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A TRIANGULAR FINITE ELEMENT FOR PLANE ELASTICITY WITH IN-PLANE ROTATION

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ملخص تم في هذه الدراسة تطوير واشتقاق معادلات الإزاحة لعنصر محدد مثلث معتمد على التشوه لحل المسائل المستوية ثنائية الأبعاد للمواد المرنة. يحتوى هذا العنصر على ثلاثة درجات حرية عند كل نقطة طرفية (Node) وهي حركتي الإزاحة الأساسيتين بالإضافة إلى الدوران المستوي. تم استخدام طريقة "التشوه" حيث يتم اشتقاق العناصر المحددة بالاعتماد على افتراض معادلات كثيرة الحدود (Polynomials) للتشوه وليس للإزاحة كما في الطريقة المعتادة. لفحص أداء هذا العنصر تم استخدامه لحل مسألتين من المسائل المستوية الشائعة وهي مسألة (deep cantilever beam) و مسألة (simply supported beam) وتم مقارنة نتائج الهبوط والإجهادات مع نتائج العنصر الموجود الشهير (CST) ومع النتائج التحليلية الحقيقية من نظرية المرونة. أظهرت الدراسة أن العنصر المثلث الجديد يعطي نتائج ذات درجة عالية من الدقة بالمقارنة مع (CST) وأن العنصر الجديد يمكن الاعتماد عليه في حل المسائل المستوية.

Abstract In the present paper the strain-based approach is applied to develop a new triangular finite element for the general plane elasticity. This triangular finite element has three degrees of freedom (two general external degrees of freedom and the in-plane rotation) at each of the three corner nodes. This element is based on linear and quadratic variations of the three components of strain. This element is used to obtain solutions to two dimensional elasticity problems where the contribution of the shear stress on deformation can be significant.

The performance of the element is investigated by applying it to two well-known plane elasticity problems; a cantilever beam loaded at the free end and a simply supported beam loaded by a point load at the mid-span. Convergence curves are plotted for vertical deflection at the points of application of load as well as for bending stress at points at the upper tension fibre and shearing stress at internal points in each problem.

It is found that the solutions obtained using this element are satisfactory for results of deflection, bending stress and the shearing stress even when only a small number of elements is used in the finite element solution.

KEYWORDS: strain based approach, triangular element, plane elasticity, in-plane rotation

1. INTRODUCTION

The development of displacement fields by the use of strain-based approach was first applied to curved elements. It was revealed that to obtain satisfactory converged results, the finite elements based on independent polynomial displacement functions require the curved structures to be divided into a large number of elements, [1], [2]. This work has shown that there are two essential components to any displacement field. The first of which relates to rigid body modes of displacements while the second component is due to the straining of the element and these are approximately represented by assumed independent polynomial terms for the various component of strains in so far as it is allowed by the compatibility equations.

In reference [3] it was shown that the above approach is not confined to the development of curved elements only but also to plane elements for the analysis of plane elasticity problems. Several such rectangular elements were developed, notably a rectangular element based on linear variation of the direct strains and constant shearing strain. This element has the two essential degrees of freedom at each of the four corner nodes, and it was shown to produce converged results when applied to several elasticity problems without the use of large number of elements.

In reference [4] Sfendji developed a rectangular element based on linear variation of all the three strain components. It has two degrees of freedom at each corner node and at an internal node. A rectangular finite element with in-plane rotation was developed in reference [5]. In this paper a new triangular element with in-plane rotation is developed and its performance is compared to that of the well know Constant Strain Triangle (CST).

2. ANALYTICAL CONSIDERATIONS

2.1 Displacement Fields for Strain Based Triangular Element with In-Plane Rotation (SBTREIR)

In this section, the strain approach is used to derive the displacement fields for strain based triangular element with in-plane rotation. The element has three degrees of freedom at each of its three corner nodes shown in Figure 1.

