INCOMPLETE FACTORIALS AND SOME COMBINATORICS IDENTITIES

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ABSTRACT. We prove in this paper some identities related to incomplete factorials. We derive and prove some recurrence relations for the incomplete factorials. Then we prove that the sum of incomplete factorials turns out to be representable in terms of incomplete factorials. Then we use these identities to prove some known, as well as new combinatorics identities.

1. INTRODUCTION

Identities involving factorials, binomial coefficients, and incomplete factorials can be proved by either algebraic manipulations or combinatorial proofs. The algebraic proofs might be time consuming and involve quite tedious calculations. We here introduce the notion of incomplete factorials and develop some of its recurrence relations. We prove a theorem that relates these incomplete factorials with their finite sums. This theorem with other theorems that we also prove in this paper allow us to use a new approach for dealing with known combinatorial identities and binomial coefficients. They also work as tools to generalize and sometimes derive new identities that are easy to visualize and prove using this new approach.

While some of results we obtain in this paper are well-known in the literature, we use different approaches and techniques to prove them. The other results, especially the recurrence relations and some identities, are new.

Besides this introduction, this paper contains two sections and the references.

In Section 2, we develop the main properties and relations of the incomplete factorials. We also prove the main theorem that relates the incomplete factorials to their sums.

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In Section 3, we derive and prove some theorems involving combinatorial, factorial, and binomial coefficient identities. We also prove in this section known identities as applications to the properties and identities developed in Section 2.

We will denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of all nonnegative integers.

2. Sum of Incomplete Factorials

The partial factorials or incomplete factorials are defined as follows:

Definition 2.1. *For each $n \in \mathbb{N}$ we let $f_n(m)$ be defined by*

$$f_n(m) \stackrel{\text{def}}{=} \prod_{i=0}^{n} (m - i), \quad \forall m \in \mathbb{N}. \tag{2.1}$$

It is clear that $f_n(m) = 0$ for all $m \leq n$. One can express $f_n(m)$ in terms of incomplete factorials as

$$f_n(m) = \langle m \rangle_{n+1}, m \in \mathbb{N}, \tag{2.2}$$

where $\langle m \rangle_k \stackrel{\text{def}}{=} m(m-1)\cdots(m-k+1)$. It follows from (2.2) that $f_{m-1}(m) = \langle m \rangle_m = m!$, where we define $f_{-1}(z) \stackrel{\text{def}}{=} 1$ for any $z \in \mathbb{N}$.

The function $f_n(m)$ is closely related to the well-known factorial polynomial, $P(m, n)$, as mentioned in [1]. In fact, they are related via the equation $f_n(m) = P(m, n+1)$. We will stick to our notations here; i.e., $f_n(m)$, for the sake of simplicity and clarification.

In the following lemma, we prove some recurrence identities for the function $f_n(m)$.

Lemma 2.1. *The function $f_n(m)$ for m and n in $\mathbb{N}$ with $m \geq n+1$ and $n \geq 0$ obeys the following recurrence identities:*

(1) $f_n(m-1) = \frac{m-n-1}{m} f_n(m)$,
(2) $f_n(m) = (m-n) f_{n-1}(m)$, *and*
(3) $f_n(m) = m f_{n-1}(m-1)$, *and*
(4) $f_n(m) = f_{k-1}(m) f_{n-k}(m-k)$, *for any $k \in \mathbb{N}$ with $1 \leq k \leq \min\{m, n\}$.*

Proof. The first and second items of Lemma 2.1 follow by direct calculations:

$$\begin{aligned} f_n(m-1) &= \langle m-1 \rangle_{n+1} = (m-1)(m-2)\cdots(m-n)(m-n-1) \\ &= \frac{(m-n-1)}{m} m(m-1)\cdots(m-n) \\ &= \frac{(m-n-1)}{m} f_n(m) \end{aligned}$$

Combining (1) and (2), we get (3).

