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Some Properties of Centralizing in a Unital Complex Banach Algebra

As'ad Y. As'ad Department of Mathematics, Islamic University of Gaza
B.O Box : 108 , E-mail: aasad@mail.iugaza.edu

بعض خواص المركزية في جبر بناخ العقدي

ملخص: في هذا البحث تم إثبات أنه إذا كان J هو مثالي مغلق لجبر بناخ الوحدوت العقدى وأنَّ a هي عنصــر شبه ممركزى لــ J وأنَّ b عنصر في J فإنَّ ab و ba يعتمدان خطياً على بعضهما البعض إذا وفقط إذا كانت .ab=ba

كذلك تم الحصول على تعميمات لبعض نتائج دينسون"

Abstract In this paper we show that, for a closed ideal J of a unital complex Banach algebra A, if a is a quasi centralizing element of J and b belongs to J, then ab and ba are linearly dependent if and only if ab = ba. Also we get generalizations of some of Renneson's results.

1. Introduction

In this paper we study some properties of centralizing in a unital complex Banach algebra, where we generalize some results related to centrality in a unital complex Banach algebra that was obtained by Rennison in [7] and Sarsour and As'ad in [8].

Throughout this paper all linear spaces and algebras are assumed to be defined over $\mathbb{C}$, the field of complex numbers.

Let A be any complex normed algebra, then we denote the center of A by
$Z(A) = \{ a \in A : ax = xa \text{ for all } x \in A \}$, and the centralizer of a subset B of A by $C(B) = \{ a \in A : ax = xa \text{ for all } x \in B \}$. For $a \in A$, the spectrum in A of a will be denoted by $\sigma_A(a)$ and the resolvent set, its complement in $\mathbb{C}$, will be denoted by $\rho_A(a)$.

In [6] Rennison defined the set of all quasi central elements in a complex Banach algebra A by $Q(A) = \bigcup_{k\geq 1} Q(k,A)$, where
$Q(k, A) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \mathbb{C} \}$.
Also he defined the set of all σ-quasi central elements in A by $Q_\sigma(A) = \bigcup_{k\geq 1} Q_\sigma(k,A)$,
where $Q_\sigma(k, A) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \rho_A(a)\}$.

In [5] the set of all ρ-quasi central elements in A was defined by
$Q_\rho(A) = \bigcup_{k\geq 1} Q_\rho(k,A)$, where $Q_\rho(k,A) = \{ a \in A: \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \sigma_A(a) \}$.

Similarly in [1] the following three concepts were defined as folow: let B be a subset of a complex nomed algebra A, then

1) The quasi centralizer (quasi-commutant) of B is QC(B) = $\bigcup_{k\geq 1} QC(k, B)$, where QC(k, B) = { $a\in A$: $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x\in B$ and all $\lambda\in ¢$ }.

2) The σ-quasi centralizer (σ -quasi-commutant) of B is $QC_\sigma(B)$ = $\bigcup_{k\geq 1} QC_\sigma(k, B)$, where $QC_\sigma(k, B)$ = {$a\in A$: $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x\in B$ and all $\lambda\in \rho_A(a)$}.

3) The ρ-quasi centralizer (ρ-quasi commutant) of B is QC_ρ (B) = $\bigcup_{k\geq} QC_\rho(k, B)$, where $QC_\rho(k, B)$ ={$a\in A$: $\| x (\lambda - a) \| \leq k \| (\lambda - a) x \|$ for all $x\in B$ and all $\lambda\in \sigma_A(a)$}.

Remark:- *In this remark we state [1 , Theorem 2.1] that will be used frequently in this paper, the theorem states that:*

If A is a complex normed algebra and $D \subseteq B \subseteq A$. Then for $k \geq 1$,

(i) $C(B) \subseteq QC(k, B) = QC_\sigma (k, B) \cap QC_\rho (k, B)$.

(ii) $Q(k, A) = QC(k, A) \subseteq QC(k, B) \subseteq QC(k, D)$.

(iii) $Q_\sigma (k, A) = QC_\sigma (k, A) \subseteq QC_\sigma (k, B) \subseteq QC_\sigma (k, D)$.

(iv) $Q_\rho (k, A) = QC_\rho (k, A) \subseteq QC_\rho (k, B) \subseteq QC_\rho(k, D)$.

