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Compactness on Upper Bounded T_0 - Alexandroff Spaces

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Abstract

The aim of this paper is to characterize several types of compactness on upper bounded T_o- Alexandroff spaces. We discuss the relations between these types under certain conditions. It is mainly shown that an Upper Bounded T_o- Alexandroff space X is $p-$closed if and only if it is quasi H-closed. In addition, it is semi-compact iff it is finite.

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1. Introduction:

An *Alexandroff space* [2] (briefly $A-$space) (or a *smallest neighborhood space*) X is a topological space in which the arbitrary intersection of open sets is open. In this space, each element x possesses a smallest open neighborhood $V(x)$ which is the intersection of all open sets containing x. For every T_o $A-$space (X,τ), there is a corresponding poset $(X,\leq_\tau)$ in one to one and onto way, where each one of them is completely determined by the other. If (X,τ) is a T_0 $A-$space, we define the corresponding partial order $\leq_\tau$, called (*Alexandroff*) *specialization order*, by: $a\leq_\tau b$ iff $a\in\overline{\{b\}}$ iff $b\in V(a)$. On the other hand, if $(X,\leq)$ is a poset, then the collection $\mathrm{B}=\{\uparrow x : x\in X\}$ forms a base for a T_0 $A-space$ on X, denoted by $\tau_\leq$. So throughout this paper, we consider $(X,\tau(\leq))$ to be a T_o $A-$space (X,τ) together with its corresponding poset $(X,\leq)$. A space is an *upper bounded* T_0 $A-space$ (briefly a UB T_0 $A-$space) (resp. a *lower bounded* T_0 $A-space$ (briefly an LB T_0 $A-$space)) if every chain of points in the corresponding poset $(X,\leq)$ is bounded above (resp. bounded below). It is *bibounded* (briefly BB) if it is both UB and LB [22]. Given a poset $(X,\leq)$, the set of all maximal elements is denoted by $M(X)$ (or simply by M) and the set of all minimal elements is denoted by $m(X)$ (or simply by m). For $x\in X$, $\uparrow x=\{y\in X : x\leq y\}$, and $\downarrow x=\{z\in X : z\leq x\}$. We define $Max=M-m$. Moreover, for each $x\in X$, we define $\hat{x}$ to be the set of all maximal elements grater than or equal to x and $\breve{x}$ the set of all minimal elements less than or equal to x. If X is a UBT_0 $A-$space, then $M\neq\varnothing$ and $\hat{x}\neq\varnothing$ $\forall x\in X$. Similarly in an LBT_0 $A-$spaces, $m\neq\varnothing$ and $\breve{x}\neq\varnothing$ $\forall x\in X$. For a T_o $A-$space $(X,\tau(\leq))$, we have the following: A set A is open (resp. closed) iff A is up (resp. down) set in the corresponding poset. For each $x \in X$, $V(x) = \uparrow x$, $\hat{x} = \uparrow x \cap M$ and $\check{x} = \downarrow x \cap m$ [11].

A subset A of a topological space (X, τ) is called a $semi-open$ [10] (resp. a *preopen* [15], an α *-open* [18]) set if $A \subseteq Cl(Int(A))$ (resp. $A \subseteq Int(Cl(A))$, $A \subseteq Int(Cl(Int(A)))$). It is called a $semi-closed$ [4] (resp. a *preclosed* [6], an α *-closed* [5]) set if A^c is semi-open (resp. preopen, α -open). The family of all semi-open (resp. preopen, α -open) sets is denoted by $SO(X)$ (resp. $PO(X)$, τ_α). We have the following facts: The collection τ_α forms a topology on X [18]. $\tau_\alpha = PO(X) \cap SO(X)$ [19]. $\tau \subseteq \tau_\alpha$ (resp. $\tau \subseteq SO(X)$, $\tau \subseteq PO(X)$).

1. Preliminary Notes

Theorem 2.1 *[12] Let $(X, \tau(\leq))$ be a UB T_0 $A-space$. Then we have the following:*

1. $PO(X) = \tau_\alpha$.
2. $PO(X) \subseteq SO(X)$.
3. For $A \subseteq X$, A is semi-open set if and only if $\hat{x} \cap A \neq \varnothing \ \forall x \in A$.

