# The Cosgrove's SD-I equation and Bäcklund transformations of Painlevé equations

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**Abstract**

In this article, we studied the usage of the one-to-one correspondence between a given Painlevé equation and a certain second-order second-degree Painlevé-type equation to derive the Bäcklund transformations for the given Painlevé equation. We showed that all basic Bäcklund transformations of the second, third, and fourth Painlevé equations can be derived by using the one-to-one correspondences between these equations and the Cosgrove's SD-I equation.

## 1 Introduction

The Painlevé equations, PI-PVI [1], are the most well known second-order first-degree equations of Painlevé type. That is, equations whose solutions are free from movable critical points. The equations PI-PVI can not be solved in terms of the known functions. Thus they define new functions called Painlevé transcendents.

It is well known [2], ...,[13] that PII-PVI admits Bäcklund transformations that map solutions of a given Painlevé equation to solutions of the same equation but with different values of the parameters.

There are several methods to derive these transformations. One of these methods is the using of second-order second-degree Painlevé-type equation.

Assume that there is a one-to-one correspondence between solutions $v(z;\alpha)$ of a given Painlevé equation and solutions $u(z;a)$ of a second-order second-degree Painlevé-type equations. The basic idea is to find two sets of parameters $\alpha$ and $\bar{\alpha}$ which give the same solution $u(z;a)$. Thus we have two one-to-one correspondences between $u(z;a)$ and both $v(z;\alpha)$ and $\bar{v}(z;\bar{\alpha})$, where $\bar{v}(z;\bar{\alpha})$ is a solution of the given Painlevé equation with parameter $\bar{\alpha}$. Now eliminating $u(z;a)$ between theses two one-to-one correspondences, we obtain Bäcklund transformation for the given Painlevé equation. This method was first used in [4] and only two examples were given, one for PIII and another one for PVI.

An important second-order second-degree equations is the Cosgrove's SD-I equation[14]

$$\begin{aligned}(u'')^2 = -4(c_1z^3+c_2z^2+c_3z+c_4)^{-2}\Big[&c_1(zu'-u)^3\\ +c_2u'(zu'-u)^2+c_3(u')^2(zu'-u)&\\ +c_4(u')^3+c_5(zu'-u)^2+c_6u'(zu'-u)&\\ +c_7(u')^2+c_8(zu'-u)+c_9u'+c_{10}&\Big].\end{aligned} \tag{1}$$

Equation (1) splits naturally into six canonical subcases SD-I.a,..,SD-I.f. The first five subcases can be solved in terms of the Painlevé transcendents while the last one can be solved in terms of elliptic function [14].

In this article, we will use the subcases SD-I.d, SD-I.c, and SD-I.b of SD-I to derive all the basic Bäcklund transformations of PII, PIII, and PIV respectively. We will study all possible cases and explain how one can obtain the inverse of the transformation in a very simple way.

# 2 Bäcklund transformations of PII

Let $v(z;\alpha)$ be a solution of the second Painlevé equation, PII,

$$v'' = 2v^3 + zv + \alpha, \tag{2}$$

and let $u(z;a)$ be a solution of the SD-I.d equation [14, 15]

$$(u'')^2 = -4(u')^3 - 2u'(zu'-u) + a. \tag{3}$$

Then there is a one-to-one correspondence between $v$ and $u$ given by

$$2u = (v')^2 - v^4 - zv^2 - 4\mu v - \frac{z^2}{4}, \tag{4}$$

and

$$4u' = -(2\epsilon v' + 2v^2 + z), \tag{5}$$

where $a=\mu^2$, $4\mu=2\alpha+\epsilon$, and $\epsilon=\pm1$. Substituting $v'$ from (5) into (4), we obtain the following quadratic equation for $v$