2. Some Properties of Centralizing

For a Banach algebra A we well denote A^{-1} for the set of all invertible elements in A. The following propositions generalise some results in [7] and [8] which have similar proofs.

Proposition 2.1:- *Let J be an ideal of a unital complex Banach algebra A and $k \geq 1$.*

(i) If $a \in QC_\sigma(k, J)$ and p is a complex polynomial of degree d then $p(a) \in QC_\sigma(k^d, J)$.

(ii) If $a \in QC_\sigma(k, J) \cap A^{-1}$, then $a^{-1} \in QC_\sigma(k \| a \| \| a^{-1} \|, J)$.

(iii) If $a \in QC_\sigma(k, J)$ and $b \in A^{-1}$, then $b^{-1}ab \in QC_\sigma(k \| b \|^2 \| b^{-1} \|^2, J)$.

Proof:-

(i) Let $x\in J$ and $\lambda \in \rho_A(p(a))$ be arbitrarty.

However, for any $\lambda \in \rho_A(p(a))$ there exist α_1, α_2, ……..,α_{d-1}, α_d and β in ¢ such that λ - p (a) = β (α - a) (α_2 - a) ....

(α_{d-1} - a) (α_d - a). [See 3, page 21]. Then α_1, α_2, ……….., α_{d-1}, α_d belong to $\rho_A(a)$.

Now, since a $\in QC_\sigma(k,J)$ and J is an ideal of A, then

$\|$ x (λ - p(a)) $\|$ = $\|$ x β (α_1 - a) (α_2 - a) ………. (α_{d-1} - a) (α_d - a) $\|$
$\leq$ k $\|$ (α_d - a) x β (α - a) (α_2 - a) ………… (α_{d-1} - a) $\|$ $\leq$ k^2
$\|$ (α_{d-1} - a) (α_d - a) x β (α_1 - a) (α_2 - a) ………… (α_{d-2} - a) $\|$
and so on after d-steps we get that
$\|$ x (λ - p(a))$\|$ $\leq$ k^d $\|$ (α -a) (α_2-a) ………… (α_{d-1} - a) (α_d - a) x β $\|$
= k^d $\|$ (λ - p(a)) x $\|$.

Therefore, p(a) $\in QC_\sigma(k^d, J)$ □

(ii) Let a $\in QC_\sigma(k,J) \cap A^{-1}$, then $\|$ x (λ - a)$\|$ $\leq$ k $\|$ (λ - a) x$\|$ for all x∈J and all λ∈ $\rho_A(a)$. Since a^{-1} ∈A then 0∈ $\rho_A(a)$ and 0∈ $\rho_A(a^{-1})$.

Note that, for λ ≠ 0,
λ∈ $\rho_A(a^{-1})$ if and only if $(\lambda - a^{-1})^{-1}$ exists if and only if $(\lambda^{-1}a)^{-1}$ $(\lambda - a^{-1})^{-1}$ exists if and only if (λ^{-1} - a) ^{1} exists if and only if $\lambda^{-1} \in \rho_A(a)$.

Now for any λ∈ $\rho_A(a^{-1})$, λ ≠ 0 and all x∈J we have
$\|$ x $(\lambda - a^{-1})$ $\|$ = $\|$ x (λ^{-1} - a) $(\lambda^{-1}a)^{-1}$ $\|$ $\leq$ $\|$ x (λ^{1} - a) $\|$ $|$ λ $|$ $\|a^{-1}\|$
$\leq$ k $\|$ (λ^{1} – a) x $\|$ $|$λ$|$ $\|a^{-1}\|$ = k $\|$ λ^{-1}a (λ - a^{-1}) x $\|$ $|$ λ $|$ $\|a^{-1}\|$
$\leq$ k $\|$ (λ - a^{-1}) x $\|$ $\|$ a $\|$ $\|a^{-1}\|$.
However, $\|$ x a^{-1} $\|$ = $\|$ aa^{-1}x $a^{-1}$$\|$ $\leq$ $\|$ a^{-1} x $\|$ $\|$ a $\|$ $\|a^{-1}\|$
$\leq$ k $\|$ a^{-1} x $\|$ $\|$ a $\|$ $\|a^{-1}\|$.
Therefore, $\|$ x (λ - a^{-1}) $\|$ $\leq$ k $\|$ a $\|$ $\|a^{-1}\|$ $\|$ (λ - a^{-1}) x $\|$ for all x∈J and all λ∈ $\rho_A(a^{-1})$. This means that $a^{-1} \in$ $QC_\sigma(k\|a\|\|a^{-1}\|, J)$ □

