Nikodym-Vitali-Hahn-Saks Theorem on an Effect Algebra

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Abstract

A noncommutative version of the Nikodym-Vitali-Hahn-Saks theorem for group-valued countably additive measures defined on a σ-effect algebra is proved. This result generalizes the previously known Nikodym-Vitali-Hahn-Saks theorems for σ-orthocomplete orthomodular posets and for σ-orthoalgebras.

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1 Introduction

The effects of a quantum-mechanical system can be represented by self-adjoint operators A on a separable complex Hilbert space $\mathcal{H}$ such that $O \leqslant A \leqslant I$, where O, I are respectively the zero and identity operators on $\mathcal{H}$ ([8]). The set $\mathcal{E}(\mathcal{H})$ of all such operators A forms an algebraic structure, known as effect algebra, which generalizes orthoalgebras and Boolean algebras. Thus $\mathcal{E}(\mathcal{H})$ is the prototypical example of the effect algebras discussed in this paper and originally introduced in ([8]), and it provides a mathematical model for the study of unsharp quantum logics ([8]).

The Nikodym-Vitali-Hahn-Saks Theorem ([2]) is one of the most useful results in commutative (i.e., classical) measure theory. It is intimately related to three classical and fundamental results: the Brooks-Jewett Theorem, the Nikodym Convergence Theorem and the Nikodym Boundedness Theorem (see [1, 2, 3, 4, 11, 12, 16]). For more about the development of these theorems and their applications to vector measure theory, we refer the reader to ([5]).

In this paper, we present a version of the Nikodym-Vitali-Hahn-Saks Theorem in the noncommutative setting of effect algebras. More precisely, our theorem is formulated for countably additive uniform group-valued functions defined on a σ-complete effect algebra. This result improves earlier versions for uniform group-valued countably additive functions defined on a σ-orthocomplete orthomodular poset (see [2]) or defined on a σ-orthoalgebra (see [11]).

Throughout this paper, the symbol $\mathcal{F}(X)$ denotes the collection of all finite subsets of X. The symbols $\mathbb{R}$, $\mathbb{Z}$ and ω denote, respectively, the set of all real numbers, all integers, and all nonnegative integers. The notation $:=$ means "equals by definition".

2 Definitions and Preliminaries

Foulis and Bennett ([8]) have introduced the following definition.

2.1 Definition. An *effect algebra* is a system $(L, \oplus, 0, 1)$ consisting of a set L containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions $\forall\, a, b, c \in L$:

(EA1) (*Commutative Law*) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(EA2) (*Associative Law*) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(EA3) (*Orthocomplementation Law*) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

(EA4) (*Zero-One Law*) If $1 \oplus a$ is defined, then $a = 0$.

We shall write $L = (L, \oplus, 0, 1)$ for an effect algebra. Let L be an effect algebra and $a, b \in L$. Following [8], we say that a is *orthogonal* to b in L and write $a \perp b$ if and only if $a \oplus b$ is defined in L. We define $a \leq b$ to mean that there exists $c \in L$ such that $a \perp c$ and $b = a \oplus c$. The unique element $b \in L$ corresponding to a in Condition (EA3) above is called the *orthocomplement* of a and is written as $a' = b$. For any effect algebra L, it can be easily proved (see [8]) that $0 \leq a \leq 1$ holds for all $a \in L$, that $a \perp b$ iff $a \leq b'$, that, with $\leq$ as defined above, $(L, \leq, 0, 1)$ is a partially ordered set (poset).

2.2 Example. Consider the set $\mathcal{E}(\mathcal{H})$ of all self-adjoint operators A on a Hilbert space $\mathcal{H}$ with $O \leq A \leq I$, where O and I are the zero and identity operators, respectively, on $\mathcal{H}$. For $A, B \in \mathcal{E}(\mathcal{H})$, define

$$A \oplus B = A + B \quad \text{iff} \quad A + B \leq I.$$

It is not difficult to show that, under this $\oplus$, the system $(\mathcal{E}(\mathcal{H}), \oplus, O, I)$ forms an effect algebra [8].

