AN EXTENSION OF COCYCLES TO βS

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ABSTRACT. Let $F : S \times S \to G$ be a cocycle where S is a semigroup and G is a group. If S has the discrete topology and G is compact and if F is a cocycle, then the extension of $F, \hat{F} : \beta S \times \beta S \to G$ is also a cocycle. (Where βS is the stone č ech compactification of S.) Also we show that the extension of a coboundary is a coboundary.

1. INTRODUCTION

In [1], it was shown that every symmetric cocycle of a cancellative abelian semigroup to a divisible group is a coboundary. The same result was proved for several classes of semigroups such as periodic semigroups [2], idempotent semigroups [3].

Let $(S, +)$ be a discrete semigroup and βS be the stone čech compactification of S. Then $(+)$ can be extended to βS such that $(\beta S, +)$ is a compact Hausdorff right topological semigroup.

Although $(S, +)$ is abelian, $(\beta S, +)$ is far from being abelian. For example, $(\beta \mathbb{N}, +)$, the stone čech compactification of the discrete semigroup of natural numbers under usual addition, contains a large number of free subsemigroups which makes it nonabelian. See [4],[5]. Therefore, if $F : S \times S \to G$ is a symmetric cocycle then

$\hat{F} : \beta S \times \beta S \to G$, where $\hat{F}$ is the extension of F need not be symmetric.

In this papaer we show that if F is a cocycle on S to G, where G is a compact group, then F can be extended to a cocycle on βS to G which is also a coboundary.

Key words and phrases. cocycles,coboundary, stone čech compactification.

2. Preliminaries

Definition 1. Let $(S,+)$ be a semigroup and $(G,+)$ be an abelian group. The function $F : S \times S \to G$ is called a cocycle if

$$F(x,y) + F(x+y,z) = F(x,y+z) + F(y,z)$$

for all $x,y,z \in S$. In addition, if $F(x,y) = F(y,x)$ then F is called a symmetric cocycle.
A cocycle F on S to G is a coboundary if there exists a function $f : S \to G$ such that

$$F(x,y) = f(x) + f(y) - f(x+y).$$

Remark 2. Note that every coboundary is a cocycle. In addition, if S is abelian, then every coboundary is a symmetric cocycle.

Definition 3. Let S be a discrete space then βS will denote the stone čech compactification of S.
Note that for each element $p \in \beta S \setminus S$ we can find a net (x_α) in S such that $x_\alpha \to p$ in βS.

The properties of βS can be given in the following theorem:

Theorem 4. *Let S be an infinite discrete space. Then*
(1) *βS is a compact Hausdorff space.*
(2) *$S \subseteq \beta S$.*
(3) *S is dense in βS.*
(4) *given any compact space Y and any function $f : S \to Y$, then there exists a unique continuous function $\tilde{f} : \beta S \to Y$ such that $\tilde{f}|_S = f$.*

Proof. : See [6]. ■

Definition 5. Let $(S,+)$ be a semigroup. For each $s \in S$ we define the maps:

$$\lambda_s : S \to S, \quad where\ \lambda_s(t) = s + t \quad \forall t \in S,$$

$$\rho_s : S \to S, \quad where\ \rho_s(t) = t + s \quad \forall t \in S.$$

A semigroup S with a topology is called right topological semigroup if $\rho_s : S \to S$ is continuous for all $s \in S$.

Theorem 6. *Let $(S,+)$ be a discrete semigroup. Then $(+)$ can be extended to a binary associative operation $(+)$ to βS such that $(\beta S,+)$ is a compact Hausdorff right topological semigroup and $\lambda_s : \beta S \to \beta S$ is continuous for any $s \in S$.*

Proof. : See [6]. ∎

3. Main Result

In this section we prove the following result:

Theorem 7. *Let S be a discrete semigroup and G be a compact group. Let $F : S \times S \to G$ be a function. Assume there is a function $f : S \to G$ such that*

$$F(x,y) = f(x) + f(y) - f(x+y).$$

Then F can be extended to a function $\hat{F} : \beta S \times \beta S \to G$ such that

$$\hat{F}(p,q) = \hat{f}(p) + \hat{f}(q) - \hat{f}(p+q)$$

where $\hat{f}$ is the unique contiunous extension of f to βS.

Proof. : Let $f : S \to G$ be a function such that

$$F(x,y) = f(x) + f(y) - f(x+y), \quad x,y \in S.$$

Since S is discrete then f is continuous and by theorem 4 it has a continuous extension $\hat{f} : \beta S \to G$. We extend F to $\beta S \times \beta S$ as follows:

<u>Step1.</u>
Let $x \in S$ and $p \in \beta S \setminus S$. Choose a net (s_α) in S such that $s_\alpha \to p$ in βS. Let

$$\begin{aligned} \hat{F}(x,p) &= \lim_\alpha F(x,s_\alpha) \\ &= \lim_\alpha [f(x) + f(s_\alpha) - f(x+s_\alpha)] \\ &= \lim_\alpha [\hat{f}(x) + \hat{f}(s_\alpha) - \hat{f}(x+s_\alpha)] \end{aligned}$$

Now, the limits on the right hand side exist since $\hat{f}(x)$ is a single point and by continuity of $\hat{f}, \lim_\alpha \hat{f}(s_\alpha) = \hat{f}(p)$. Also $\hat{f}(x+s_\alpha) = \hat{f}(\lambda_x(s_\alpha)) = \hat{f} \circ \lambda_x(s_\alpha) \to \hat{f} \circ \lambda_x(p) = \hat{f}(x+p)$ since $\hat{f} \circ \lambda_x$ is continuous being the composition of two continuous functions. Hence, for any $x \in S$ and $p \in \beta S$

$$\hat{F}(x,p) = \hat{f}(x) + \hat{f}(p) - \hat{f}(x+p) \qquad (1).$$

Step2.

Let $p \in \beta S$ and $q \in \beta S \setminus S$ and choose a net (y_β) in S with $y_\beta \to q$ in βS. Let

$$\hat{F}(q,p) = \lim_{\beta} F(y_\beta, p)$$

Then from (1) we have

$$\hat{F}(q,p) = \lim_{\beta} [\hat{f}(y_\beta) + \hat{f}(p) - \hat{f}(y_\beta, p)]$$

As in step1, the limits in the right hand side exist with the only difference that $\hat{f}(y_\beta, p) = \hat{f}(\rho_p(y_\beta)) = \hat{f} \circ \rho_p(y_\beta) \to \hat{f} \circ \rho_p(q) = \hat{f}(q + p)$ since ρ_p is continuous.

Therefore, for any $p, q \in \beta S$,

$$\hat{F}(q,p) = \hat{f}(q) + \hat{f}(p) - \hat{f}(q + p).$$

■

Corollary 8. *Let S be a discrete semigroup and G be a compact group. Then if $F : S \times S \to G$ is a cocycle, which is a coboundary, then $\hat{F} : \beta S \times \beta S \to G$ is a cocycle.*

Proof. It follows from remark 2 and theorem 7.

■

4. References

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