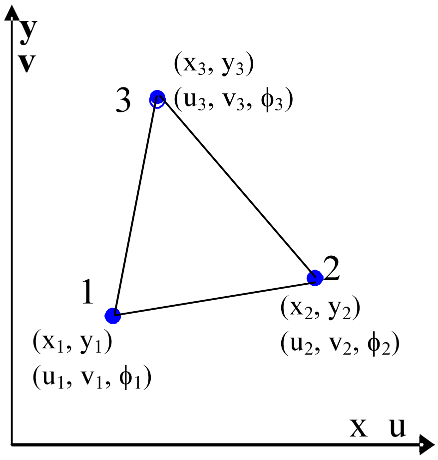

Figure 1: Coordinates and nodal points of triangular element with in-plane rotation (3 DOF per node)

The displacement fields are required to satisfy the requirement of strain-free rigid body mode of displacement in addition to the straining within the element. To get the first part of the displacement fields (U_1, V_1)

corresponding to the first mode, we begin by writing the strain/displacement relationships and make them equal to zero. By integration, we obtain the expressions for the displacements that give zero strains as follows.

$$\varepsilon_x = \frac{\partial U}{\partial x} = 0 \quad ===> U = a + f\ (y)$$

$$\varepsilon_y = \frac{\partial V}{\partial y} = 0 \quad ===> V = a + f\ (x)$$

$$\gamma_{xy} = \frac{\partial U_1}{\partial y} + \frac{\partial V_1}{\partial x} = 0 \quad ===> f_1'(y) + f_2'(x) = 0$$

where $f_1'(y)$ and $f_2'(x)$ are constants that can be taken as

$$f_1'(y) = -a_3 \quad ===> \quad f_1(y) = -a_3\, y$$

$$f_2'(x) = a_3 \quad ===> \quad f_2(x) = a_3\, x$$

thus

$$U_1 = a_1 - a_3\, y \qquad \textbf{Eq. 1}$$

$$V_1 = a_2 + a_3\, x$$

The in-plane rotation is calculated using the relation

$$\varphi = \frac{1}{2}\left(\frac{\partial V_1}{\partial x} - \frac{\partial U_1}{\partial y}\right) = a_3 \qquad \textbf{Eq. 2}$$

where

U_1, V_1: displacements in the x and y directions respectively.

ε_x, ε_y : direct strains in the x and y directions respectively.

γ_{xy} : shear strain.

a , a_2: translations of the element in the x & y directions respectively.

a_3 : rigid body in-plane rotation of the element.

The displacements within the element have to be defined by nine constants, a through a_9, (no. of constants should equal the no. of nodes times the no. of DOF per node). Three constants (a , a_2 & a_3) have already been defined while the remaining six have to be used to describe the deformation straining of the element. A first attempt to do so is to assume that

$$\varepsilon_x = a_4 + a_5\, y$$

$$\varepsilon_y = a_6 + a_7\, x \qquad \textbf{Eq. 3}$$

$$\gamma_{xy} = a_8\, x + a_9\, y$$

This arrangement of strains does not contain a constant term in the expression for γ_{xy} and it is expected that it wouldn't give good results. Also it was found that it leads to a singular displacement transformation matrix and hence it can't be used to derive a stiffness matrix.
Several other arrangements were tried to avoid this. A good arrangement that gives none-singular displacement transformation matrix is found to be as follows:

$$\varepsilon_x = a_4 + a\ \ y + a\ \ \frac{y}{4}$$

$$\varepsilon_y = a_6 + a\ \ x - a\ \ \frac{x}{4} \qquad \textbf{Eq. 4}$$

$$\gamma_{xy} = a_8 + a_9(x+y) - a_5\frac{x^2}{4} + a_7\frac{y^2}{4}$$

The constants a_4, a_6 and a_8 are the terms corresponding to state of constant strain that ensures the convergence of the solution with mesh refinement. The constants a_5, a_7 and a_9 are the terms corresponding to linear strain behaviour within the element.
We observe that, if the terms of this equation are twice differentiated, they satisfy the general compatibility equation of strains, namely:

$$\frac{\partial^2\varepsilon}{\partial y^2} + \frac{\partial^2\varepsilon}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\,\partial y} \qquad \textbf{Eq. 5}$$

To get the second part of the displacement fields (U_2, V_2), we first integrate the fist two equations as follows.

