ON THE LAGRANGIAN FORMALISM OF QED

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Abstract

The QED theory is treated as a constrained system using the Euler-Lagrange equation for field system. It is shown that a singular Lagrangian as a field system for QED is in agreement with the general Lagrangian approach and the canonical Hamiltonian approach of constrained systems.

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1 Introduction

Singular Lagrangian systems represent a special case of a more general dynamics called constrained systems. A general feature of constrained systems is the existence of constraints in its classical configurations.

The basic ideas of the classical treatment and the quantization of such systems were initiated and developed by Dirac [1]. He distinguished between two types of constraints; first and second-classes. In the case of unconstrained systems, the Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics. The first study of Hamilton-Jacobi equations for arbitrary first-order actions was carried out by Santilli [2]. Gitman and Tyutin [3] discussed the canonical quantization of singular theories as well as the Hamiltonian formalism of gauge theories in an arbitrary gauge. Recently, the canonical method based on Hamilton-Jacobi formulation was developed to investigate singular systems. In this formalism there is no need to distinguish between first and second-class constraints as in the Dirac theory [4],[5]. Also, in the canonical method which has been developed by Güler [6-8], the equations of motion were written as total differential equations. In ref.[7], the discrete singular system was treated as a continuous system. Hamiltonian and Lagrangian formulations are used together. Our aim in this work is to use the latter treatment to study the quantum electrodynamic (QED) system.

The paper is arrange as follows: In section 2, a brief discussion of the canonical Hamiltonian method is given, together with the treatment of a singular system as a continuous system. Next, in section 3, QED is treated as a singular field system. Finally, several concluding remarks follow.

2 Theoretical Framework

The Lagrangian $L = L\left(q_i, \dot{q_i}, t\right)$ is singular if the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q_i} \dot{q_j}}, \quad i, j = 1, ..., n, \tag{1}$$

is $r = n - m, m < n$. Otherwise the Lagrangian will be regular.

The generalized momenta p_i corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q_a}}, \qquad a = 1, 2, ..., n - r, \tag{2}$$

$$p_\mu = \frac{\partial L}{\partial \dot{x_\mu}}, \qquad \mu = 1, 2, ..., r, \quad x_\mu \equiv q_\mu. \tag{3}$$

Equation (2) enables us to obtain $\dot{q_a}$ as

$$\dot{q_a} = \dot{q_a}\left(q_i, \dot{x_\mu}, p_a, t\right). \tag{4}$$

Substituting eq. (4) into eq.(3), we get

$$p_\mu = -H_\mu \left(q_i, \dot{x_\mu}, p_a, t\right). \tag{5}$$

The usual Hamiltonian H_0 is defined as

$$H_0 = -L\left(q_i, \dot{x_\mu}, p_a, t\right) + p_a \dot{q_a} - \dot{x_\mu} H_\mu, \quad \mu = 1, 2, ..., r. \tag{6}$$

The set of Hamilton-Jacobi partial differential equations is expressed as

$$H'_\alpha \left(q_\beta; q_a; p_a = \frac{\partial S}{\partial q_a}; p_\mu = \frac{\partial S}{\partial q_\mu}\right) = 0, \qquad \alpha, \beta = 0, 1, 2, ..., r, \tag{7}$$

where

$$H'_0 = p_0 + H_0; \tag{8}$$

$$H'_\mu = p_\mu + H_\mu. \tag{9}$$

The equations of motion are written as total differential equations:

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha; \tag{10}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha; \tag{11}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha; \tag{12}$$

$$dZ = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha, \tag{13}$$

where $Z \equiv S(t_\alpha, q_a)$. These equations are integrable if the variation of H'_μ is identically zero.

