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Symmetrized Nearest Neighbor Kernel Estimator of the Conditional Quantiles

Raid B. Salha[1,*]
Esam B. Mahdi

[1]Department of mathematics, Faculty of Science, Islamic University of Gaza, Gaza Strip, Palestine.

* Corresponding author
e-mail address: rbsalha@iugaza.edu.ps

Abstract

Stute (1986) has introduced the symmetrized nearest neighbor (SNN) kernel estimator to estimate the conditional quantiles in the univariate case. This estimator is here extended to the multivariate case. Two methods were proposed for the derivation of the asymptotic normality of the proposed estimators. The first method considers two different quantiles estimated at the same conditional point. In the other one, the conditional quantile estimated at k different conditional points is considered. The construction of the confidence bands as well as the problem of bandwidth selection to avoid the boundary effects were discussed. Empirical studies are performed to assess the performance of the SNN kernel estimator in finite samples. Simulation results attested a reasonably good performance of the proposed estimator.

Keywords:
Nearest Neighbor estimator;
conditional quantile;
asymptotic normality;
confidence bands;
bandwidth.

1. Introduction:

Ordinary least-squares regression models the relationship between one or more covariates X and the conditional mean of a response variable Y given $X = x$. In contrast, quantile regression refers to the process of estimating the quantiles of a conditional distribution. It models the relationship between X and the conditional quantiles of Y given $X = x$. It is especially useful in applications where extremes are important, such as environmental studies, ecology, climatology, demography, biostatistics, econometrics, finance and insurance. Pairs of extreme conditional quantiles give a conditional prediction intervals within which one expects the majority of individual points to lie. For more details, see Cai (2002) and Yu (1998). Quantile regression also provides a more complete picture of the conditional distribution of Y given $X = x$ when both lower and upper or all quantiles are of interest. Because of their useful applications, estimation of the conditional quantiles has gained particular attention during the recent three decades. Hogg (1975) used the idea of the conditional quantile technique when he studied the percentiles regression lines using salary data, whereas Koenker (1978) was the first who introduced the conditional quantiles providing an extensive background and motivation from econometrics applications. Nonparametric conditional quantiles obtained by inverting a kernel estimator of the conditional distribution of the response are long established in statistics.

The asymptotic properties of nonparametric estimation of conditional quantiles using kernel or nearest neighbor methods have been studied by Samanta (1989), Sheather (1990), Roussas (1991) and Roussas (1969). Over the past two decades, some new methods of estimating conditional quantiles have been proposed. For example, Fan and Troung (1994), Yu and Jones (1998) and Yu and Jones (1997) developed an approach using a check function. An alternative procedure is first to estimate the conditional distribution functions using the double kernel local linear technique, and then to invert it to produce an estimator of the conditional quantile, which is called Yu and Jones estimator. Hall and Yao (1999) discussed the Reweighted Nadaraya- Watson (RNW) estimator of the conditional distribution function. Cai (2002) inverted the RNW estimator of the conditional distribution function to derive a nonparametric estimator for the conditional quantiles of time series data. Cai (2002) established the asymptotic normality and weak consistency for the conditional distribution RNW estimator for α-mixing processes, at both boundary and interior points. It was shown that, to the first order, the RNW estimator enjoys the same convergence rates as those for the double kernel local linear estimator of Schuster (1972). A generalization of the work of Yu and Jones for a time series is found in Gannoun and Yu (2003). Salha (2011) established the joint asymptotic normality of the conditional quantiles and tested their performance in constructing prediction intervals. Schuster (1972) established the joint asymptotic normality of the conditional regression function estimated at a finite number of distinct points. Stute (1986) considered the problem of estimating the conditional quantile using the SNN kernel estimator. Ducharme and Jequier (2008) used the results of Stute (1986) to construct confidence intervals. In this paper, we consider the SNN kernel estimator of the conditional quantile. We generalized the result of Stute (1986) to the multivariate case by considering two approaches. In the first one, we derive the asymptotic normality of the joint distribution of two different quantiles estimated at the same conditional point. In the other one, we derive the asymptotic normality of the conditional quantile estimated at k different conditional points, $k \geq 2$, by using the techniques of Schuster (1972). We discuss the method of constructing the confidence bands and the selection of the optimal bandwidth. The performance of the SNN kernel estimator in estimating the conditional quantiles is tested using simulated data. The paper is organized as follows: In Section 2 we introduce the SNN estimator of the conditional quantiles. We state the main assumptions and results proving our contributed lemmas and theorems in Section 3. The construction of the confidence bands is discussed in Section 4, while in Section 5 we discuss the problem of the optimal bandwidth selection. The practical performance of the SNN kernel estimator is tested in Section 6 using simulated data. Finally, we summarize some conclusion remarks in Section 7.