$$U_2 = a_4 x + a_5\ x\,y + a_9\frac{x\,y^2}{4} + f(y)$$

$$V_2 = a_6 y + a_7\ x\,y - a_9\frac{y\,x^2}{4} + f(x) \qquad \textbf{Eq. 6}$$

To get the functions f(x) and f(y), we substitute their derivatives in the shear strain equation then separate the resulting expressions for x and y respectively as follows:

$$\gamma_{xy} = \frac{\partial U_2}{\partial y} + \frac{\partial V_2}{\partial x}$$

$$a_8 + a_9(x+y) - a_5\frac{x^2}{4} + a_7\frac{y^2}{4} = a_5 x + f\ (x) + a_7 y + f'(y) \qquad \textbf{Eq. 7}$$

$$f(x)= \int [\frac{a_8}{2} + a_9 x - a_5 \frac{x^2}{4} - a_5 x]\,dx \;=\; a_8 \frac{x}{2} + a_5(-\frac{x^3}{12} - \frac{x^2}{2}) + a_9 \frac{x^2}{2}$$

$$f(y)= \int [\frac{a_8}{2} + a_9 y + a_7 \frac{y^2}{4} - a_7 y]\,dy \;=\; a_8 \frac{y}{2} + a_7(\frac{y^3}{12} - \frac{y^2}{2}) + a_9 \frac{y^2}{2} \qquad \textbf{Eq. 8}$$

Now, f(x) and f(y) are substituted in U_2, V_2. By adding the expressions for U_1& V_1 and U_2 & V_2 then calculating the in-plane rotation, (Eq. 2) the complete expressions for the displacement fields are obtained as:

$$U = a_1 - a_3 y + a_4 x + a_5\, x\, y + a_7(-\frac{y^2}{2} + \frac{y^3}{12}) + a_8 \frac{y}{2} + a_9(\frac{y^2}{2} + \frac{xy^2}{4})$$

$$V = a_2 + a_3\, x + a_5(-\frac{x^2}{2} - \frac{x^3}{12}) + a_6 y + a_7 xy + a_8 \frac{x}{2} + a_9(\frac{x^2}{2} - \frac{x^2 y}{4}) \qquad \textbf{Eq. 9}$$

$$\Phi = a_3 + a_5(-x - \frac{x^2}{8}) + a_7(y - \frac{y^2}{8}) + a_9(x - y - xy)/2$$

It is noted that we obtained quadratic terms (x^2, x^3, y^2 & y^3) without increasing the number of nodes beyond the three corner nodes. This is not achieved in the known constant strain triangular element (CST). It is expected that this increase in the degree of the polynomials will result in accurate solutions using this element as will be shown in the subsequent discussion.

Having obtained the displacement fields, the stiffness matrix of each triangular element can be evaluated using the general expression

$$\left[K^e\right] = \left[C^{-}\right]^T \left[\int [B]^T [D][B]^T \,dvol\right][C]^{-1} \qquad \textbf{Eq. 10}$$

where, the transformation matrix [C] is calculated as

$$[C] = \begin{bmatrix} U_1 @ (x_1, y_1) \\ V_1 @ (x_1, y_1) \\ \Phi_1 @ (x_1, y_1) \\ U_2 @ (x_2, y_2) \\ V_2 @ (x_2, y_2) \\ \Phi_2 @ (x_2, y_2) \\ U_3 @ (x_3, y_3) \\ V_3 @ (x_3, y_3) \\ \Phi_3 @ (x_3, y_3) \end{bmatrix}$$

and the strain matrix $[B]$ for this element is

$$[B]=\begin{bmatrix} 0 & 0 & 0 & 1 & y & 0 & 0 & 0 & \frac{y^2}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & x & 0 & -\frac{x^2}{2} \\ 0 & 0 & 0 & 0 & -\frac{x^2}{4} & 0 & \frac{y^2}{4} & 1 & x+y \end{bmatrix}$$

and [D]is the rigidity matrix given by

$$[D]=\frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \text{ for the state of plane stress}$$

and $[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$ for the state of plane strain.

The performance of the element derived above is investigated by applying it to two of the famous plane elasticity problems as described below. The results obtained by the developed element (SBTREIR) are compared to those given by the well-known constant strain triangular (CST) and the analytical value for deflection and stresses.