We note that an *orthoalgebra* [7, 9] is an effect algebra L in which the zero-one law (Condition (EA4) of Definition 2.1) is replaced by the stronger condition:

(OA4) (*Consistency Law*) $a \in L$, $a \oplus a$ defined $\Rightarrow$ $a = 0$.

Consequently, *every orthoalgebra is an effect algebra*. There are many effect algebras that are not orthoalgebras [8, 13]. The effect algebra $\mathcal{E}(\mathcal{H})$ of Example 2.2 is one such.

Recall that an *orthomodular poset* (OMP) [9] may be regarded as an effect algebra L that satisfies the following additional condition [7]:

$$a, b \in L, \ a \perp b \quad \Rightarrow \quad a \vee b \quad \text{exists and} \quad a \vee b = a \oplus b.$$

It can be shown (see [7]) that this condition is equivalent to the *coherence law*:

$$a, b, c \in L, \ a \perp b \perp c \perp a \quad \Rightarrow \quad a \oplus b \perp c.$$

It can also be shown (see [8]) that an effect algebra L is an OMP iff it satisfies the coherence law. An *orthomodular lattice* (OML) may be defined as an OMP which is also a lattice. A *Boolean algebra* may be defined as a *distributive* OML. It has been shown in ([8]) that every Boolean algebra is an effect algebra L that satisfies the coherence law and the following *law of compatibility*: For all $a, b \in L$, there exist $a_1, b_1, c \in L$ such that $b_1 \oplus c$ and $a_1 \oplus (b_1 \oplus c)$ are defined, $a = a_1 \oplus c$ and $b = b_1 \oplus c$.

2.3 Definition. Let L be an effect algebra. A subset $A \subseteq L$ is called a *sub-effect algebra* iff $0, 1 \in A$, $p' \in A$ whenever $p \in A$, and $p \oplus q \in A$ whenever $p, q \in A$ and $p \perp q$.

Note that if A is a sub-effect algebra of L, then a pair of elements of A is orthogonal in A iff it is orthogonal in L.

Let A be a sub-effect algebra of an effect algebra L. For $a, b, c \in A$, we write $c = a \vee^A b$ (resp., $c = a \wedge^A b$) to indicate that c is the least upper bound (resp., greatest lower bound) of a and b in the poset $(A, \leq)$.

2.4 Definition. Let L be an effect algebra and $A \subseteq L$ be a sub-effect algebra. Then A is called

1. a *sub-OMP* if $a, b \in A$, $a \perp b \Rightarrow a \vee^A b$ exists;

2. a *sub-OML* if $a, b \in A \Rightarrow a \vee^A b$ exists;

3. a *Boolean sub-effect algebra* if it is a distributive sub-OML;

Riečanová ([15]) and Dvurečenskij and Riečan ([6]) have introduced the following definition.

2.5 Definition. Let L be an effect algebra. A finite subset $F = \{a_1, a_2, \dots, a_n\} \subseteq L$ is called *orthogonal* iff $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ is defined in L. In this case, we shall write $\bigoplus F$ to mean $a_1 \oplus a_2 \oplus \cdots \oplus a_n$. An arbitrary subset $X \subseteq L$ is called *$\bigoplus$-orthogonal* iff for every $F \in \mathcal{F}(X)$, the element $\bigoplus F$ exists in L. If for a $\bigoplus$-orthogonal subset $X \subseteq L$, the supremum of the set $\{\bigoplus F : F \in \mathcal{F}(X)\}$ exists in L, then we define

$$\bigoplus X = \bigvee_{F \in \mathcal{F}(X)} \bigoplus F.$$

2.6 Lemma. *Let L be an effect algebra and let $X, Y \subseteq L$.*

(i) *If $X \cap Y \subseteq \{0\}$, $X \cup Y$ is $\bigoplus$-orthogonal, and $\bigoplus X$, $\bigoplus Y$ both exist, then $\bigoplus X \perp \bigoplus Y$.*