(iii) Let a ∈ $QC_\sigma(k,J)$ and b∈ A^{-1} then,
$\rho_A(b^{-1}ab)$= { λ∈¢ : (λ - b^{-1}ab) $^{-1}$ exists } = {λ∈¢ : b^{-1}(λ - a) ^{1} b exists } = $\rho_A(a)$.

Now for any λ∈ $\rho_A(b^{-1}ab)$= $\rho_A(a)$ and all x∈J we have:
$\|$ x (λ - b^{-1}ab) $\|$ = $\|$ x b^{-1} (λ - a) b$\|$ $\leq$ $\|$ b^{-1}bx b^{-1}(λ - a)$\|$ $\|$ b$\|$
$\leq$ $\|$ $b^{-1}$$\|$ $\|$ bx b^{-1}(λ - a)$\|$ $\|$ b$\|$
$\leq$ k $\|$ (λ - a) bx$b^{-1}$$\|$ $\|$ b $\|$ $\|b^{-1}\|$ since a $\in QC_\sigma(k,J)$ and bx b^{-1} ∈ J

$= \; k \| b(\lambda - b^{-1}ab)\, x b^{-1} \| \, \| b \| \, \| b^{-1} \| \; \le \; k \| b \|^2 \| b^{-1} \|^2 \, \| (\lambda - b^{-1}ab)\, x \|$

.

Therefore, $b^{-1}ab \in QC_\sigma(k \| b \|^2 \| b^{-1} \|^2, J)$ □

Similarly one can prove the following two propositions for ρ-quasi and quasi centralizer. However, Proposition 2.3 (ii) and (iii) can be proved in another way by using Proposition 2.1 and 2.2 and [1 , Theorem 2.1(i)].

Proposition 2.2:- *Let J be an ideal of a unital complex Banach algebra A and $k \ge 1$.*

(i) *If* $a \in QC_\rho(k, J) \cap A^{-1}$ *then* $a^{-1} \in QC_\rho(k \| a \| \| a^{-1} \|, J)$.

(ii) *If* $a \in QC_\rho(k, J)$ *and* $b \in A^{-1}$ *then* $b^{-1}ab \in QC_\rho(k \| b \|^2 \| b^{-1} \|^2, J)$.

Proposition 2.3:- *Let J be an ideal of a unital complex Banach algebra A and $k \ge 1$.*

(i) *If* $a \in QC(k, J)$ *and p is a complex polynomial of degree d then* $p(a) \in QC(k^d, J)$.

(ii) *If* $a \in QC(k, J) \cap A^{-1}$ *then* $a^{-1} \in QC(k \| a \| \| a^{-1} \|, J)$.

(iii) *If* $a \in QC(k, J)$ *and* $b \in A^{-1}$ *then* $b^{-1}ab \in QC(k \| b \|^2 \| b^{-1} \|^2, J)$.

Now we can remark here that Renneson's and As'ad's results in [7] and [8] respectively become corollaries of our propositions.

Corollary1 :- *Let A be a unital complex Banach algebra, $k \ge 1$ and p is a complex polynomial of degree d.*

(i) *If* $a \in Q_\sigma(k, A)$, *then* $p(a) \in Q_\sigma(k^d, A)$.

(ii) *If* $a \in Q_\sigma(k, A) \cap A^{-1}$, *then* $a^{-1} \in Q_\sigma(k \| a \| \| a^{-1} \|, A)$.

(iii) *If* $a \in Q_\sigma(k, A)$ *and* $b \in A^{-1}$, *then* $b^{-1}ab \in Q_\sigma(k \| b \|^2 \| b^{-1} \|^2, A)$

(iv) *If* $a \in Q_\rho(k, A) \cap A^{-1}$, *then* $a^{-1} \in Q_\rho(k \| a \| \| a^{-1} \|, A)$.