3. DEEP CANTILEVER BEAM LOADED AT THE FREE END

3.1 Problem Description

The first problem is a deep cantilever beam loaded by a point load at the free end. The beam has length, L=10m, height H=4m, and thickness t=0.0625m. The material properties; modulus of elasticity and Poisson's ratio are taken as E=100,000 kPa and ν=0.20 respectively. The point load at the free end of the cantilever is taken as P =100 kN. In order to achieve full fixity at the built-in end of the cantilever, all the nodes occurring at that end are assumed to be restrained in both the x and y directions as well as the in-plane rotation. The locations of the investigated points within the cantilever beam are shown in Figure 2. A sample mesh is shown in Figure 3.

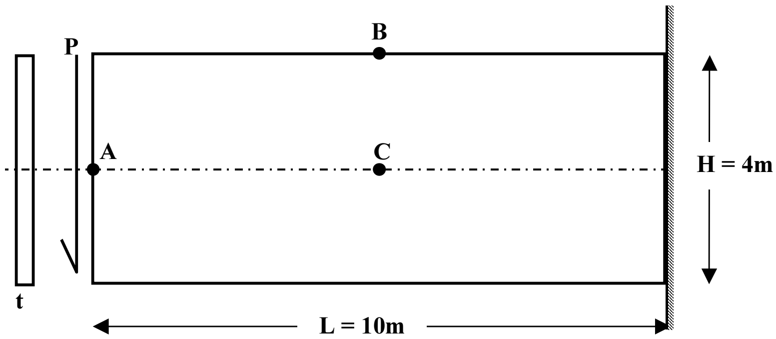

Figure 2: Dimensions of the cantilever beam problem

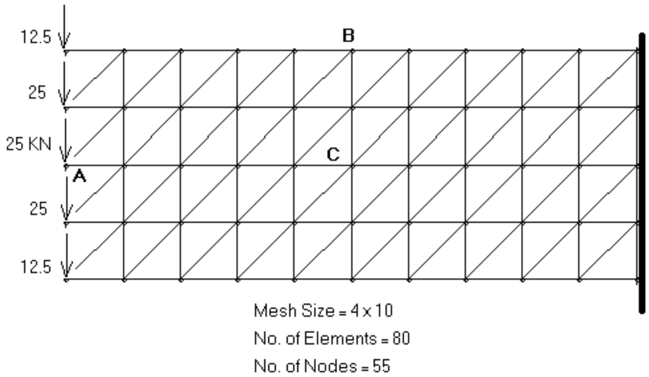

Figure 3: Sample mesh of the cantilever beam problem

3.2 Convergence Results of the Deep Cantilever Beam

Table 1 shows a summary of the used mesh sizes in the solution and results for vertical deflection, bending stress, shearing stress and the in-plane rotation at the specified points within the deep cantilever beam. Figures 4–7 show graphical comparison between the results obtained by each of the SBTREIR, the constant strain element, CST and the exact analytical solution.

Table 1: Results obtained by the SBTREIR and CST for the cantilever beam problem

Mesh Size	Vertical Deflection at A (mm)		Bending Stress at B (kPa)		Shearing Stress at C (kPa)		In-Plane Rotation at A (Rad)
	CST	SBTREIR	CST	SBTREIR	CST	SBTREIR	
2 x 5	0.632	0.786	1,096.03	2,105.46	379.00	378.632	0.112
4 x 10	0.922	0.988	2,033.33	2,624.35	512.83	586.728	0.137
5 x 12	0.978	1.024	2,250.00	2,719.65	521.00	568.396	0.143
6 x 15	1.015	1.056	2,405.00	2,795.31	559.67	595.095	0.146
8 x 20	1.053	1.086	2,583.33	2,880.09	576.5	597.552	0.149
Exact	**1.105**		**3000**		**600**		**0.156**