1. SNN Estimator of the Conditional Quantile

Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be two-dimensional random vectors which are iid as (X,Y) with joint density function $f(x,y)$ and has a marginal density function $g(x)$ of X. Let $F(y|x)$ denotes the conditional distribution of Y given $X=x$.

Definition 1 (Empirical Distribution Function)

Given an observed random sample $x_1,x_2,\ldots,x_n$, an empirical distribution function $\hat{G}_n(x)$ is the fraction of sample observations less than or equal to the value x. More specifically, if $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ are the order statistics of the observed random sample, with no two observations being equal, then the empirical distribution function is defined as:

$$\hat{G}_n(x)=\begin{cases}0, & \text{for } x<x_{(1)},\\ \dfrac{k}{n}, & \text{for } x_{(k)}\leq x<x_{(k+1)},\ k=1,2,\ldots,n-1,\\ 1, & \text{for } x\geq x_{(n)}.\end{cases}$$

That is, for the case in which no two observations are equal, the empirical distribution function is a "step" function that jumps $1/n$ in height at each observation x_k. For the cases in which two (or more) observations are equal, that is, when there are n_k observations at x_k, the empirical distribution function is a "step" function that jumps n_k/n in height at each observation x_k.

Stute (1986) has introduced the following symmetrized nearest neighbor (SNN) kernel estimator of the conditional cdf, $F(y \mid x)$.

Definition 2 (kernel estimator of $F(y \mid x)$)

The SNN kernel estimator of $F(y \mid x)$ is denoted by $\hat{F}(y \mid x)$ and defined as

$$\hat{F}(y \mid x) = \frac{\sum_{i=1}^{n} I(Y_i \le y) K\left(\frac{\hat{G}_n(x) - \hat{G}_n(X_i)}{h_n}\right)}{\sum_{i=1}^{n} K\left(\frac{\hat{G}_n(x) - \hat{G}_n(X_i)}{h_n}\right)}, \qquad (1)$$

where:

1. $K(\cdot)$ is a continuous and bounded probability kernel on [-1.1].

2. $\hat{G}_n(\cdot)$ denotes the empirical distribution function of values of X.

3. $I(\cdot)$ is the indicator function.

4. The bandwidth h_n is a sequence of positive number.

This estimator has many interesting statistical properties. The SNN kernel estimator possesses better stability properties at the boundaries than the ordinary kernel estimators. (See Ducharme and Jequier (2008)).

Definition 3 (kernel estimator of a conditional quantile)

The α - conditional quantile of $F(y \mid x)$ is denoted by $q_\alpha(x)$ and is defined by

$$q_\alpha(x) = F^{-1}(\alpha \mid x) = inf\ \{y \mid F(y \mid x) \ge \alpha\}. \qquad (2)$$

The SNN kernel estimator of $q_\alpha(x)$ is denoted by $\hat{q}_\alpha(x)$ and it is defined by

$$\hat{q}_\alpha(x) = \hat{F}^{-1}(\alpha \mid x) = inf\ \{y \mid \hat{F}(y \mid x) \ge \alpha\}. \qquad (3)$$

In this paper, we generalize the results of Stute (1986) to the multivariate case by considering two approaches. In the first, we will derive the asymptotic normality of the joint distribution of two different quantiles $q_{\alpha_1}(x)$ and $q_{\alpha_2}(x)$, $0 < \alpha_1 < \alpha_2 < 1$, estimated at the same conditional point x. Secondly, we will derive the asymptotic normality of $q_\alpha(x_1)$, $q_\alpha(x_2), \ldots$ and $q_\alpha(x_k)$ the $\alpha-$conditional quantile, $q_\alpha(.)$, estimated at k different conditional points, $x_1, x_2, \ldots, x_k$. For simplicity, we will consider the bivariate case, i.e. when $k = 2$.