$$(2u'v-\mu)^2=-4(u')^3-2u'(zu'-u)+a. \tag{6}$$

The aim is to find two values $\alpha$ and $\bar{\alpha}$ that give the same value of $a$ and hence the same value of $u$. Firstly, it should be noted that the two values of $\epsilon$ give two one-to-one correspondence between $u$ and $v$. That is, in addition to the one-to-one correspondence (5,6), there is a one-to-one correspondence given by

$$4u'=-(-2\epsilon v'+2v^2+z), \tag{7}$$

and

$$(2u'v-\hat{\mu})^2=-4(u')^3-2u'(zu'-u)+a., \tag{8}$$

where $4\hat{\mu}=2\alpha-\epsilon$. Therefore, in order to derive Bäcklund transformations for PII, we should find two sets of parameters $\{\alpha,\epsilon\}$ and $\{\bar{\alpha},\bar{\epsilon}\}$ , where $\bar{\epsilon}=\pm\epsilon$, which give the same value for $a$.

Assume that there is two sets of parameters $\{\alpha,\epsilon\}$ and $\{\bar{\alpha},\bar{\epsilon}\}$ that give the same value for $a$ , and let $\bar{v}=v(z;\bar{\alpha})$. Then, there is one-to-one correspondence between $u(z,a)$ and $\bar{v}=v(z;\bar{\alpha})$ given by

$$(2u'\bar{v}-\bar{\mu})^2=-4(u')^3-2u'(zu'-u)+a, \tag{9}$$

and

$$4u'=-(2\bar{\epsilon}\bar{v}'+2\bar{v}^2+z). \tag{10}$$

Subtracting (6) from (9), we get

$$[2u'(\bar{v}+v)-\mu-\bar{\mu}][2u'(\bar{v}-v)-\mu+\bar{\mu}]=0. \tag{11}$$

Solving equation (11) for $\bar{v}$ and substituting $u'$ from (5), gives transformations from $v$ to $\bar{v}$. Moreover, solving equation (11) for $v$ and substituting $u'$ from (10), gives transformations from $\bar{v}$ to $v$.

Since $a=\mu^2$, two sets of parameters $\{\alpha,\epsilon\}$ and $\{\bar{\alpha},\bar{\epsilon}\}$ give the same value of $a$ if and only if $\bar{\mu}^2=\mu^2$, where $\bar{\mu}=2\bar{\alpha}+\bar{\epsilon}$. Thus, we have $\mu=\bar{\mu}$ or $\mu=-\bar{\mu}$.

**Case (1) $\bar{\mu}=\mu$:**

In this case, equation (11) becomes

$$(\bar{v}-v)[u'(\bar{v}+v)-\mu]=0. \tag{12}$$

If $\bar{v}=v$, then equations (5) and (10) give $\bar{\epsilon}=\epsilon$ and hence the transformation is trivial. If $\bar{v}\neq v$, then we have

$$\bar{v}+v=\frac{\mu}{u'}. \tag{13}$$

Compatibility with equations (5) and (10) gives $\bar{\epsilon} = -\epsilon$. Solving (13) for $\bar{v}$ and using (5) to substitute $u'$ , we obtain the following Bäcklund transformations for PII

$$\bar{v} = -v - \frac{2\alpha + \epsilon}{2\epsilon v' + 2v^2 + z}, \qquad \bar{\alpha} = \alpha + \epsilon. \tag{14}$$

Solving (13) for $v$ and using (10) to substitute $u'$ , we obtain the inverse of the transformation (14)

$$v = -\bar{v} - \frac{2\bar{\alpha} - \epsilon}{2\bar{\epsilon}\bar{v}' + 2\bar{v}^2 + z}, \qquad \alpha = \bar{\alpha} - \epsilon. \tag{15}$$