A topological space (X, τ) is said to be *extremally disconnected* if the closure of every open set is open. X is *submaximal* [21] if each dense subset is open. It is *resolvable* [9] if and only if $X = D \bigcup D^c$ where both D and D^c are dense. A subset $A \subseteq X$ is *resolvable* if the subspace (A, τ_A) is resolvable. X is *irresolvable* if it is not resolvable and X is *strongly irresolvable* [8] if no nonempty open set is resolvable.

Theorem 2.2 *[12] Let X be a UB T_0 $A-space$, then*

1. X is strongly irresolvable.
2. The space (X, τ_α) is submaximal.
3. X is submaximal iff $\tau = \tau_\alpha$.

Theorem 2.3 *[12] Let X be a UB T_0 $A-space$, then X is extermally disconnected iff $|\hat{x}| = 1$ for each $x \in X$ iff $\tau_\alpha = SO(X)$.*

Theorem 2.4 *[11] Let $(X, \tau(\leq))$ be a T_0 A-space. Then X is submaximal iff each element set in X is either open or closed.*

3. Compactness of T_o – Alexandroff Spaces

Let (X, τ) be a topological space then X is *compact* iff each open cover of X has a finite subcover. X is *countably compact* iff each countable open cover of X has a finite subcover. X is *Lindelöf* iff each open cover of X has a countable subcover. X is *locally compact* iff each point in X has a nhood base consisting of compact sets. A collection $\mathcal{U}$ of subsets of X is *locally finite* iff each $x \in X$ has a nhood meeting only finitely many $U \in \mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are covers of X, then we say $\mathcal{U}$ is a *refinment* of $\mathcal{V}$ iff each $U \in \mathcal{U}$ is contained in some $V \in \mathcal{V}$. X is *paracompact* iff each open cover of X has an open locally finite refinement. X is *separable* iff it contains a countable dense subset. X is *orthocompact* iff every open cover has an open refinment $\mathcal{V}$ such that for all $x \in X, \cap\{V \in \mathcal{V} : x \in V\}$ is open.

Theorem 3.1 *[13] Let $(X, \tau(\leq))$ be a T_o – Alexandroff space. Then*

1. X is compact iff X contains a finite subset N of minimal elements such that $\uparrow N = X$.
2. X is locally compact.
3. If X contains a countable subset C of minimal elements in which $\uparrow C = X$, then X is Lindelöf (but not conversely).

Corollary 3.2 *[22] Let $(X,\tau(\leq))$ be a T_o $A-$space. Then X is compact iff X is an LB and the set of all minimal elements m is finite.*

Corollary 3.3 *Let $(X,\tau(\leq))$ be an LB T_o $A-$space. Then X is Lindelöf iff the set of minimal elements m is countable.*

Theorem 3.4 *Let $(X,\tau(\leq))$ be an LB T_o $A-$space. Then X is compact if and only if it is countably compact.*

Proof. A compact space is countably compact. Conversely, If X is not compact, then the set of minimal elements m is infinite. Let $x_1, x_2, x_3, \cdots$ be elements in m and let $m' = m - \{x_1, x_2, x_3, \cdots\}$. Then $\{V(x_1), V(x_2), V(x_3), \cdots\} \cup \{V(m')\}$ is a countable open cover without finite subcover.

Theorem 3.5 *[7] If $(X,\tau(\leq))$ is a UB $T_\circ$ A-space, then the space (X,τ_α) is an Alexandroff space (necessarily $T_\circ$). Moreover, $x \leq_\alpha y$ in $(X,\leq_\alpha)$ if and only if $y \in \{x\} \bigcup \hat{x}$.*

Corollary 3.6 *Let $(X,\tau(\leq))$ be a UB T_o $A-$space. Then (X,τ_α) is compact (resp. Lindelöf) if and only if the set $X - Max$ is finite (resp. countable).*

Proof. Using Theorem 3.5, each element in $(X,\leq_\alpha)$ is either minimal or maximal. So, the space (X,τ_α) is a BB T_o $A-$space with a set of minimal elements $m_\alpha = X - Max$. Hence, the result follows directly from Corollary 3.2 and Corollary 3.3.