(v) *If* $a \in Q_\rho(k, A)$ *and* $b \in A^{-1}$, *then* $b^{-1}ab \in Q_\rho(k \| b \|^2 \| b^{-1} \|^2, A)$.

(vi) *If* $a \in Q(k, A)$ *then* $p(a) \in Q(k^d, A)$.

(vii) *If* $a \in Q(k, A) \cap A^{-1}$, *then* $a^{-1} \in Q(k \| a \| \| a^{-1} \|, A)$.

(viii) If $a \in Q(k, A)$ *and* $b \in A^{-1}$, *then* $b^{-1}ab \in Q(k \| b \|^2 \| b^{-1} \|^2, A)$.

Proof:

Let J = A in the above propositions, then use [1, Theorem 2.1] □

3. Linerly Dependence in Quasi Centralizing Sets

If A is a complex normed algebra and $a \in A$, then the inner derivation corresponding to a is denoted by D_a, which is a bounded linear operator on A defined by $D_a x = ax - xa$. The bounded linear operators L_a and R_a on A are defined by $L_a x = ax$ and $R_a x = xa$. D_a is called topologically nilpotent if $\lim_{\to\infty} \| Da \|^{\frac{1}{n}} = 0$.

The following theorem generalises proposition 3.4 in [7] (which has a similar proof), so the proposition becomes a corollary of our theorem. In the proof of the following theorem we shall use [2, Theorem2.3] which states that " Let A be a complex Banach algebra with unity, J a closed ideal of A, and $a \in QC(J)$. Then D_a is topologically nilpotent. Where D_a is restricted to a domain J.

Theorem 3.1:- *Let J be a closed ideal of a unital complex Banach algebra A, and let* $a \in QC(J)$ *and* $b \in J$. *Then ab and ba are linearly dependent if and only if ab = ba.*

Proof:

Suppose that ab and ba are linearly dependent. Then there exist $\alpha, \beta \in ¢$ such that $\alpha \neq 0$ or $\beta \neq 0$ and $\alpha ab + \beta ba = 0$. Then two cases.

Case1: If $\alpha + \beta = 0$, it is clear that ab = ba.

Case 2: If $\alpha + \beta \neq 0$, then

$0 = \alpha ab + \beta ba = (\alpha L_a + \beta R_a)b = [(\alpha + \beta)L_a - \beta(L_a - R_a)] b = [(\alpha + \beta)L_a - \beta D_a] b$.

Hence $ab = L_a b = \gamma D_a b$, where $\gamma = \frac{\beta}{\alpha + \beta} \in ¢$. However, L_a and D_a commute, then by induction we have, $L_a^n b = \gamma^n D_a^n b$ for all natural numbers n, and so $a^n b = \gamma^n D^n b$. Since $a \in QC(J)$ then by [2, Theorem2.3] D_a is topologically nilpotent; that is $\lim_{\to\infty} \| D_a \|^{\frac{1}{n}} = 0$. However, $\| a\ b \|^{\frac{1}{}} = \| \gamma\ D_a b \|^{\frac{1}{}} \leq | \gamma | \ \| D_a \|^{\frac{1}{}} \ \| b \|^{\frac{1}{n}}$, then for any $\lambda \in ¢\backslash\{0\}$,

$\lim_{\to\infty} \| \lambda^{-\,-1} a \; b \|^{\frac{1}{n}} = 0,$. Hence the series $f(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} a^n b$ is absolutely convergent, then by the completeness of A the series converges. But J is a closed ideal of A and $a^n b \in J$ for all natural numbers n, then f (λ) ∈J and (λ-a) f (λ) = b.

However, a∈ QC(J), then there exists k ≥ 1 such that ||f (λ) (λ - a)|| ≤ k ||(λ - a) f(λ)|| = k || b ||. Therefore, f(λ) (λ - a) = b + λ^{-1}(ab – ba) + λ^{-2}(a^2 b – bab) + …… is a bounded J-valued function on ¢\{0} which can easily be seen analytic there. But {0} is a countable compact subset of ¢, then by [9] it has zero analytic capacity and so by [4, Theorem 1.10VIII], f extends to be analytic on ¢. Hence, by Liouville's Theorem f(λ) (λ - a) is constan, which implies that ab = ba.

Therefore, ab = ba whenever ab and ba are linearly dependent.