In Refs. [7,8] the singular Lagrangian systems are treated as continuous systems. The Euler-Lagrange equation of a singular-Lagrangian system is given as

$$\frac{\partial}{\partial x_\alpha}\left[\frac{\partial L'}{\partial\left(\partial_\alpha q_a\right)}\right] - \frac{\partial L'}{\partial q_a} = 0, \quad \partial_\alpha q_a = \frac{\partial q_a}{\partial x_\alpha}, \tag{14}$$

with constraints

$$dG_\alpha = -\frac{\partial L'}{\partial x_\alpha} dt, \tag{15}$$

where L' is the "modified Lagrangian" defined as

$$L'\left(x_\mu, \partial_\mu q_a, \dot{x_\mu}, q_a\right) \equiv L\left(x_\mu, q_a, \dot{q_a} = \left(\partial_\mu q_a\right) \dot{x_\mu}\right); \tag{16}$$

and

$$G_\alpha = H_\alpha \left(x_\mu, q_a, p_a = \frac{\partial L}{\partial q_a}\right). \tag{17}$$

The solution of eq. (14), together with the constraints equation (15), gives us the solution of the system.

3 Treatment of QED as a Singular Field System

Let us write the Lagrangian density [9]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\left[\overline{\psi}\gamma^{\mu}\left(\partial_{\mu}\psi\right) - \left(\partial_{\mu}\overline{\psi}\right)\gamma^{\mu}\psi\right] - m\overline{\psi}\psi + j_{\mu}A^{\mu} \tag{18}$$

in the form

$$\mathcal{L} = -\frac{1}{4}\left(\frac{\partial A_{\nu}}{\partial x_{\mu}} - \frac{\partial A_{\mu}}{\partial x_{\nu}}\right)\left(\frac{\partial A^{\nu}}{\partial x^{\mu}} - \frac{\partial A^{\mu}}{\partial x^{\nu}}\right) + \frac{i}{2}\left[\overline{\psi}\gamma^{\mu}\left(\partial_{\mu}\psi\right) - \left(\partial_{\mu}\overline{\psi}\right)\gamma^{\mu}\psi\right]$$

$$-m\overline{\psi}\psi - e\overline{\psi}\gamma_{\mu}\psi A^{\mu}. \tag{19}$$

One can define the canonical momenta as follows:

$$\pi^{\nu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}A^{\nu}\right)} = -F^{\mu\nu}; \tag{20}$$

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\psi\right)} = \frac{i}{2}\overline{\psi}\gamma^{\mu} = -H_{1}; \tag{21}$$

$$\overline{\pi} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\overline{\psi}\right)} = -\frac{i}{2}\gamma^{\mu}\psi = -H_{2}. \tag{22}$$

The singular Lagrangian in eq.(18) can be treated as a continuous system by introducing

$$A_{\nu} = A_{\nu}\left(x_{\mu},\psi,\overline{\psi}\right),\ \psi = \psi(x_{\mu}),\ \overline{\psi} = \overline{\psi}(x_{\mu}). \tag{23}$$

Let us define the four-dimensional derivative of A_{ν} as

$$\frac{\partial A_{\nu}}{\partial x_{\mu}} \equiv \frac{dA_{\nu}}{dx_{\mu}} = \partial_{\mu}A_{\nu} + \frac{\partial A_{\nu}}{\partial\psi}\frac{\partial\psi}{\partial x_{\mu}} + \frac{\partial A_{\nu}}{\partial\overline{\psi}}\frac{\partial\overline{\psi}}{\partial x_{\mu}}. \tag{24}$$

The modified Lagrangian $\mathcal{L}'$ becomes

$$\mathcal{L}' = -\frac{1}{4}\left[\partial_{\mu}A_{\nu} + \frac{\partial A_{\nu}}{\partial\psi}\partial_{\mu}\psi + \frac{\partial A_{\nu}}{\partial\overline{\psi}}\partial_{\mu}\overline{\psi} - \partial_{\nu}A_{\mu} - \frac{\partial A_{\mu}}{\partial\psi}\partial_{\nu}\psi - \frac{\partial A_{\mu}}{\partial\overline{\psi}}\partial_{\nu}\overline{\psi}\right]$$

$$\times\left[\partial^{\mu}A^{\nu} + \frac{\partial A^{\nu}}{\partial\psi}\partial^{\mu}\psi + \frac{\partial A^{\nu}}{\partial\overline{\psi}}\partial^{\mu}\overline{\psi} - \partial^{\nu}A^{\mu} - \frac{\partial A^{\mu}}{\partial\psi}\partial^{\nu}\psi - \frac{\partial A^{\mu}}{\partial\overline{\psi}}\partial^{\nu}\overline{\psi}\right]$$

$$+\frac{i}{2}\left[\overline{\psi}\gamma^{\mu}\left(\partial_{\mu}\psi\right) - \left(\partial_{\mu}\overline{\psi}\right)\gamma^{\mu}\psi\right] - m\overline{\psi}\psi - e\overline{\psi}\gamma_{\mu}\psi A^{\mu}. \tag{25}$$