(ii) *If, in addition to the hypotheses of* (i), $\oplus(X \cup Y)$ *exists, then*

$$(\oplus X) \oplus (\oplus Y) = \oplus(X \cup Y).$$

Proof: See the proof of Lemma 3.2 of ([12]). ∎

2.7 Definition. An effect algebra L is called *complete* (resp.,σ-*complete*) if for every (resp., for every countable) $\oplus$-orthogonal subset $X \subseteq L$, we have

$$\oplus X = \bigvee_{F\in\mathcal{F}(X)} \oplus F$$

exists (in L). A σ-*effect algebra* is simply a σ-complete effect algebra.

For more about completeness and σ-completeness of lattice effect algebras (resp., orthoalgebras), the reader may consult ([15]) (resp., [10]).

3 The Main Result

Before we state and prove the main result (Theorem 3.7), we need to establish a few more definitions.

3.1 Definition. A quadruple $(S, +, 0, \mathcal{U})$ where S is a set, $+$ is a binary operation on S, 0 is a distinguished element of S, and $\mathcal{U}$ is a *uniformity* on S is said to be a *uniform semigroup* if the following axioms are satisfied:

S1) The binary operation $+$ is associative and commutative.

S2) $\forall x \in S, x + 0 = x$.

S3) The function $(x, y) \longmapsto x + y : S \times S \to S$ is uniformly continuous.

It is well-known (see [14]) that the uniformity $\mathcal{U}$ can be generated by a set $\mathcal{D}$ of continuous pseudometrics d on S that are *semi-invariant*; that is, $\forall\, s, t, v \in S$, we have

$$d(s + v, t + v) \leq d(s, t). \tag{1}$$

Two typical examples of uniform semigroups are $\mathbb{R}$ and $[0, \infty)$ under the usual addition.

3.2 Definition. Let L be an effect algebra and let S be a Hausdorff uniform semigroup. A function $\mu : L \longrightarrow S$ is called

(i) *additive* if $\mu(0) = 0$, and for every finite $\oplus$-orthogonal subset $\{a_i : i = 1, ..., n\} \subseteq L$,

$$\mu\Big(\bigoplus_{i=1}^{n} a_i\Big) = \sum_{i=1}^{n} \mu(a_i),$$

(ii) *countably additive* (or *σ-additive*) if $\mu(0) = 0$, and for every $\oplus$-orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$ the infinite series

$$\mu\Big(\bigoplus_{i\in\omega} a_i\Big) = \sum_{i\in\omega} \mu(a_i) \quad \text{converges in } S,$$

(iii) *s-bounded* if for every $\oplus$-orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$ we have

$$\lim_{i\to\infty} \mu(a_i) = 0.$$

Since any pair of orthogonal elements in L is $\oplus$-orthogonal, it is easy to see that if $\mu : L \to S$ is additive, then

$$a, b \in L \text{ and } a \perp b \quad \Rightarrow \quad \mu(a \oplus b) = \mu(a) + \mu(b).$$

Henceforth, unless otherwise stated, we assume that L is a σ-effect algebra, S is a Hausdorff uniform semigroup with a fixed set $\mathcal{D}$ of continuous pseudometrics that generate its uniformity, $sa(L, S)$ denotes the set of all additive and s-bounded functions on L with values in S, and $ca(L, S)$ denotes the set of all countably additive functions on L with values in S.

