, and define $(\| \ \|_r)$ on F such that

$$\|h\|_r = \sum_{n \notin \cup_{m=1}^{\infty} H_{i_m}} |f'_n(h)| \, \|f_n\|_r + \sum_{n \in \cup_{m=1}^{\infty} H_{i_m}} |f'_n(h)| \, \overline{\|f_n\|}_r$$

Then we have $|||h|||_r \leq \|h\|_{r+4} \leq |||h|||_{r+8}, \forall h \in F$. So

$$(|||\ |||_r) = (|\ |_r)$$

since $\forall e \in E$,

$$|Qe|_r \leq (2^4)^{-r}\|Qe\|_{r+1} \leq (2^4)^{-r}|||Qe|||_{r+5} \leq (2^4)^{r}\|Qe\|_{r+9} \leq (2^4)^{-2(r+1)}|e|_{r+10}$$

Then we can choose a suitable subsequences of $(|||\ |||_r)$, $(|\ |_r)$ call it $(|||\ |||_r)$, $(|\ |_r)$ defining the topologies of F, E and satisfies all the conditions in proposition (1). Since G_i is unstable Köthe space, then by lemmas (3 and 4) and by the definition of E_i, we have for all subsequence (s_t) of $\mathbb{N}$ there exists $M \in \mathbb{N}$ such that for all $m \geq M$

$$A_m = \left\{n : \frac{|e_{k_m}|_r}{|e_{k_m}|_{s_t+4}} \leq \frac{|e_n|_r}{|e_n|_{s_t+2}} \leq \frac{|e_{k_m}|_r}{|e_{k_m}|_{s_t}}, s_t \geq r\right\} \geq \left\{n : \frac{|||f_n|||_r}{|||f_n|||_{s_t}} = \frac{|e_{k_m}|_r}{|e_{k_m}|_{s_t}} \forall s_t \geq r\right\}$$

since

$$\frac{|||f_n|||_r}{|||f_n|||_s} = \frac{|e_{k_m}|_r}{|e_{k_m}|_s} \quad \forall n \in H_m, \forall m \in \mathbb{N}$$

Then

$$H_{i_m} \leq \left\{n : \frac{|||f_n|||_r}{|||f_n|||_s} = \frac{|e_{k_{i_m}}|_r}{|e_{k_{i_m}}|_s} \forall s \geq r\right\}$$

Therefore

$$A_{i_m} \leq \left\{n : \frac{|||f_n|||_r}{|||f_n|||_s} = \frac{|e_{k_{i_m}}|_r}{|e_{k_{i_m}}|_s} \forall s \geq r\right\}$$

Since E is d_1, then we have a contradiction with the result in proposition (1). So

$$\exists M \in \mathbb{N}, \forall n \geq M, |A_n| \geq |H_n|.$$

Hence for each distinct indices $m_1, m_2 \ldots, m_k$ we have

$$\left|\bigcup_{i=1}^{k} A_{m_i}\right| \geq k$$

Therefore we can apply the Hall'Koeng theorem [4] to choose distinct indices (n_m) with $n_m \in A_m$.
Let $t_m = \frac{\|f_m\|_r}{|e_{n_m}|_r}$, we get

$$\|f_m\|_r \leq t_m |e_{n_m}|_{r+2} \leq \|f_m\|_{r+4} \forall t \geq r, \text{and } n \geq M$$

Therefore $(f_m)_{m>M}$ QE to (e_{n_m}).

Since E is not isomorphic to any of it's proper subspace of E, then

$$(e_m) \setminus (e_{n_m}) \geq M.$$

Therefore we can find (e_{i_n}) subsequence of (e_n), and QE to (f_n).

Corollary 2. If E_i is d_1 unstable Köthe space, and $E = E_1 \oplus E_2$ is not isomorphic to any proper subspace of E then E satisfies Bessaga's conjecture.

Proof:- Let $p_n = q_n = n$ for all n in $\mathbb{N}$ in the proof of the theorem

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