2. Main Results

In this section, the two main theorems in this paper will be stated and proved. We consider the following assumptions that will be used in proving the main results of this paper.

Assumption (A1) The kernel function K is probability

$$\begin{aligned}\sup_{y \in R} |\hat{f}(y \mid x) - f(y \mid x)| &= \sup_{y \in R} \left| \frac{\hat{f}(x,y)}{\hat{g}(x)} - \frac{f(x,y)}{g(x)} \right| \\ &= \sup_{y \in R} \left| \frac{\hat{f}(x,y)}{\hat{g}(x)} - \frac{f(x,y)}{\hat{g}(x)} + \frac{f(x,y)}{\hat{g}(x)} - \frac{f(x,y)}{g(x)} \right| \\ &\le \sup_{y \in R} \left| \frac{\hat{f}(x,y)}{\hat{g}(x)} - \frac{f(x,y)}{\hat{g}(x)} \right| + \sup_{y \in R} \left| \frac{f(x,y)}{g(x)} \right| \left| \frac{g(x)}{\hat{g}(x)} - 1 \right| \\ &\le \frac{\sup_{y \in R} |\hat{f}(x,y) - f(x,y)|}{\hat{g}(x)} + V(x) \left| \frac{g(x)}{\hat{g}(x)} - 1 \right|.\end{aligned}$$

density function satisfying the following:

i. K has a compact support.

ii. K is symmetric probability density function.

iii. K is Lipschitz continuous.

Assumption (A2) The bandwidth $\{h_n\}$ satisfies

$\lim_{n \to \infty} h_n = 0$ and $\lim_{n \to \infty} n h_n = \infty$.

Assumption (A3) For fixed y and x, $0 < F(y \mid x) < 1$, $F^{(2,0)}(y \mid x) = \dfrac{\partial^2 F(y \mid x)}{\partial x^2}$ exist and

$F^{(0,2)}(y\mid x)=\dfrac{\partial^2 F(y\mid x)}{\partial y^2}$ exists in neighborhoods of x and y respectively. Now consider the following

$$S_{nr}=\hat{F}(y_r\mid x)-\mathrm{E}[\hat{F}(y_r\mid x)],\ r=1,2, \tag{4}$$

$$W_{ni}(x)=\frac{K\left(\dfrac{\hat{G}_n(x)-\hat{G}_n(X_i)}{h_n}\right)}{\sum_{i=1}^{n}K\left(\dfrac{\hat{G}_n(x)-\hat{G}_n(X_i)}{h_n}\right)},\ i=1,\dots,n \tag{5}$$

$$Z_{ri}=W_{ni}(x)I_{(Y_i\le y_r)}-\mathrm{E}[W_{ni}(x)I_{(Y_i\le y_r)}],\ i=1,..,n,\ r=1,2 \tag{6}$$

$$S_{nr}=\sum_{i=1}^{n}Z_{ri}=\hat{F}(y_r\mid x)-\mathrm{E}[\hat{F}(y_r\mid x)]. \tag{7}$$

Define S_n as follows: $S_n=c_1S_{n1}+c_2S_{n2}$, (8)

where c_1 and c_2 are constants. Now we state two lemmas that will help us in proving the two main results, Theorem 1 and Theorem 2.

Lemma 1. Under the assumptions A1-A3 and as $n\to\infty$, the following holds.