Note that when $\epsilon = 1$, (14) reduces to the well known Bäcklund transformation of PII [4] and when $\epsilon = -1$, (14) gives the inverse of this transformation.
**Case (2) $\bar{\mu} = -\mu$:**
In this case, equation (11) becomes

$$(\bar{v} + v)[u'(\bar{v} - v) + \mu] = 0. \tag{16}$$

When $\bar{v} = -v$, equations (5) and (10) give $\bar{\epsilon} = -\epsilon$. Therefore we get the well known Lie-point symmetry of PII [4]

$$\bar{v} = -v, \qquad \bar{\alpha} = -\alpha. \tag{17}$$

When $\bar{v} + v \neq 0$, equation (16) gives

$$\bar{v} = v - \frac{\mu}{u'}, \tag{18}$$

where $u'$ is given by (5). In this case, equations (5) and (10) give $\bar{\epsilon} = \epsilon$ and hence $\bar{\alpha} = -\alpha - \epsilon$. It should be noted that the transformation (18) can be obtained by combining the transformations (14) and (17).

# 3 Bäcklund transformations of PIII

Let $v(z; \alpha, \beta, \gamma, \delta)$ be a solution of the PIII equation

$$v'' = \frac{1}{v}(v')^2 - \frac{1}{z}v' + \gamma v^3 + \frac{1}{z}(\alpha v^2 + \beta) + \frac{\delta}{v}. \tag{19}$$

Then, the equations [14, 15]

$$u = \frac{1}{16v^2}[(zv' - v)^2 - \gamma z^2 v^4 - 2\mu z v^3 + 2\beta z v + \delta z^2], \tag{20}$$

and

$$\dot{u} = -\frac{1}{8z}[\sqrt{\gamma}(zv' - v) + \gamma z v^2 + \mu v], \tag{21}$$

give a one-to-one correspondence between solutions $v(z;\alpha,\beta,\gamma,\delta)$ of PIII and solutions $u(x;a,b,c)$ of the SD-I.b equation

$$x^2(\ddot{u})^2 = -4(\dot{u})^2(x\dot{u}-u) - a(x\dot{u}-u) + b\dot{u} + c, \tag{22}$$

where $x = z^2$, $a = \dfrac{\gamma\delta}{16}$, $b = \dfrac{\beta\mu}{16}$, $c = \dfrac{1}{256}(\gamma\beta^2 - \delta\mu^2)$, $\mu = \alpha + 2\sqrt{\gamma}$. Eliminating $v'$ between (20) and (21) we obtain the following quadratic equation for $v$

$$(2Av + B)^2 = \Delta, \tag{23}$$

where

$$\begin{aligned} &A = \gamma(x\dot{u}-u) + \tfrac{\mu^2}{16}, \qquad B = z(\mu\dot{u} + \tfrac{\gamma\beta}{8}), \\ &\Delta = 4\gamma x[-4(\dot{u})^2(x\dot{u}-u) - a(x\dot{u}-u) + b\dot{u} + c]. \end{aligned} \tag{24}$$

In order to find Bäcklund transformations for PIII, one has to find two sets of parameters $\{\alpha,\beta,\gamma,\delta\}$ and $\{\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta}\}$ which give the same values for $a$, $b$, $c$ and hence the same value of $u$. Thus, we should have

$$\bar{\gamma}\bar{\delta} = \gamma\delta, \qquad \bar{\mu}\bar{\beta} = \beta\mu, \qquad \bar{\gamma}\bar{\beta}^2 - \bar{\delta}\bar{\mu}^2 = \gamma\beta^2 - \delta\mu^2. \tag{25}$$

Assume that $\bar{\gamma}\bar{\mu} \neq 0$. Then solving (25.a) and (25.b) for $\bar{\delta}$ and $\bar{\beta}$ and substituting in (25.c), we get

$$(\gamma\bar{\mu}^2 - \bar{\gamma}\mu^2)(\beta^2\bar{\gamma} + \delta\bar{\mu}^2) = 0. \tag{26}$$