3.3 Definition.

(i) A nonempty subset $M \subseteq sa(L, S)$ is called *uniformly s-bounded* if for every $\oplus$-orthogonal sequence $(a_i)_{i\in\omega}$ in L we have

$$\lim_{i\to\infty} \mu(a_i) = 0 \quad \text{uniformly in } \mu \in M.$$

(ii) A nonempty subset $M \subseteq ca(L, S)$ is called *uniformly countably additive* iff for every $\oplus$-orthogonal sequence $(a_i)_{i\in\omega}$ in L, we have

$$\mu(\oplus_{i\in\omega} a_i) = \sum_{i\in\omega} \mu(a_i) \quad \text{uniformly in } \mu \in M.$$

3.4 Lemma. *If S is a Hausdorff topological Abelian group, then* $ca(L,S) \subseteq sa(L,S)$.

Proof: Given a $\oplus$-orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$, $d \in \mathcal{D}$ and $\epsilon > 0$, the countable additivity of μ implies that , $\exists k_0 \in \omega$ such that $\forall k \geq k_0$, we have

$$d(\textstyle\sum_{i=0}^{k} \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) < \frac{\epsilon}{2}.$$

Since S is a group, d is invariant. So this and the triangle inequality imply that $\forall k > k_0$, we have

$$\begin{aligned} d(\mu(a_k), 0) &= d(\mu(a_k) + \textstyle\sum_{i=0}^{k-1} \mu(a_i), \sum_{i=0}^{k-1} \mu(a_i)) \\ &= d(\textstyle\sum_{i=0}^{k} \mu(a_i), \sum_{i=0}^{k-1} \mu(a_i)) \\ &\leq d(\textstyle\sum_{i=0}^{k} \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) + d(\sum_{i=0}^{k-1} \mu(a_i), \mu(\bigoplus_{i\in\omega} a_i)) \\ &< \epsilon. \end{aligned}$$

Thus μ is s-bounded. ■

The following useful lemma generalizes Lemma 2.2 of [3] and generalizes Lemma 4.8 of [11] to effect algebras.

3.5 Lemma. *Let L be a σ-effect algebra and $(\mu_n)_{n\in\omega} \subseteq ca(L,S)$ be uniformly s-bounded. If $(a_i)_{i\in\omega} \subseteq L$ is $\oplus$-orthogonal, then $(\mu_n(a_i))_{i\in\omega}$ is summable uniformly in $n \in \omega$.*

Proof: We need to show that for every $d \in \mathcal{D}$ and every $\epsilon > 0$ $\exists F^* = F^*(d,\epsilon) \in \mathcal{F}(\omega)$ such that $\forall F \in \mathcal{F}(\omega)$ with $F \supseteq F^*$ and $\forall n \in \omega$,

$$d(\textstyle\sum_{i\in F} \mu_n(a_i), \mu_n(\bigoplus_{i\in\omega} a_i)) < \epsilon.$$

Suppose that this is false. Then $\exists d \in \mathcal{D}$ and $\exists \epsilon > 0$ such that for every $F \in \mathcal{F}(\omega)$, $\exists K(F) \in \mathcal{F}(\omega)$ with $K(F) \supseteq F$ and

$$\sup_{n\in\omega} d(\textstyle\sum_{i\in K(F)} \mu_n(a_i), \mu_n(\bigoplus_{i\in\omega} a_i)) > \epsilon.$$

By the finite additivity of each μ_n and by Lemma 2.6, we have

$$\mu_n(\textstyle\bigoplus_{i\in\omega} a_i) = \mu_n(\bigoplus_{i\in K(F)} a_i) + \mu_n(\bigoplus_{i\in\omega\setminus K(F)} a_i);$$

so, by the semi-invariance of d, we obtain

$$\begin{aligned} \epsilon &\leq \sup_n d(\textstyle\sum_{i\in K(F)} \mu_n(a_i) + 0, \mu_n(\bigoplus_{i\in K(F)} a_i) + \mu_n(\bigoplus_{i\in\omega\setminus K(F)} a_i)) \\ &\leq \sup_n d(0, \mu_n(\textstyle\bigoplus_{i\in\omega\setminus K(F)} a_i)). \end{aligned}$$