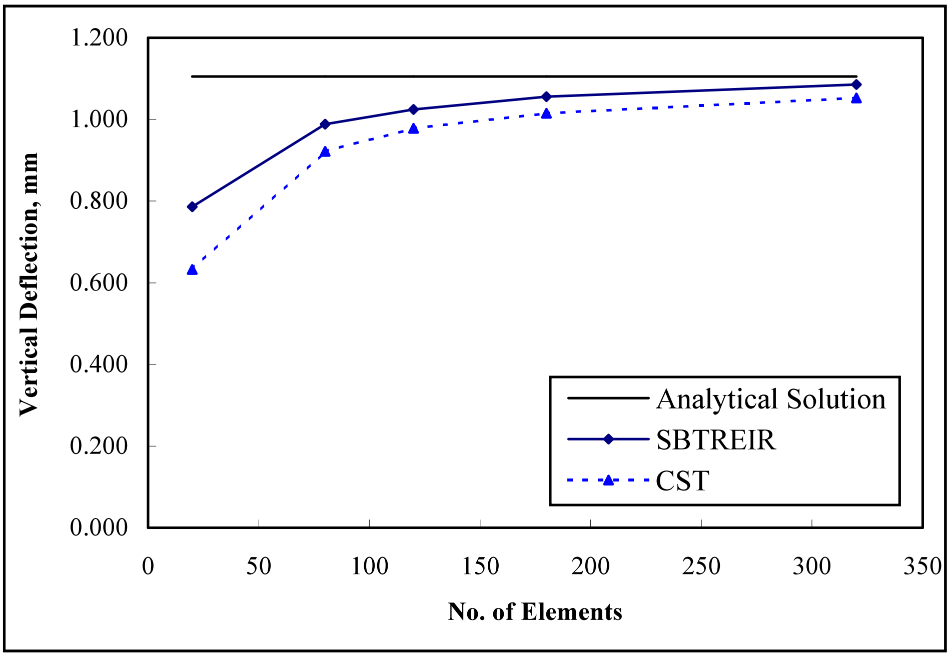

Figure 4: Vertical deflection at "A", (mm) in the cantilever beam

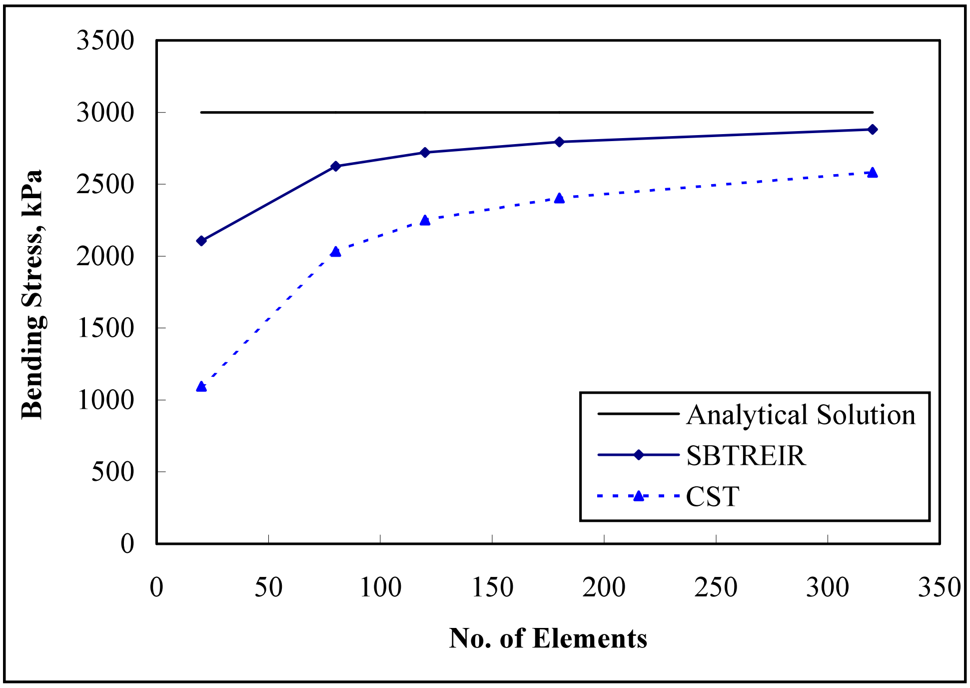

Figure 5: Bending stress at "B", (kPa) in the cantilever beam

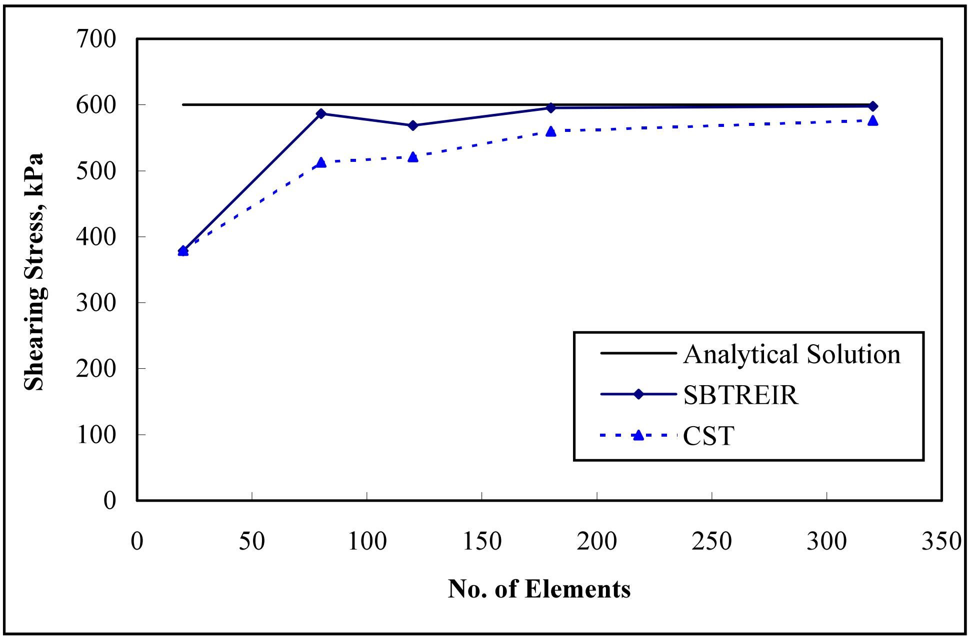

Figure 6: Shearing stress at "C", (kPa) in the cantilever beam

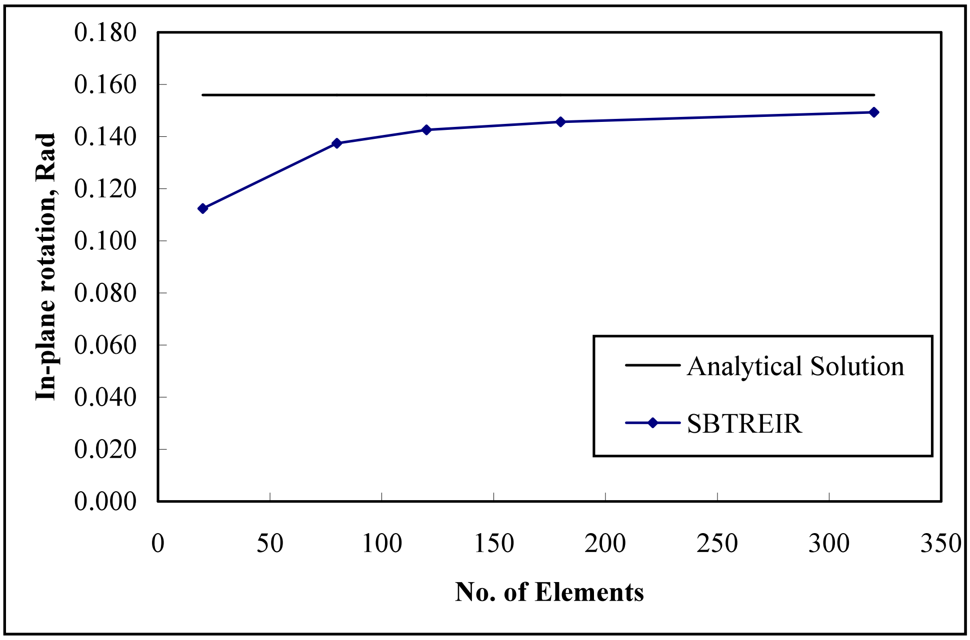

Figure 7: In-plane rotation at "A", (Rad) in the cantilever beam

4. SIMPLY SUPPORTED BEAM

4.1 Problem Description

The second problem is a simply supported beam loaded by a point load at the middle of its upper face. The beam has length, L=4m, height H=1m, and thickness t=0.5m. The material properties are taken as E=20,000 kPa and ν=0.20 respectively. The point load is taken as P =4.2 kN. The locations of the investigated points within the simply supported beam are shown in Figure 8 below.