1. $\hat{f}(y\mid x)\xrightarrow{d} f(y\mid x)$,

2. $\hat{q}_\alpha(x)\xrightarrow{d} q_\alpha(x)$, where $\xrightarrow{d}$ denotes convergence in distribution.

Proof

(i) let $V(x)$ be an upper bound for $\dfrac{f(x,y)}{g(x)}$, $y\in\boldsymbol{R}$

.

Now, an application of Theorem 1 in Samanta (1989) implies that

$$\hat{f}(y\mid x)\xrightarrow{d} f(y\mid x).$$

(ii) Follows from Theorem 3 in Stute (1986).

Lemma 2. Under the assumptions A1-A3, and as $n\to\infty$, the following holds

$$\left(\hat{F}(y_1\mid x)-\mathrm{E}[\hat{F}(y_1\mid x)],\hat{F}(y_2\mid x)-\mathrm{E}[\hat{F}(y_2\mid x)]\right)^T \xrightarrow{d} N(0,\Sigma), \tag{9}$$

where $0=(0,0)^T$ and Σ is the covariance matrix with the $(i,j)^{th}$ element $\sigma_{ij}, 1\le i\le j\le 2$ is given by

$$\sigma_{ij}=\frac{\hat{F}(y_i\mid x)[1-\hat{F}(y_j\mid x)]}{nh_n}\int_{-\infty}^{\infty}K^2(u)du,$$

Proof

Following the same technique as in Abberger (1997), we get that

$$S_n=\sum_{r=1}^{2}c_rS_{nr}\xrightarrow{d} N(0,\sigma^2).$$

where

$$\sigma^2=\sum_{r=1}^{2}\frac{\hat{F}(y_r\mid x)[1-\hat{F}(y_r\mid x)]}{nh_n}\int_{-\infty}^{\infty}K^2(u)du.$$

This implies to

$$\sum_{r=1}^{2}c_r(\hat{F}(y_r\mid x)-\mathrm{E}[\hat{F}(y_r\mid x)])\xrightarrow{d} N(0,\sigma^2).$$

Now, an application of the Cramér-Wold theorem completes the proof of the lemma. In Theorem 1 we derive the joint asymptotic normality of the regression quantile for different values of α, $0<\alpha<1$.

Theorem 1. Suppose that $f(q_{\alpha_i}(x)\mid x)>0$, $i=1,2$, $0<\alpha_1<\alpha_2<1$, then under the assumptions A1 - A3, the following holds.

$$\sqrt{nh_n}\left[\left(\hat{q}_{\alpha_1}(x)-q_{\alpha_1}(x),\hat{q}_{\alpha_2}(x)-q_{\alpha_2}(x)\right)^T\right]\xrightarrow{d} N(0,\Lambda) \tag{10}$$

where $0=(0,0)^T$ and the covariance matrix Λ is defined as $\Lambda=\dfrac{\int_{-\infty}^{\infty}K^2(u)du}{g(x)}\begin{bmatrix}\lambda_{11} & \lambda_{12}\\ \lambda_{21} & \lambda_{22}\end{bmatrix}$, (11)

where

$$\lambda_{ij} = \frac{\alpha_i(1-\alpha_j)}{f(q_{\alpha_i}(x)\mid x)\, f(q_{\alpha_j}(x)\mid x)}, 1\le i\le j\le 2. \quad (12)$$

Proof

Now, by expanding $\hat{F}(\hat{q}_{\alpha_i}(x)\mid x)$ around $q_{\alpha_i}(x)$, $i=1,2,$ we get

$$F(q_{\alpha_i}(x)\mid x) = \hat{F}(\hat{q}_{\alpha_i}(x)\mid x) = \hat{F}(q_{\alpha_i}(x)\mid x) + (\hat{q}_{\alpha_i}(x_i) - q_{\alpha_i}(x))\hat{f}(\hat{q}^*_{\alpha_i}(x)\mid x)$$

where $\hat{q}^*_{\alpha_i}(x)$ is some random point between $\hat{q}_{\alpha_i}(x)$ and $q_{\alpha_i}(x)$. This implies

$$\hat{q}_{\alpha_i}(x) - q_{\alpha_i}(x) = \frac{F(q_{\alpha_i}(x)\mid x) - \hat{F}(q_{\alpha_i}(x)\mid x)}{\hat{f}(\hat{q}^*_{\alpha_i}(x)\mid x)}. \quad (13)$$