Thus for each $F \in \mathcal{F}(\omega)$ we may (and do) choose an $n = n(F) \in \omega$ such that

$$d(\mu_{n(F)}(\bigoplus_{i\in\omega\setminus K(F)} a_i), 0) \geq \epsilon \tag{2}$$

Note that the countable additivity of each $\mu_{n(F)}$ implies that there exists $H(F) \in \mathcal{F}(\omega \setminus K(F))$ such that

$$d(\mu_{n(F)}(\bigoplus_{i\in H(F)} a_i), \mu_{n(F)}(\bigoplus_{i\in\omega\setminus K(F)} a_i)) \leq \frac{\epsilon}{2}.$$

So the triangle inequality and inequality (2) yield that $\forall F \in \mathcal{F}(\omega)$, we have

$$\begin{aligned} d(\mu_{n(F)}(\bigoplus_{i\in H(F)} a_i), 0) &\geq d(\mu_{n(F)}(\bigoplus_{i\in\omega\setminus K(F)} a_i), 0) \\ &\quad - d(\mu_{n(F)}(\bigoplus_{i\in H(F)} a_i), \mu_{n(F)}(\bigoplus_{i\in\omega\setminus K(F)} a_i)) \\ &\geq \epsilon - \frac{\epsilon}{2} \geq \frac{\epsilon}{2}. \end{aligned}$$

Thus we can successively choose sets $F_0, F_1, F_2, \ldots$ in $\mathcal{F}(\omega)$ and corresponding triples $(K(F_0), H(F_0), n(F_0))$, $(K(F_1), H(F_1), n(F_1))$, $(K(F_2), H(F_2), n(F_2)), \ldots$ such that

- $n(F_i) \in \omega \; \forall i \in \omega$,
- $F_i \subseteq K(F_i) \in \mathcal{F}(\omega) \; \forall i \in \omega$,
- $H(F_0) \in \mathcal{F}(\omega \setminus K(F_0))$, $H(F_i) \in \mathcal{F}(\omega \setminus (K(F_i) \cup \bigcup_{j=0}^{i-1} H(F_j))) \; \forall i \geq 1$, and
- $d(\mu_{n(F_j)}(\bigoplus_{i\in H(F_j)} a_i), 0) \geq \frac{\epsilon}{2} \; \forall j \in \omega$.

Now for $j = 0, 1, 2, \ldots$, set $c_j = \bigoplus_{i\in H(F_j)} a_i$. Since $(a_i)_{i\in\omega}$ is $\bigoplus$-orthogonal and $(H(F_j))_{j\in\omega} \subseteq \mathcal{F}(\omega)$ is pairwise disjoint, we see that $(c_j)_{j\in\omega}$ is $\bigoplus$-orthogonal and that $\forall j \in \omega$,

$$d(\mu_{n(F_j)}(c_j), 0) \geq \frac{\epsilon}{2}.$$

This contradicts the uniform s-boundedness of $(\mu_n)_{n\in\omega}$. ■

3.6 Theorem (Brooks-Jewett Theorem for σ-Effect Algebras). *Let L be a σ-effect algebra and let S be a Hausdorff topological Abelian group. If $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq sa(L, S)$ is such that*

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \quad \forall a \in L,$$

then $\{\mu_n : n \in \omega\}$ *is uniformly s-bounded.*

Proof: See Theorem 4.4 and Remark 4.8(1) of [12]. ■

Now, we are ready to state and prove a Nikodym-Vitali-Hahn-Saks Theorem for σ-effect algebras.

3.7 Theorem (Nikodym-Vitali-Hahn-Saks Theorem for σ-effect algebras). *Let* L *be a* σ*-effect algebra and let* S *be a Hausdorff topological Abelian group. If* $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq ca(L,S)$ *is such that*

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \quad \forall a \in L, \tag{3}$$

then $\mu_0 \in ca(L,S)$ *and* $(\mu_n)_{n\in\omega}$ *is uniformly countably additive.*

Proof: Let $(\mu_n)_{n\in\omega\setminus\{0\}}$ be as in the statement of the theorem. Then, by Lemma 3.4, $(\mu_n)_{n\in\omega\setminus\{0\}} \subseteq sa(L,S)$ and satisfies the hypothesis (3). Hence, by Theorem 3.6, $(\mu_n)_{n\in\omega}$ is uniformly s-bounded. Now, by Lemma 3.5, we infer that $(\mu_n)_{n\in\omega\setminus\{0\}}$ is uniformly countably additive.