Therefore, there are two cases to be considered: (1) $\gamma\bar{\mu}^2 - \bar{\gamma}\mu^2 = 0$ and (2) $\beta^2\bar{\gamma} + \delta\bar{\mu}^2 = 0$. If $\bar{v} = v(\bar{\alpha},\bar{\beta},\bar{\gamma},\bar{\delta})$, then we have

$$(2\bar{A}\bar{v} + \bar{B})^2 = \frac{\bar{\gamma}}{\gamma}\Delta, \tag{27}$$

where

$$\bar{A} = \frac{\bar{\gamma}}{\gamma}A + \frac{1}{16\gamma}(\gamma\bar{\mu}^2 - \bar{\gamma}\mu^2), \qquad \bar{B} = \frac{\bar{\mu}}{\mu}B - \frac{\beta z}{8\mu\bar{\mu}}(\gamma\bar{\mu}^2 - \bar{\gamma}\mu^2) \tag{28}$$

Therefore, (23) and (27) give

$$\gamma(2\bar{A}\bar{v} + \bar{B})^2 = \bar{\gamma}(2Av + B)^2. \tag{29}$$

**Case** (1) $\gamma\bar{\mu}^2 - \bar{\gamma}\mu^2 = 0$ **:**
Let $k = \dfrac{\mu}{\bar{\mu}}$. Then

$$\bar{A} = \frac{1}{k^2}A, \qquad \bar{B} = \frac{1}{k}B, \qquad \bar{\gamma} = \frac{\gamma}{k^2}, \qquad \bar{\delta} = k^2\delta, \qquad \bar{\beta} = k\beta, \tag{30}$$

and (29) gives

$$2\bar{A}\bar{v}+\bar{B}=\pm\frac{1}{k}(2Av+B). \tag{31}$$

Transformation (31) with positive sign is compatible with (21) if $\sqrt{\gamma}=k\sqrt{\bar{\gamma}}$. Thus we obtain the transformation

$$\bar{v}=kv, \qquad \bar{\alpha}=\frac{\alpha}{k}, \quad \bar{\gamma}=\frac{\gamma}{k^2}, \quad \bar{\delta}=k^2\delta, \quad \bar{\beta}=k\beta. \tag{32}$$

If $k=1$, then the transformation is trivial. If $k=-1$, then (32) give the well known Lie-point symmetry of PIII [4].

Transformation (31) with negative sign is compatible with (21) if $\sqrt{\gamma}=-k\sqrt{\bar{\gamma}}$. Thus we obtain the transformation

$$\begin{aligned}&\bar{v}=-k(v+\frac{B}{A}),\\&\bar{\alpha}=k\alpha+4k\sqrt{\gamma}, \qquad \bar{\gamma}=\frac{\gamma}{k^2}, \qquad \bar{\delta}=k^2\delta, \qquad \bar{\beta}=k\beta,\end{aligned} \tag{33}$$

where $A$ and $B$ are given by (24).

When $k=-1$, this transformation is reduced to the well known Bäcklund transformations for PIII [6]

$$\begin{aligned}&\bar{v}=v+\frac{2v^2[\sqrt{\gamma}\mu(zv'-v)+\mu\gamma zv^2+\mu^2v-\gamma\beta z]}{[\sqrt{\gamma}(zv'-v)+\gamma zv^2]^2-\mu^2v^2+2\gamma\beta zv+\delta\gamma z^2},\\&\bar{\alpha}=-(\alpha+4\sqrt{\gamma}), \quad \bar{\beta}=-\beta, \quad \bar{\gamma}=\gamma, \quad \bar{\delta}=\delta,\end{aligned} \tag{34}$$

The general case $k\neq-1$ can be obtained by combining the transformations (32) and (34).