To prove (4), note that

$$
\begin{aligned}
f_n(m) &= \{m(m-1)\cdots(m-k+1)\}\{(m-k)(m-k-1)\cdots(m-n)\} \\
&= f_{k-1}(m)f_{n-k}(m-k).
\end{aligned}
$$

∎

Remark **2.1.** By using Item (2) of Lemma 2.1, Item (4) of Lemma 2.1 can be rewritten as

$$f_n(m) = (m-n)f_{k-1}(m)f_{n-k-1}(m-k).$$

We now introduce another function that involves the sum of incomplete factorials.

Definition 2.2. *For each $n \in \mathbb{N}$ we let $g_n(m)$ be defined by*

$$g_n(m) \stackrel{\text{def}}{=} \sum_{i=1}^{m} \prod_{j=1}^{n} (i-j), \quad \forall m \in \mathbb{N}. \tag{2.3}$$

It follows from Definition 2.2 that $g_n(m) = 0$ for all $m \leqslant n$. An alternative expression for $g_n(m)$ is

$$g_n(m) = \begin{cases} \sum_{i=1}^{m} \langle i-1 \rangle_n, & \text{if } m > n, \\ 0, & \text{if } m \leqslant n. \end{cases} \tag{2.4}$$

The next Lemma gives us a recurrence identity that relates the two functions $f_n(m)$ and $g_n(m)$ for each $n \in \mathbb{N}$.

Lemma 2.2. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be such that $m \geqslant n+1$. Then*

$$g_n(m) = g_n(m-1) + \frac{1}{m} f_n(m), \quad m \geqslant n+1. \tag{2.5}$$

Proof. By using (2.4) and Item (3) of Lemma 2.1, we may write for $m \geqslant n+1$

$$
\begin{aligned}
g_n(m) &= \sum_{i=1}^{m} \langle i-1 \rangle_n \\
&= \sum_{i=1}^{m-1} \langle i-1 \rangle_n + \langle m-1 \rangle_n \\
&= g_n(m-1) + f_{n-1}(m-1) \\
&= g_n(m-1) + \frac{1}{m} f_n(m).
\end{aligned}
$$

∎

The next Theorem gives us an explicit identity that relates the two functions $f_n(m)$ and $g_n(m)$ for each $n \in \mathbb{N}$.

Theorem 2.1. *Let $f_n(m)$ and $g_n(m)$, for n and m in $\mathbb{N}$, be defined by (2.1) and (2.3), respectively. Then $(n+1)g_n(m) = f_n(m)$.*

Proof. Let $n \in \mathbb{N}$ be fixed. Since $f_n(m) = g_n(m) = 0$ for all $m \le n$, we start our induction at $m = n+1$. It follows from (2.4) that $g_n(n+1) = \langle n+1-1\rangle_n = \langle n\rangle_n$. Similarly, $f_n(n+1) = \langle n+1\rangle_{n+1} = (n+1)\langle n\rangle_n = (n+1)g_n(n+1)$.

Assume now that $f_n(k-1) = (n+1)g_n(k-1)$ for some $k \in \mathbb{N}$ with $k \ge n+2$. We need to show that $f_n(k) = (n+1)g_n(k)$. By Item (1) of Lemma 2.1, we see that $f_n(k-1) = \frac{k-n-1}{k} f_n(k)$. Now, by Lemma 2.2, we see that

$$
\begin{aligned}
(n+1)g_n(k) &= (n+1)g_n(k-1) + \frac{n+1}{k} f_n(k) \\
&= f_n(k-1) + \frac{n+1}{k} f_n(k), \quad \text{(by assumption of induction)} \\
&= \frac{k-n-1}{k} f_n(k) + \frac{n+1}{k} f_n(k) \\
&= f_n(k).
\end{aligned}
$$

■

Lemma 2.3. *Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be such that $m \ge n+1$. Then*

$$f_n(m) = f_n(m-1) + (n+1)f_{n-1}(m-1), \quad m \ge n+1. \tag{2.6}$$

Proof. The proof follows by multiplying both sides of (2.5) by $n+1$. Then by using Theorem 2.1 and noting that $\frac{1}{m} f_n(m) = f_{n-1}(m-1)$, the result follows. ■

In the next lemma, we prove some recurrence identities for $g_n(m)$.