From Equation (13), we obtain that

$$\sqrt{nh_n}\begin{bmatrix} \hat{q}_{\alpha_1}(x) - q_{\alpha_1}(x) \\ \hat{q}_{\alpha_2}(x) - q_{\alpha_2}(x) \end{bmatrix} = \begin{bmatrix} \sqrt{nh_n}\left(\dfrac{F(q_{\alpha_1}(x)\mid x) - \hat{F}(q_{\alpha_1}(x)\mid x)}{\hat{f}(\hat{q}^*_{\alpha_1}(x)\mid x)}\right) \\ \sqrt{nh_n}\left(\dfrac{F(q_{\alpha_2}(x)\mid x) - \hat{F}(q_{\alpha_2}(x)\mid x)}{\hat{f}(\hat{q}^*_{\alpha_2}(x)\mid x)}\right) \end{bmatrix} = \mathbf{W_n X_n},$$

where

$$\mathbf{W_n} = \begin{bmatrix} \dfrac{1}{\hat{f}(\hat{q}^*_{\alpha_1}(x)\mid x)} & 0 \\ 0 & \dfrac{1}{\hat{f}(\hat{q}^*_{\alpha_2}(x)\mid x)} \end{bmatrix},$$

$$\mathbf{X_n} = \begin{bmatrix} \sqrt{nh_n}F(q_{\alpha_1}(x)\mid x) - \hat{F}(q_{\alpha_1}(x)\mid x) \\ \sqrt{nh_n}F(q_{\alpha_2}(x)\mid x) - \hat{F}(q_{\alpha_2}(x)\mid x) \end{bmatrix}.$$

From Corollary 1 in Abberger (1997), the following holds

$$\sqrt{nh_n}(\mathrm{E}[\hat{F}(y\mid x)] - F(y\mid x)) = o(1). \quad (14)$$

Now, a combination of Lemma 2 and (14) implies that

$$\mathbf{X_n} \xrightarrow{d} \mathbf{X} \sim N(\mathbf{0}, \Delta),$$

where $\mathbf{0} = (0,0)^T$ and Δ is the covariance matrix with the $(i,j)^{th}$ element $\delta_{ij}, 1\le i\le j\le 2$ is given by

$$\delta_{ij} = F(y_i\mid x)[1-F(y_j\mid x)]\int_{-\infty}^{\infty}K^2(u)d$$
$$= \begin{bmatrix} \alpha_1(1-\alpha_1) & \alpha_1(1-\alpha_2) \\ \alpha_1(1-\alpha_2) & \alpha_2(1-\alpha_2) \end{bmatrix}\int_{-\infty}^{\infty}K^2(u)du.$$

Let

$$\mathbf{W} = \begin{bmatrix} \dfrac{1}{f(q_{\alpha_1,n}(x)\mid x)} & 0 \\ 0 & \dfrac{1}{f(q_{\alpha_2,n}(x)\mid x)} \end{bmatrix}.$$

From Lemma 1, we have

$$\hat{f}(\hat{q}^*_{\alpha_i}(x)\mid x) \xrightarrow{d} f(q_{\alpha_i}(x)\mid x), i=1,2. \quad (15)$$

Now, (15) implies that

$$tr\{(\mathbf{W_n} - \mathbf{W})^T(\mathbf{W_n} - \mathbf{W})\} \xrightarrow{d} 0.$$

Now, using Slutsky theorem (See Theorem 3.4.3 in Pranab and Singert (1993)), we obtain

$$\mathbf{W_n X_n} \xrightarrow{d} \mathbf{WX},$$

which completes the proof of the Theorem. In Theorem 2 we derive the joint asymptotic normality of the regression quantile for k different points $x_1, x_2, ..., x_k$.