**Case**(2) $\beta^2\bar{\gamma}+\delta\bar{\mu}^2=0$**:**

Let $k=\pm\frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}$. Then we have

$$\bar{\mu}=\frac{\epsilon\beta\sqrt{\bar{\gamma}}}{\sqrt{-\delta}}, \qquad \bar{\beta}=\frac{\epsilon\mu\sqrt{-\delta}}{\sqrt{\bar{\gamma}}}, \qquad \bar{\delta}=\frac{\delta}{k^2}, \tag{35}$$

where $\epsilon=\mp1$. Moreover, (29) gives

$$2\bar{A}\bar{v}+\bar{B}=k(2Av+B). \tag{36}$$

This is the known transformation

$$\begin{aligned}&\bar{v}=\frac{1}{k}v+\frac{[k(\alpha+2\sqrt{\gamma})-(\epsilon\beta\sqrt{\bar{\gamma}}/\sqrt{-\delta})]v^2}{k^2\sqrt{\gamma}[zv'-v+\sqrt{\gamma}zv^2+(\beta/\sqrt{-\delta})v-\sqrt{-\delta}]},\\&\bar{\alpha}=\tfrac{1}{\sqrt{\bar{\gamma}}}(2-\epsilon\beta/\sqrt{-\delta}), \qquad \bar{\beta}=\frac{\epsilon\sqrt{-\delta}(\alpha+2\sqrt{\gamma})}{\sqrt{\bar{\gamma}}}, \qquad \bar{\delta}=\frac{\gamma\delta}{\bar{\gamma}},\end{aligned} \tag{37}$$

of PIII [4].

If $\bar{\mu} = \bar{\gamma} = 0$, then PIII is solved explicitly [4]. If $\bar{\gamma} = 0$ and $\bar{\mu} \neq 0$ or $\bar{\gamma} \neq 0$ and $\bar{\mu} = 0$, then one obtains special cases of the transformation (32).

# 4 Bäcklund transformations of PIV

In this section, we will study BT for the Painlevé IV equation, PIV,

$$v'' = \frac{1}{2v}(v')^2 + \frac{3}{2}v^3 + 4zv^2 + 2(z^2 - \alpha)v + \frac{\beta}{v}. \tag{38}$$

Let $3\mu = \alpha - \epsilon$, where $\epsilon = \pm 1$. Then, the transformations

$$u = \frac{1}{8v}[(v')^2 - v^4 - 4zv^3 - 4(z^2 - 3\mu)v^2 + 8\mu zv + 2\beta], \tag{39}$$

and

$$u' = -\frac{1}{2}(\epsilon v' + v^2 + 2zv - 2\mu), \tag{40}$$

give one-to-one correspondence between PVI and the so called SD-I.c equation [14]

$$(u'')^2 = -4(u')^3 + 4(zu' - u)^2 + au' + b, \tag{41}$$

where $a = 2(6\mu^2 - \beta)$ and $b = -4\mu(2\mu^2 + \beta)$. Equation (39) can be replaced by

$$(2Av + B)^2 = \Delta, \tag{42}$$

where

$$\begin{aligned} &A = 2(u' + 2\mu), \qquad B = 4(zu' - u), \\ &\Delta = -16(u')^3 + 16(zu' - u)^2 + 4au' + 4b. \end{aligned} \tag{43}$$

Two sets of parameters $\{\alpha, \beta\}$ and $\{\bar{\alpha}, \bar{\beta}\}$ give the same solution $u(z, a, b)$ of (41) if

$$6\bar{\mu}^2 - \bar{\beta} = 6\mu^2 - \beta, \qquad \bar{\mu}(2\bar{\mu}^2 + \bar{\beta}) = \mu(2\mu^2 + \beta). \tag{44}$$