Lemma 2.4. *The function $g_n(m)$ for m and n in $\mathbb{N}$ with $m \ge n+1$ obeys the following recurrence relations:*

(1) $g_n(m-1) = \frac{m-n-1}{m} g_n(m)$,
(2) $g_n(m) = (m-n)g_{n-1}(m)$, *and*
(3) $g_n(m) = m g_{n-1}(m-1)$, *and* $g_n(m) = g_{k-1}(m) g_{n-k}(m-k)$, *for any $k \in \mathbb{N}$ with $k \le m$ and $k \le n+1$, where we define $g_{-1}(z) \overset{\text{def}}{=} 1$ for any $z \in \mathbb{N}$.*

Proof. The proof follows directly from Lemma 2.1 and Theorem 2.1. ■

Remark 2.2. One may rewrite (2.5) as

$$g_n(m) = (m - n + 1)g_{n-1}(m - 1) + f_{n-1}(m - 1), m \geqslant n + 1.$$

To see this, note that item (2) of (2.4) may be expressed as $g_n(m-1) = (m - n + 1)g_{n-1}(m - 1)$ for $m \geqslant n + 1$.

Corollary 2.1.

$$\begin{aligned}&(1)(2)\cdots(m)+\\&(2)(3)\cdots(m+1)+\\&\cdots+\\&(n)(n+1)\cdots(m+n-1)\\&= m!\binom{n+m}{m+1}.\end{aligned}$$

Proof. The left hand side of the above equation has the following form $\sum_{j=1}^{n}\prod_{i=1}^{m}(j - i + m)$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n}\prod_{i=1}^{m}(j - i + m) &= \sum_{k=m+1}^{n+m}\prod_{i=1}^{m}(k - i)\\&= g_m(n + m) = m!\binom{n+m}{m+1}.\end{aligned}$$

■

Example 2.1. Let $n \in \mathbb{N}$. Then

$$1 \times 2 + 2 \times 3 + \cdots + n \times (n + 1) = 2\binom{n+2}{3}.$$

Example 2.2. Let $n \in \mathbb{N}$. Then

$$1 \times 2 \times 3 + 2 \times 3 \times 4 + \cdots + n \times (n + 1) \times (n + 2) = 6\binom{n+3}{4}.$$

3. Combinatorial Identities

In this section, we show that both functions f_n and g_n when summed reproduce themselves. We also prove some theorems that can be used to prove some combinatorial identities like the Pascal's identity, the hexagon property, the hypergeometric identity, etc. We also generalize some of the known identities and derive new ones.

Although $f_n(m)$ and $g_n(m)$ seem to be unrelated, it turns out that they related in a very interesting way. The equation that relates them has many applications in probability like moments of nonnegative integer-valued random variables and the birthday problem.

Theorem 3.1. *Let* $f_n(m)$ *and* $g_n(m)$, *for* n *and* m *in* $\mathbb{N}$, *be defined by* (2.1) *and* (2.3), *respectively. Then*

(1) $g_n(m) = \sum_{k=1}^{m} k^{-1} f_n(k) = \sum_{k=1}^{m} f_{n-1}(k-1)$,
(2) $f_n(m) = (n+1)\sum_{k=1}^{m} k^{-1} f_n(k) = (n+1)\sum_{k=1}^{m} f_{n-1}(k-1)$,
(3) $g_n(m) = (n+1)\sum_{k=1}^{m} k^{-1} g_n(k) = (n+1)\sum_{k=1}^{m} g_{n-1}(k-1)$.

Proof. To prove the first item, note that $f_n(m) = \langle m\rangle_{n+1}$. Then we may write $g_n(m) = \sum_{k=1}^{m}\langle k-1\rangle_n = \sum_{k=1}^{m} f_{n-1}(k-1) = \sum_{k=1}^{m} k^{-1} f_n(k)$, by Item (3) of (2.1).

If we use Item (1) and Theorem 2.1, we get $f_n(m) = (n+1)g_n(m) = (n+1)\sum_{k=1}^{m} k^{-1} f_n(k) = f_{n-1}(k-1)$. Next, by using Item (2) and Theorem 2.1, we get Item (3). ■

Next, we will use Theorem 3.1 to prove some factorials identities.