Theorem 2. Suppose that $f(q_{\alpha,n}(x_i)\mid x_i) > 0$, $i=1,2,...,k, 0<\alpha<1$ then under the assumptions A1-A3, the following holds.

$$\sqrt{nh_n}\left[(\hat{q}_\alpha(x_1)-q_\alpha(x_1),\ldots,\hat{q}_\alpha(x_k)-q_\alpha(x_k))^T\right] \xrightarrow{d} N(0,\Delta),$$

where $0=(0,0,\ldots,0)^T$ and Δ is a diagonal covariance matrix with the $(i,i)th$ element

$$\delta_{ii}=\frac{\alpha(1-\alpha)}{f^2(q_\alpha(x_i)\,|\,x_i)}\int_{-\infty}^{\infty}K^2(u)du, i=1,2,\ldots,k.$$

Proof

It is enough to prove the theorem for two different conditional points as in Schuster (1972). The proof can be obtained by using the same techniques as in Schuster (1972) and Salha (2011), which has been used in the proof of Theorem 1.

3. Confidence Bands

A natural way of constructing a confidence band for $q_\alpha(x)$, $0<\alpha<1$, is as follows. Suppose $\hat{q}_\alpha(x)$ is an estimator of $q_\alpha(x)$, then a $100(1-\beta)\%$, $0<\beta<1$, confidence band is of the form

$$P\left[|\hat{q}_\alpha(x)-q_\alpha(x)|\le d\right]\ge 1-\beta, \infty\,\forall x\in[0,1]. \quad (16)$$

To find a good solution to (16), we must derive the asymptotic distribution of $\hat{q}_\alpha(x)-q_\alpha(x)$, then a good estimator of the bandwidth h_n must be computed. Now, using the asymptotic normality results from Theorem 2 for univariate random design case, we have the following result, which is the same result from Theorem 3 in Stute (1986).

$$\sqrt{nh_n}[\hat{q}_\alpha(x)-q_\alpha(x)]\xrightarrow{d}N(0,R(K)\sigma_\alpha^2(x)) \quad (17)$$

where

$$\sigma_\alpha^2(x)=\frac{\alpha(1-\alpha)}{f^2(q_\alpha(x)\,|\,x)},$$

$$R(K)=\int_{-\infty}^{\infty}K^2(x)dx<\infty.$$

From (17), we have

$$\frac{\sqrt{nh_n}[\hat{q}_\alpha(x)-q_\alpha(x)]}{\sqrt{R(K)\sigma_\alpha^2(x)}}\xrightarrow{d}N(0,1). \quad (18)$$

Dharmasena and Silva (2008) have used (18) to produce the following $(1-\beta)100\%$ confidence interval for $q_\alpha(x)$

$$\left(\hat{q}_\alpha(x)-z_{\frac{\beta}{2}}\hat{\sigma}_\alpha(x)\sqrt{\frac{R(K)}{nh_n}},\hat{q}_\alpha(x)+z_{\frac{\beta}{2}}\hat{\sigma}_\alpha(x)\sqrt{\frac{R(K)}{nh_n}}\right), \quad (19)$$

where $\hat{\sigma}_\alpha(x)=\frac{\sqrt{\alpha(1-\alpha)}}{\hat{f}(q_\alpha(x)\,|\,x)}$ is the kernel estimator of $\sigma_\alpha(x)$.

4. Bandwidth Selection

In this section, we discuss the problem of the selection of the optimal bandwidth h_n, which plays an important role in kernel estimation. As in Dharmasena and Silva (2008), we take $h_n=n^{-r}$. Using the assumption (A2), we have $r\in(0,1)$. Also to avoid boundary effects, the conditional point x_0 at which the estimation is taking place is selected such that $h_n<x_0$ and $x_0<1-h_n$. This means that the bandwidth must satisfies,

$$h_n<\min\{x_0,1-x_0\}.$$

Now, $n^{-r}<\min\{x_0,1-x_0\}$ then solving for r, we get

$$r>\left\{\frac{-ln[\min\{x_0,1-x_0\}]}{lnn}\right\}=r_0. \quad (16)$$

Since $0<r<1$, from the above we have that $r\in(\max\{0,r_0\},1)=(r_{\min},1)$, where $r_{\min}=\max\{0,r_0\}$.