The Euler-Lagrange equation for the continuous system (14), for $q_{\alpha} \equiv x_{\mu},\psi,\overline{\psi}$ and $q_{a} \equiv A_{\nu}$, becomes

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial \mathcal{L}'}{\partial\left(\partial_\mu A_\nu\right)}\right]+\frac{\partial}{\partial \psi}\left[\frac{\partial \mathcal{L}'}{\partial\left(\frac{\partial A_\nu}{\partial \psi}\right)}\right]+\frac{\partial}{\partial \overline{\psi}}\left[\frac{\partial \mathcal{L}'}{\partial\left(\frac{\partial A_\nu}{\partial \overline{\psi}}\right)}\right]-\frac{\partial \mathcal{L}'}{\partial A_\nu}=0. \quad (26)$$

With the modified Lagrangian $\mathcal{L}'$, eq.(26) takes the form

$$\partial_\mu F^{\mu\nu}-e\overline{\psi}\gamma^\nu\psi=0. \quad (27)$$

Equation (27) is the first set of QED Euler-Lagrange equations obtained from the standard Lagrangian formulation; it is equivalent to the equations of motion obtained from the canonical method [9]. The second set of Euler-Lagrange equations of the standard Lagrangian formulation is obtained by using the constraint equations (15), that is,

$$dG_1=-\frac{\partial \mathcal{L}'}{\partial \psi}dx_\mu. \quad (28)$$

G_1 is obtained from the Hamiltonian formulation, eq.(5):

$$G_1\equiv H_1=-\frac{i}{2}\overline{\psi}\gamma^\mu; \qquad \textit{and} \qquad d\overline{\psi}=\frac{\partial\overline{\psi}}{\partial x_\mu}\,dx_\mu.$$

Thus, eq. (28) becomes

$$\overline{\psi}\left[i\gamma^\mu\partial_\mu+m+eA_\mu\gamma^\mu\right]=0. \quad (29)$$

Similarly, from eq. (5), we have $G_2\equiv H_2=\frac{i}{2}\gamma^\mu\psi$. Then, by using eq. (15), we get

$$dG_2=-\frac{\partial \mathcal{L}'}{\partial \overline{\psi}}\,dx_\mu; \quad (30)$$

or

$$\left[i\gamma^\mu\partial_\mu-m-e\gamma_\mu A^\mu\right]\psi=0. \quad (31)$$

Equations (29)and (31) are the second set of Euler-Lagrange equations of the standard formulation.

Conclusion

Quantum Electrodynamics (QED) is perhaps the best fundamental physical theory we have. The theory is formulated as a set of simple equations (Maxwell's and Dirac's equations) whose form is essentially determined by relativistic invariance. The quantum mechanical solutions of these equations give detailed predictions of electromagnetic phenomena starting from macroscopic distances down to regions several hundred times smaller than the proton. Fermionic fields are of fundamental nature in the physical world.

In this work we have studied the Lagrangian formulation of the QED system as singular Lagrangian. The system is treated as a continuous field system with constraints. It is shown that this treatment is in exact agreement with the general approach. Our formalism is a mixture of the Hamiltonian and Lagrangian formulations. In the general approach, the constraint equations can be obtained from the Euler-Lagrange equations; whereas in the treatment of singular Lagrangians as fields, the constraints can be determined from eq.(15), which is obtained with the help of the canonical Hamiltonian formalism. The equations of motion are partial differential equations, which are equivalent to those equations obtained from the canonical Hamiltonian approach.

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