Theorem 3.7 *[3] Let X be a T_o-Alexandroff space.*

1. X is orthocompact.
2. X is paracompact if and only if for every $x \in X$, $V(x)$ meets only a finite number of $V(y)$.

Theorem 3.8 *Let X be a T_o-Alexandroff space. Then X is paracompact iff $\forall x \in X$ and $\forall z \in V(x)$, both $V(z)$ and $\overline{z}$ are finite.*

Proof. If there exist infinite different numbers $y_1, y_2, y_3, \cdots$ in $V(z)$, then $y_i \in V(y_i) \subseteq V(x)$ $\forall i$. Hence $V(x)$ meets infinite numbers of $V(y_i)$. By Theorem 3.7 X is not paracompact. In addition, if there exist infinite different numbers $z_1, z_2, z_3, \cdots$ in $\overline{z}$, then $z \in V(x) \cap V(z_i)$ $\forall i$. Hence a gain, $V(x)$ meets infinite number of $V(z_i)$. Conversely, suppose to contrary that there exists $V(x)$ meets infinitely many $V(w_i), i = 1,2,\cdots$. Then there exists $x_i \in V(x) \cap V(w_i)$. Set $A = \{x_1, x_2, \cdots\}$. If A is infinite, then $V(x)$ is infinite which is a contradiction. If A is finite, then there exist x_j and an infinite subset $\{w_{j_k} : k \in \mathbb{N}\}$ of $\{w_i : i \in \mathbb{N}\}$ such that $x_j \in V(w_{j_k})$ $\forall k$. Equivalently, $w_{j_k} \in \overline{x_j}$. Hence, $x_j \in V(x)$ and $\overline{x_j}$ is infinite which is a contradiction.

4. Compactness Based on Generalized Open Sets

Definitions 4.1 *A topological space (X,τ) is called:*

1. $p-$closed [1] if every preopen cover of X has a finite subfamily whose preclosures cover X.
2. *quasi-H-closed* (QHC) [20] if every open cover of X has a finite subfamily whose closures cover X (=every open cover of X has a finite subfamily whose union is dense),
3. *strongly compact* [14] if every preopen cover of X has a finite subcover,
4. $\alpha-$*compact* [17] if every $\alpha-$open cover of X has a finite subcover,
5. *semi-compact* [16] if every semi-open cover of X has a finite subcover,

The following implications hold in general:

strongly compact $\Rightarrow$ $p-$closed

$\Downarrow$ $\Downarrow$

$\alpha-$compact $\Rightarrow$ compact $\Rightarrow$ QHC

$\Uparrow$

semi-compact

Remark 4.2 *There is no guarantee that the reverse of these implications hold. Nevertheless, we have the following:*

6. If X is strongly irresolvable, then $\tau_\alpha = PO(X)$, and hence $\alpha-$compact $\Leftrightarrow$ stronly compact.
7. If X is submaximal, then $\tau = \tau_\alpha$, and hence compact $\Leftrightarrow$ $\alpha-$compact.
8. If X is extremally disconnected, then $\tau_\alpha = SO(X)$, and hence $\alpha-$compact $\Leftrightarrow$ semi-compact.

As we mentioned above, if X is $p-$closed then it is QHC. The following theorem provides a condition when a QHC space is $p-$closed.

Theorem 4.3 *[15] Let (X,τ) be a strongly irresolvable space, then (X,τ) is $p-$closed if and only if it is QHC.*

Theorem 4.4 *Let X be a $UB\,T_o$ $A-$space. Then the following implications hold for X :*

strongly compact $\Rightarrow$ $p-$closed

$\Updownarrow$ $\Updownarrow$

$\alpha-$compact $\Rightarrow$ compact $\Rightarrow$ QHC

$\Uparrow$

semi-compact

Proof. By Theorem 2.2, X is strongly irresolvable. So, using Remark 4.2 part 1 and Theorem 4.3 we get the result.