Theorem 3.2. *Let* n *and* m *be in* $\mathbb{N}$. *Then*

$$\sum_{i=0}^{m}\frac{(n+i)!}{i!} = \frac{n!}{m!}\prod_{i=1}^{m}(n+i+1). \tag{3.1}$$

Proof. First, note that $g_n(m) = \sum_{k=1}^{m}\langle k-1\rangle_n$. But the term $\langle z\rangle_n = \frac{z!}{(z-n)!}$ for each $z \in \mathbb{N}$. Therefore,

$$g_n(n+m+1) = \sum_{k=n+1}^{n+m+1}\langle k-1\rangle_n = \sum_{k=n+1}^{n+m+1}\frac{(k-1)!}{(k-n-1)!}, \tag{3.2}$$

where we started the sum at $k = n+1$, since the preceding terms are zeros. Now let $i = k-n-1$ in the last term of (3.2) to get

$$g_n(n+m+1) = \sum_{i=0}^{m}\frac{(n+i)!}{i!}. \tag{3.3}$$

Using Item (1) of (2.1) (with m replaced by $n+m+1$) iteratively m times, we get

$$f_n(n+m+1) = \frac{n+m+1}{m}\cdot\frac{n+m}{m-1}\cdots\frac{n+2}{1}\cdot f_n(n+1). \tag{3.4}$$

But note that $f_n(n+1) = \langle n+1\rangle_{n+1} = (n+1)!$. Then (3.4) becomes

$$f_n(n+m+1) = \frac{(n+1)!}{m!}\prod_{i=1}^{m}(n+i+1). \tag{3.5}$$

Combining (3.3) and (3.5) and using Theorem 2.1, we get (3.1). ■

We use Theorem 3.2 to derive some identities that involve finite sums of binomial coefficients.

Corollary 3.1. *Let n and m be in $\mathbb{N}$. Then*

$$\sum_{i=0}^{m} \binom{n+i}{i} = \binom{n+m+1}{m}. \tag{3.6}$$

In particular, we have

$$\sum_{i=0}^{n} \binom{n+i}{i} = \binom{2n+1}{n}. \tag{3.7}$$

Proof. It follows from (3.1) that

$$\sum_{i=0}^{m} \binom{n+i}{i} = \sum_{i=0}^{m} \frac{(n+i)!}{i!\,n!} = \frac{1}{m!} \cdot \frac{(n+m+1)!}{(n+1)!}$$
$$= \binom{n+m+1}{m}.$$

This gives us (3.6). To get (3.7), we simply let $m = n$ in (3.6). ■

Theorem 3.3. *Let m, n, and r be in $\mathbb{N}$ such that $r \leqslant \min\{m, n\}$. Then*

$$f_{r-1}(m+n) = \sum_{k=0}^{r} \binom{r}{k} f_{k-1}(m) f_{r-k-1}(n). \tag{3.8}$$

Proof. First, we note that $\frac{d^i}{dt^i} t^j = f_{i-1}(j) t^{j-i}$. In other words, evaluating the ith derivative of both sides at $t = 1$ yields

$$\frac{d^i}{dt^i} t^j |_{t=1} = f_{i-1}(j). \tag{3.9}$$

Using (3.9) when taking the rth derivative of both sides of the equation $t^{m+n} = t^m \cdot t^n$, using the rule of differentiating the product of two functions of t (t^m and t^n), and then letting $t = 1$, we get

$$f_{r-1}(m+n) = \sum_{k=0}^{r} \binom{r}{k} f_{k-1}(m) f_{r-k-1}(n),$$

as required. ■

The following corollary gives an easy proof of the fact that the sum of the hypergeometric probability mass function over all of its possible values is equal to 1.