It remains to show that $\mu_0 \in ca(L,S)$. Let $(a_i)_{i\in\omega} \subseteq L$ be $\oplus$-orthogonal, $d \in \mathcal{D}$ and $\epsilon > 0$ be given. By the uniform countable additivity of $(\mu_n)_{n\in\omega\setminus\{0\}}$, $\exists F_0 \in \mathcal{F}(\omega)$ such that $\forall F \in \mathcal{F}(\omega)$ with $F_0 \subseteq F$ and $\forall n \in \omega \setminus \{0\}$, we have

$$d(\mu_n(\textstyle\bigoplus_{i\in\omega} a_i), \sum_{i\in F} \mu_n(a_i)) \leq \epsilon. \tag{4}$$

Hence, by continuity of d and of addition in S and by using inequality (4), we have $\forall F \in \mathcal{F}(\omega)$ with $F_0 \subseteq F$ that

$$\begin{aligned} d(\mu_0(\textstyle\bigoplus_{i\in\omega} a_i), \sum_{i\in F} \mu_0(a_i)) &= d(\lim_{n\to\infty} \mu_n(\textstyle\bigoplus_{i\in\omega} a_i), \sum_{i\in F} \lim_{n\to\infty} \mu_n(a_i)) \\ &= d(\lim_{n\to\infty} \mu_n(\textstyle\bigoplus_{i\in\omega} a_i), \lim_{n\to\infty} \sum_{i\in F} \mu_n(a_i)) \\ &= \lim_{n\to\infty} d(\mu_n(\textstyle\bigoplus_{i\in\omega} a_i), \sum_{i\in F} \mu_n(a_i)) \\ &\leq \epsilon. \end{aligned}$$

Therefore $\mu_0 \in ca(L,S)$. ■

4.8 Remarks. (1) If P is an orthomodular poset, then it is not difficult to show (see Habil, 1994a, Lemma(4.6)) that P is σ-orthocomplete iff P is a σ-effect algebra. In this case,

$$\textstyle\bigoplus_{i\in\omega} a_i = \bigvee_{i\in\omega} a_i$$

for all pairwise orthogonal sequences $(a_i)_{i\in\omega} \subseteq P$. Therefore, our definitions of countable additivity and uniform countable additivity coincide with the ones that are given in the literature (see [2, 3]).

(2) Recall (see[10]) that a subset X in an effect algebra L is *jointly orthogonal* iff X is pairwise orthogonal and is contained in a Boolean sub-effect algebra of L. According to [10], an effect algebra L is called a σ-effect algebra if for every jointly orthogonal sequence $(a_i)_{i\in\omega} \subseteq L$,

$$\bigoplus_{i\in\omega} a_i = \bigvee_{F\in\mathcal{F}(\omega)} \bigoplus_{i\in F} a_i$$

exists in L. Evidently, every jointly orthogonal subset of L is $\bigoplus$-orthogonal. Thus every σ-effect algebra algebra in the sense of ([11]) is a σ-effect algebra in our sense. Since every orthoalgebra is an effect algebra, it follows that every σ-orthoalgebra is a σ-effect algebra. Furthermore, every σ-orthocomplete OMP is a σ-effect algebra. Taking these notes into account, we see that our Theorem 3.7 contains Proposition 6.4 of ([3]) and Theorem 4 of ([2]) in the special case when the σ-effect algebra L is assumed to be a σ-orthocomplete orthomodular poset. It also contains Theorem 4.10 of ([11]).

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