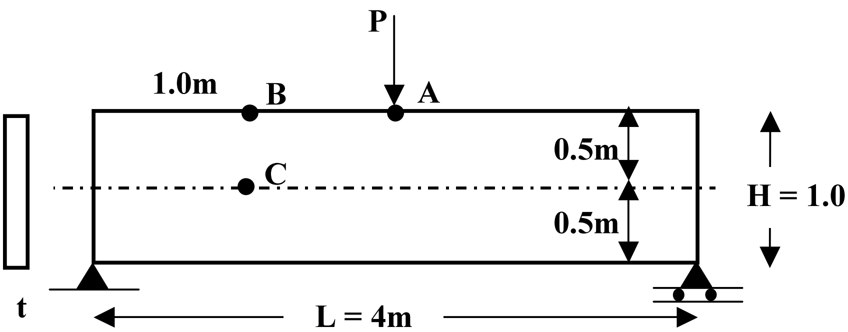

Figure 8: Dimensions of the simply supported beam problem

4.2 Convergence Results of the Simply Supported Beam

Table 2 shows a summary of the used mesh sizes in the solution and results for vertical deflection, bending stress and shearing stress at the specified points within the simply supported beam. Figures 9-11 show graphical comparison between the results obtained by each of the SBTREIR, the constant strain element CST and the exact analytical solution.

Table 2: Results obtained by the SBTREIR and CST for the simply supported beam problem

Mesh Size	Vertical Deflection at A (mm)		Bending Stress at B (kPa)		Shearing Stress at C (kPa)	
	CST	SBTREIR	CST	SBTREIR	CST	SBTREIR
1 x 4	2.138	3.389	3.21	19.75	3.97	4.105
2 x 8	3.248	4.208	8.98	22.02	3.91	5.151
3 x 12	3.897	4.655	13.48	23.07	4.38	5.537
4 x 16	4.306	4.955	16.39	23.64	5.44	6.139
5 x 20	4.586	5.176	18.23	24.01	5.50	6.208
Exact	**6.720**		**25.20**		**6.3**	