Eliminating $\bar{\beta}$ between (44.a) and (44.b), we get

$$(\bar{\mu} - \mu)[\beta + 2(2\bar{\mu} + \mu)^2] = 0. \tag{45}$$

Therefore, we should distinguish between two cases: $\bar{\mu} - \mu = 0$ and $\beta + 2(2\bar{\mu} + \mu)^2 = 0$. Since the two sets of parameters $\{\alpha, \beta\}$ and $\{\bar{\alpha}, \bar{\beta}\}$ give the same value for $u$, there is a one-to-one correspondence between $u$ and $\bar{v} = v(z; \bar{\alpha}, \bar{\beta})$ given by

$$u' = -\frac{1}{2}(\epsilon v' + v^2 + 2zv - 2\mu), \tag{46}$$

and

$$(2\bar{A}\bar{v}+\bar{B})^2=\bar{\Delta}. \tag{47}$$

Since $\bar{\Delta}=\Delta$, equations (42) and (47) give

$$(2\bar{A}\bar{v}+\bar{B})^2=(2Av+B)^2, \tag{48}$$

and hence

$$(2\bar{A}\bar{v}+\bar{B}-2Av-B)(2\bar{A}\bar{v}+\bar{B}+2Av+B)=0. \tag{49}$$

Now, using $\bar{A}=A+4(\bar{\mu}-\mu)$ and $\bar{B}=B$, we obtain

$$[A(\bar{v}-v)+4(\bar{\mu}-\mu)\bar{v}][A(\bar{v}+v)+4(\bar{\mu}-\mu)\bar{v}+B]=0. \tag{50}$$

**Case(1)** $\bar{\mu}-\mu=0$**:**
In this case, (50) becomes

$$(\bar{v}-v)[A(\bar{v}+v)+B]=0. \tag{51}$$

Therefore, in addition to the trivial transformation $\bar{v}=v$, we have the transformation

$$\bar{v}=-v-\frac{A}{B}. \tag{52}$$

Compatibility with (40) implies $\bar{\epsilon}=-\epsilon$. Therefore, we have the following well known transformation of PIV [3]

$$\bar{v}=-\frac{(\epsilon v'+v^2+2zv)^2+2\beta}{2v(\epsilon v'+v^2+2zv-2\alpha+2\epsilon)}, \qquad \bar{\alpha}=\alpha-2\epsilon, \qquad \bar{\beta}=\beta. \tag{53}$$

**Case(2)** $\bar{\mu}-\mu\neq 0$**:**
In this case, (50) becomes

$$[A(\bar{v}-v)+4(\bar{\mu}-\mu)\bar{v}][A(\bar{v}+v)+4(\bar{\mu}-\mu)\bar{v}+B]=0. \tag{54}$$

When $A(\bar{v}-v)+4(\bar{\mu}-\mu)\bar{v}=0$, compatibility with (40) gives $\epsilon=\bar{\epsilon}$. Thus we obtain the transformation

$$\begin{aligned}&\bar{v}=v+\frac{4(\bar{\mu}-\mu)v}{\epsilon v'+v^2+2zv-4\bar{\mu}-2\mu},\\ &\bar{\alpha}=\tfrac{1}{4}(6\epsilon-2\alpha\pm 3\sqrt{-2\beta}), \qquad \bar{\beta}=-\tfrac{1}{8}(2\alpha-2\epsilon\pm\sqrt{-2\beta})^2.\end{aligned} \tag{55}$$

This transformation can be found in [3, 5]

When $A(\bar{v}+v)+4(\bar{\mu}-\mu)\bar{v}+B=0$, compatibility with (40) gives $\epsilon=-\bar{\epsilon}$. Thus we get

$$\begin{aligned}&\bar{v}=-\frac{\epsilon v'+v^2+2zv+4\bar{\mu}+2\mu}{2v},\\ &\bar{\alpha}=-\tfrac{1}{4}(2\epsilon+2\alpha\pm 3\sqrt{-2\beta}), \quad \bar{\beta}=-\tfrac{1}{8}(2\alpha-2\epsilon\pm\sqrt{-2\beta})^2.\end{aligned} \tag{56}$$

The case $\epsilon=-1$ is given in [4] while the two cases $\epsilon=\pm 1$ are given in [3].

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