Theorem 4.5 *If X is a T_o $A-$space such that each element set $\{x\}$ in X is either open or closed, then the following statements are equivalent:*

1. X is compact.
2. X is $\alpha-$compact
3. X is strongly compact.
4. The set of minimal elements m is finite.

Proof. By Theorem 2.4, X is submaximal. So X is $BB\,T_0$ $A-$space. By Remark 4.2 part 2, $(1 \Leftrightarrow 2)$. From Theorem 4.4, $(2 \Leftrightarrow 3)$. Finally, $(4 \Leftrightarrow 1)$ comes from Theorem 3.2 part 1.

Theorem 4.6 *Let X be a UBT_o $A-$space. Then X is strongly compact ($\equiv$ α-compact) iff $X-Max$ is finite set.*

Proof. Equivalent form of Corollary 3.6.

Theorem 4.7 *If X is a UBT_o $A-$space such that $|\hat{x}|=1$ for each $x \in X$, then the following statements are equivalent:*

1. X is semi-compact.
2. X is $\alpha-$compact
3. X is strongly compact.

Proof. Direct from Theorem 2.3 and Remark 4.2 part 3.

Theorem 4.8 *Let X be a UBT_o $A-$space. Then X is semi-compact if and only if X is finite.*

Proof. Since X is $\alpha-$compact, $X-Max$ is finite. Suppose to contrary that M is infinite. Let $X-M=\{x_1,x_2,x_3,\cdots,x_n\}$. Pick $y_i \in \hat{x}_i$ and set $S_i=\{x_i,y_i\}$. Then by Theorem 2.1 part 3, S_i is semi-open set. Moreover, $\mathrm{A}=\{S_i : i=1,2,\cdots,n\}\cup\{\{z\}: z\in M \text{ and } z\neq y_i\}$ is a collection of semi-open cover without finite subcover which is a contradiction. Conversely, since the power set $\mathcal{P}(X)$ is finite, any collection of semi-open cover is finite.

Theorem 4.9 *Let* $(X,\tau(\leq))$ *be a* UB T_0 $A-$*space. Then the following statements are equivalent:*

1. X is $p-$closed.
2. X is QHC.
3. There exists a finite subset N such that $V(N)(=\uparrow N)$ is dense.

Proof. From Theorem 4.4, $(1 \Leftrightarrow 2)$

$(2 \Rightarrow 3)$ The family $\{V(x): x \in X\}$ has a finite subfamily $\{V(x_1), V(x_2), \cdots, V(x_n)\}$ such that $X = \cup_{i=1}^{n} \overline{V(x_i)} = \overline{\cup_{i=1}^{n} V(x_i)}$. Set $N = \{x_1, x_2, \cdots, x_n\}$, so $X = \overline{V(N)}$.

$(3 \Rightarrow 2)$ Suppose that $N = \{x_1, x_2, \cdots, x_n\}$ and $X = \overline{V(N)}$. Let $\{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. Take $U_{\alpha_1}, U_{\alpha_2}, \cdots, U_{\alpha_n}$ to be such that $x_i \in U_{\alpha_i}$. Then $V(x_i) \subseteq U_{\alpha_i}$ for each $i = 1, 2, \cdots, n$. Therefore $X = \overline{V(N)} = \overline{\cup_{i=1}^{n} V(x_i)} = \cup_{i=1}^{n} \overline{V(x_i)} \subseteq \cup_{i=1}^{n} \overline{U_{\alpha_i}}$.

Corollary 4.10 *If* X *is a compact* $BB\,T_0$ $A-$*space, then* X *is* QHC.

Proof. Take $N = m$.

Corollary 4.11 *If* X *is a* $UB\,T_0$ $A-$*space such that* M *is finite, then* X *is* QHC.

Proof. Take $N = M$.

The reverse of the previous two corollaries need not be true. Consider the space $X = \{\top\} \cup \mathbb{N}$ where the order on $\mathbb{N}$ is anti-chain and $\forall n \in \mathbb{N}$, $n < \top$. Then X is QHC. Moreover, X is not compact since $m = \mathbb{N}$ is infinite. Consider the space $Y = \{\bot\} \cup \mathbb{N}$ where the order on $\mathbb{N}$ is anti-chain and $\forall n \in \mathbb{N}$, $n > \bot$. Then Y is compact and hence QHC, while $M = \mathbb{N}$ is infinite.

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