The converse is straitforword □

Corollary:- *[7, Proposition 3.4]*

Suppose that A is a unital complex Banach algebra, a ∈ Q(A) and b ∈ A. Then ab and ba are linearly dependent if and only if ab = ba.

Proof:

Let J = A in the above Theorem, then use [1, Theorem 2.1] □

At the end we shall write the following paragraph which consists of the proofs of Lemma 2.2 and Theorem 2.3 from [2] which are not a part of this paper, but they are written here because [2] is not published until the moment of writing this paper.

2.2 Lemma in [2] *Let A be a complex Banach algebra with unity, J a closed ideal of A, and k ≥ 1 such that a ∈ QC(k, J). If M is a closed commutative subalgebra of BL(J) containing the identity operator I, L_a and R_a, then $||D_aT|| \le (k + 1)|| (\lambda – L_a)T ||$ for all T ∈ M and all λ ∈ ¢.* Where L_a *and* R_a are restricted here to a domain J.

Proof

Since a ∈ QC(k, J), for all x in J and λ∈ ¢ we have, || x (λ - a) || ≤ k || (λ - a) x ||. However, || x (λ - a) || = || (λ – R_a) x || and || (λ – a) x || = || (λ – L_a) x ||, then || (λ - R_a) x || ≤ k || (λ – L_a) x ||. So that || D_ax || = || (λ - R_a) x - (λ - L_a) x || ≤ (k + 1) || (λ - L_a) x ||. Finally, since Tx ∈ J for all x ∈ J, then the result follows by replacing x by Tx in the above inequality and taking the supremum over all x in J with || x || = 1 □

In the proof of the following theorem we need [6, Theorem 4.2], which states that " Suppose that M is a commutative complex Banach algebra with unity, that u and v are elements of M, and that for some c ≥ 0,

$\| ux \| \leq c \| (\lambda - v) x \|$ for all $x \in M$ and all $\lambda \in ¢$. Then $u \in$ Rad(M), the radical of M.

2.3 Theorem in [2] *Let A be a complex Banach algebra with unity, J a closed ideal of A, and $a \in QC(J)$. Then D_a is topologically nilpotent. Where D_a is restricted to a domain J.*

Proof

Since J is closed in the Banach algebra A, then J is complete and so BL(J) is a complex Banach algebra. Since $a \in QC(J)$, then there exists $k \geq 1$ such that $a \in QC(k, J)$. Now, let M be a closed commutative subalgebra of BL(J) containing the identity operator I, L_a and R_a, then by Lemma 2.2, $\| D_a T \| \leq (k + 1) \| (\lambda - L_a) T \|$ for all $T \in M$ and all $\lambda \in ¢$. But M is closed in BL(J) and BL(J) is complete, then M is complete, then by [6, Theorem 4.2] we have $D_a \in$ Rad(M) and so $\lim \| D_a^n \|^{\frac{1}{n}} = 0$.

That means, D_a is topologically nilpotent □

References

[1] A . As'ad Y., Centralizing In a Complex Banach Algebra, *The Islamic University Journal,* v.10, n.2, (2002), 1-12.

[2] A . As'ad Y., Quasi Centralizers and Inner Derivation in a closed Ideal of a Complex Banach Algebra, preprint.

[3] F. F. Bonsal and J. Duncan, *Complete normed algebras*, (Springer, Berlin, 1973).

[4] T. Gamelin, *Uniform Algebra,* Prentice-Hall, Eng-Lewood Cliffs, 1969.

[5] D. Hussein and A.Y. As'ad, The ρ-quasi center of a Banach algebra, *Arabian. Journal for Science and Engineering*, 16, (1991), no. 4, 471-474.

[6] J. F. Rennison, Conditions related to centrality in a Banach algebra. *J. London Math. Soc.*, 26, (1982), 155-168.

[7] J.F. Rennison, Closure Properties of the Quasi Center of a Banach algebra, Math. *Proc. Camb. Phil. Soc.*, 108 (1990), 355-364.

[8] J. Sarsour and A .Y. As'ad, Some properties of Centrality in the complex Banach algebra, *the Journal of Islamic universit Journal, V.9, no.2, (2001).*, 1-9 .

[9] M. Tsuji, *Potential Theory in Modern Function Theory,* Tokyo, Maruzen, 1959.