Corollary 3.2. *Let m, n, and r be in $\mathbb{N}$ such that $r \leqslant \min\{m, n\}$. Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}. \tag{3.10}$$

Proof. The proof follows by using (3.8) and the fact that

$$f_n(m) = \begin{cases} (n+1)!\binom{m}{n+1}, & \text{if } m \geq n, \\ 0, & \text{if } m < n. \end{cases} \tag{3.11}$$

■

By dividing both sides of (3.10) by $\binom{m+n}{r}$, we get

$$1 = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \Big/ \binom{m+n}{r}.$$

Corollary 3.3. *If k and n are in $\mathbb{N}$ such that $n \geq k$, then*

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. The proof follows from (3.10) by letting $m = n$ and $r = n$. ■

Theorem 3.4. *Let m and n be in $\mathbb{N}$. Then*

$$f_k(n) + a_{k-1}f_{k-1}(n) + \cdots + a_1f_1(n) + a_0f_0(n) = n^{k+1}. \tag{3.12}$$

Here the coefficients a_is can be obtained for $0 \leq i \leq k$ from the equation

$$a_i = \sum_{j=i+1}^{k} (-1)^{j-i+1}a_j\beta_{j,j-i}, \quad \textit{where} \tag{3.13}$$

$$\beta_{j,l} \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq j} i_1 i_2 \cdots i_l. \tag{3.14}$$

Proof. Note that the $f_k(n) = n(n-1)\cdots(n-k)$ is a polynomial of n of degree $k+1$. The coefficient of n^i in the expansion of $f_k(n)$ is the sum of the product of the numbers $\{1, 2, \ldots, k\}$ taken $k-i$ at a time. This means that we can write the expansion of $f_k(n)$ as

$$f_k(n) = \sum_{i=0}^{k-1} (-1)^i\beta_{k-1,i}n^{k-i}, where$$

$$\beta_{j,l} = \sum_{1 \leq i_1 < i_2 < \cdots < i_l \leq j} i_1 i_2 \cdots i_l \quad \text{and}$$

$\beta_{j,0} \stackrel{\text{def}}{=} 1$, for $0 \leq j \in \mathbb{N}$.

We need to find constants $a_0, a_1, \ldots, a_{m-1}$ such that

$$f_k(n) + a_{k-1}f_{k-1}(n) + \cdots + a_0f_0(n) = n^{k+1}.$$

The coefficients for each ith power of n must vanish for $1 \leqslant i \leqslant k$. Therefore, we should have

$$
\begin{aligned}
a_{k-1} &= \beta_{k,1}\\
a_{k-2} &= -\beta_{k,2} + a_{k-1}\beta_{k-1,1}\\
a_{k-3} &= \beta_{k,3} - a_{k-1}\beta_{k-1,2} + a_{k-2}\beta_{k-2,1}\\
&\vdots
\end{aligned}
$$

In general, we have

$$a_i = \sum_{j=i+1}^{k} (-1)^{j-i+1} a_j \beta_{j,j-i},$$

as required. Incidently, one can check that

$$a_0 = a_1\beta_{1,1} - a_2\beta_{2,2} + \cdots + \beta_{k,k} = 1,$$

since $\beta_{i,i} = i!$. ■

Example 3.1. We have the following equations

$$
\begin{aligned}
&f_0(n) = n\\
&f_1(n) + f_0(n) = n^2\\
&f_2(n) + 3f_1(n) + f_0(n) = n^3\\
&f_3(n) + 6f_2(n) + 7f_1(n) + f_0(n) = n^4\\
&f_4(n) + 10f_3(n) + 25f_2(n) + 15f_1(n) + f_0(n) = n^5\\
&f_5(n) + 15f_4(n) + 65f_3(n) + 90f_2(n) + 31f_1(n) + f_0(n) = n^6.
\end{aligned}
$$

Corollary 3.4. *Let m and n be in $\mathbb{N}$. Then*

$$\sum_{k=1}^{m+1} k! \binom{n}{k} a_{k-1} = n^{m+1}, \tag{3.15}$$

where the a_is are defined as in Theorem 3.4.

Proof. The proof follows by using (3.11) and Theorem 3.4. ■

Example 3.2. If we let m take the values 1, 2, and 3 in Theorem 3.4, we get, respectively

$$\binom{n}{1}+2\binom{n}{2}=n^2$$
$$\binom{n}{1}+6\binom{n}{2}+6\binom{n}{3}=n^3$$
$$\binom{n}{1}+12\binom{n}{2}+42\binom{n}{3}+24\binom{n}{4}=n^4.$$

Theorem 3.5. *Let m and n be in $\mathbb{N}$ such that $n \leqslant m$. Then*

$$2^n f_{n-1}(m)=\sum_{k=0}^{n}\binom{n}{k} f_{k-1}(m) f_{n-k-1}(m-k). \qquad (3.16)$$