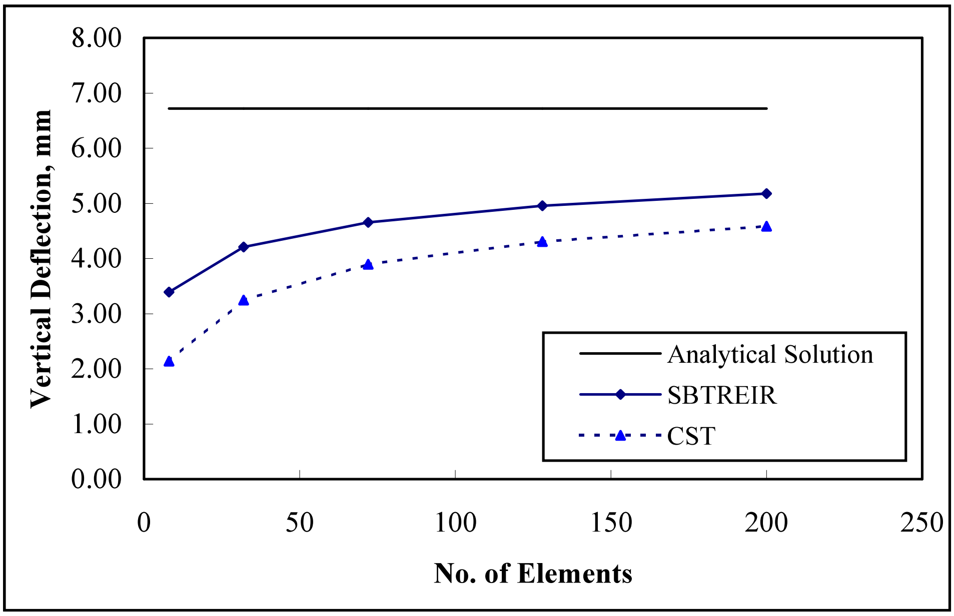

Figure 9: Vertical deflection at "A", (mm) in the simple beam

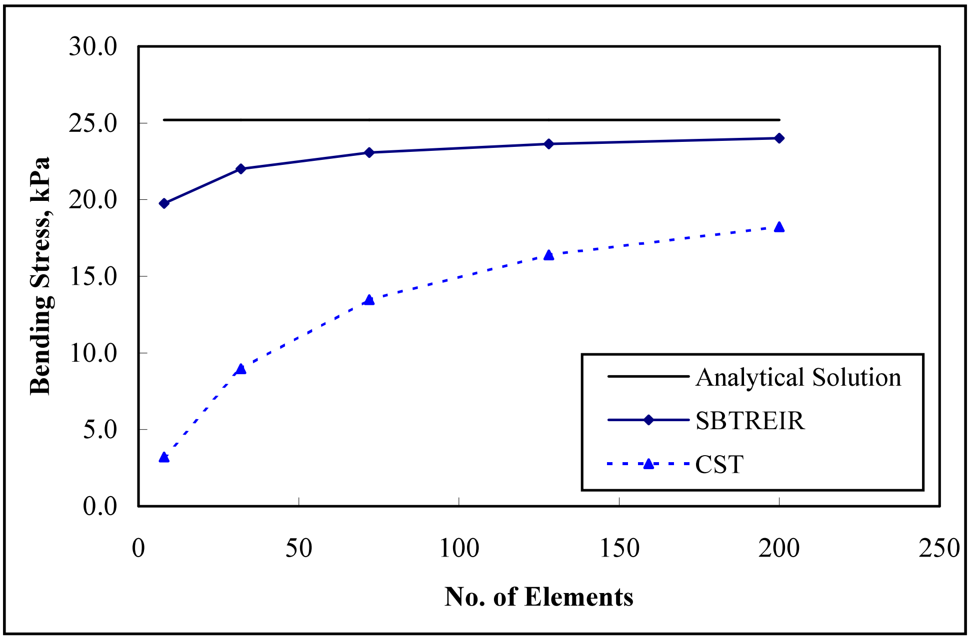

Figure 10: Bending Stress at "B", (kPa) in the simple beam

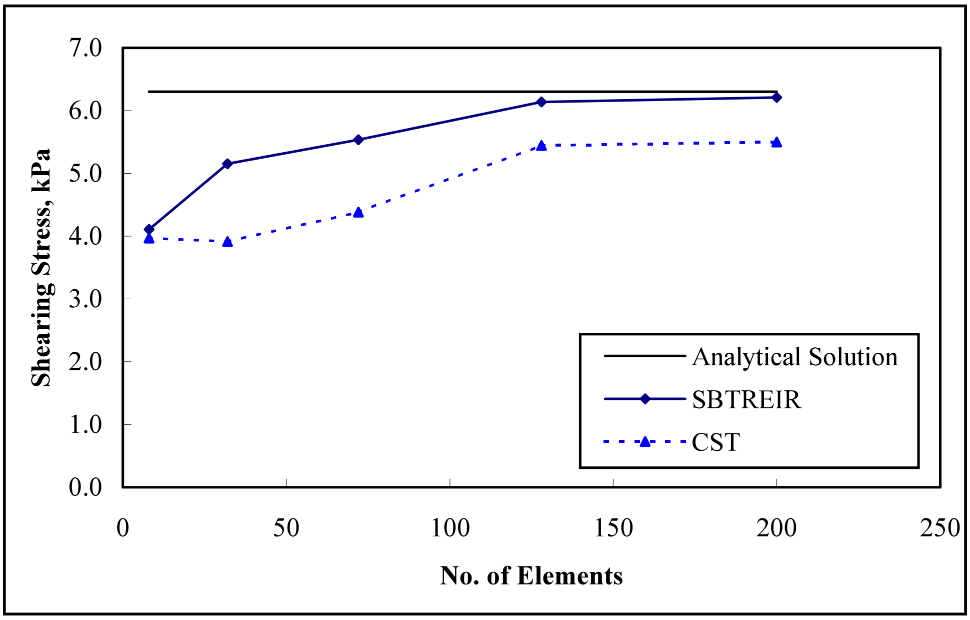

Figure 11: Shearing Stress At "C", (kPa) in the simple beam

5. CONCLUSIONS

The solutions obtained using the proposed (SBTREIR) element are satisfactory for results of deflection, bending stress, shearing stress and the in-plane rotation even when only a small number of elements are used in the finite element solution. The new element gives good results and convergence to the analytical solution. Also, it has fewer discontinuities in the corner stresses than the constant strain element

6. REFERENCES

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