Proof. It follows from Item (4) of Lemma 2.1 that

$$f_{n-1}(m)=f_{k-1}(m) f_{n-k-1}(m-k). \qquad (3.17)$$

Multiplying both sides of (3.17) by $\binom{n}{k}$ and take the sum as k ranges from 0 to n, we get

$$\sum_{k=0}^{n}\binom{n}{k} f_{n-1}(m)=\sum_{k=0}^{n}\binom{n}{k} f_{k-1}(m) f_{n-k-1}(m-k).$$

But the left hand side is equal to $f_{n-1}(m)\sum_{k=0}^{n}\binom{n}{k}=2^n f_{n-1}(m)$. Therefore, we have

$$2^n f_{n-1}(m)=\sum_{k=0}^{n}\binom{n}{k} f_{k-1}(m) f_{n-k-1}(m-k).$$

∎

Corollary 3.5. *Let n, m, and k be in $\mathbb{N}$ such that $n \geqslant k$ and $m \geqslant n$. Then*

$$\binom{n}{k}\binom{m}{n}=\binom{m}{k}\binom{m-k}{n-k}. \qquad (3.18)$$

Proof. The proof follows by using Theorem 3.5 and (3.11):

$$n!\binom{m}{n}=k!\binom{m}{k}(n-k)!\binom{m-k}{n-k} \quad \text{implies}$$

$$\binom{n}{k}\binom{m}{n}=\binom{m}{k}\binom{m-k}{n-k}. \qquad (3.19)$$

∎

Remark 3.1. Corollary 3.5 has a very interesting meaning in probability. It says that the number of ways to choose n out of m objects and then k out of the chosen n objects is equal to the number of ways of choosing k objects out of m objects and then $n-k$ objects out of the remaining $m-k$ objects.

Corollary 3.6.

$$\sum_{k=0}^{n}\binom{m}{k}\binom{m-k}{n-k}=2^n\binom{m}{n}. \tag{3.20}$$

Proof. The proof follows from (3.16) and (3.11). ■

Corollary 3.7.

$$\sum_{n=k}^{m}\binom{n}{k}\binom{m}{n}=2^{m-k}\binom{m}{k}.$$

Proof. By (3.19), we see that

$$\begin{aligned}\sum_{n=k}^{m}\binom{n}{k}\binom{m}{n}&=\sum_{n=k}^{m}\binom{m}{k}\binom{m-k}{n-k}\\&=\binom{m}{k}\sum_{n=k}^{m}\binom{m-k}{n-k}\\&=\binom{m}{k}\sum_{j=0}^{m-k}\binom{m-k}{j}\\&=2^{m-k}\binom{m}{k}.\end{aligned}$$

■

Corollary 3.8.

$$\sum_{n=0}^{m}\sum_{k=0}^{n}2^{-n}\binom{m}{k}\binom{m-k}{n-k}=2^m.$$

Proof. The proof follows by taking the sum of (3.20) over n and noting that $\sum_{n=0}^{m}\binom{m}{n}=2^m$. ■

In the following Theorem, we prove the hexagon property which appears in Pascal's triangle.

Theorem 3.6. *Let n and k be in $\mathbb{N}$ such that $n-1\geq k$. Then*

$$\binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}=\binom{n-1}{k-1}\binom{n+1}{k}\binom{n}{k+1}.$$

Proof. By using Lemma 2.1, we see that

$$f_{k-1}(n-1) = (n-k) f_{k-2}(n-1)$$

$$f_k(n+1) = (n-k+1) f_{k-1}(n+1)$$

$$f_{k-2}(n) = \frac{1}{n-k} \cdot \frac{1}{n-k+1} \cdot f_k(n).$$

Multiplying the left hand sides of the these equations yield

$$f_{k-1}(n-1) f_k(n+1) f_{k-2}(n) = f_{k-2}(n-1) f_{k-1}(n+1) f_k(n). \quad (3.21)$$

Using (3.11) and (3.21), we get

$$\binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1} = \binom{n-1}{k-1}\binom{n+1}{k}\binom{n}{k+1}.$$

∎

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