5. Simulation studies

In this section the performance of the SNN kernel estimator in estimating the conditional quantiles is tested using simulated data. The performance of the estimator has been tested using the correlation coefficients between $\hat{y}$ the predicated values and y the actual values,

$$R^2_{y,\hat{y}} = 1 - \frac{SSE}{SSTO} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2},$$

where, n is the sample size, $\bar{y}$ denotes the mean of actual values, SSE denotes the total sum of errors, and $SSTO$ denotes the total sum of squares. Also, the mean squared error, $MSE = \frac{SSE}{n}$ has been computed. Three samples each of size 200 are simulated from the following three models

$$y = x\sin(2\pi x) + e \quad (17)$$

$$y = x^3 + xe \quad (18)$$

$$y = \sin(2\pi(1-x)^2) + xe \quad (19)$$

where $x \sim N(0,1)$ and $e \sim U[0,1]$, where N and U stand for Normal and Uniform distribution respectively. For the first sample, Figure 1 presents the estimated conditional median, %5 quantile and %95 quantile. For a direct comparison of the perfect curve and the conditional median estimator, a scatter plot of the original data, the perfect curve and the estimated conditional median curve are plotted in Figure 2.

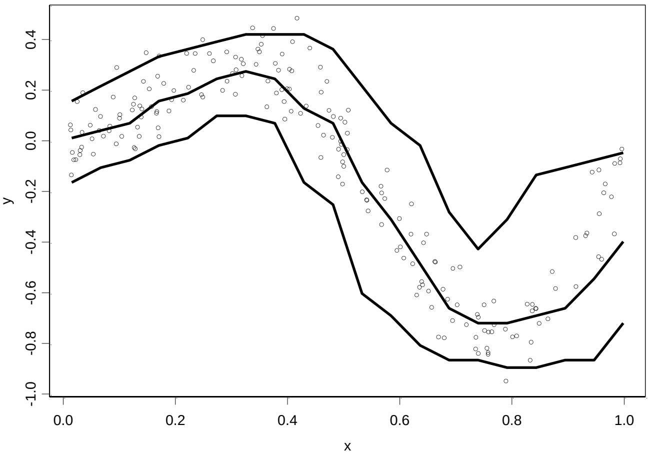

Figure 1: SNN kernel estimation of the conditional quantiles for the model $y = x\sin(2\pi x) + e$

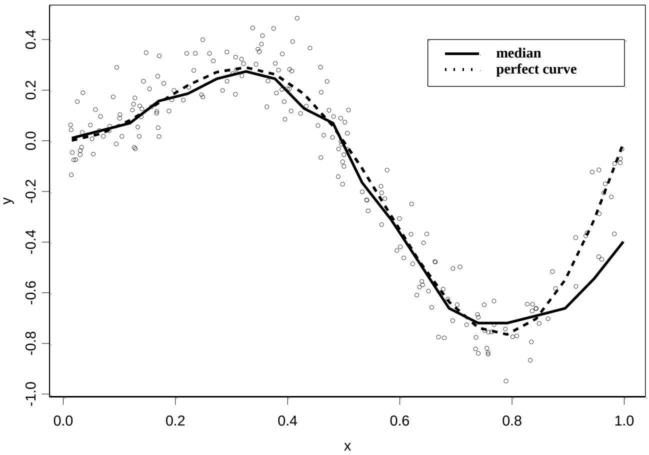

Figure 2: The true perfect curve and its median estimation for the model $y = x\sin(2\pi x) + e$

For the second sample, Figure 3 presents the estimated conditional median, %10 quantile and %90 quantile. For a direct comparison of the perfect curve and the conditional median estimator, a scatter plot of the original data, the perfect curve and the estimated conditional median curve are plotted in Figure 4.

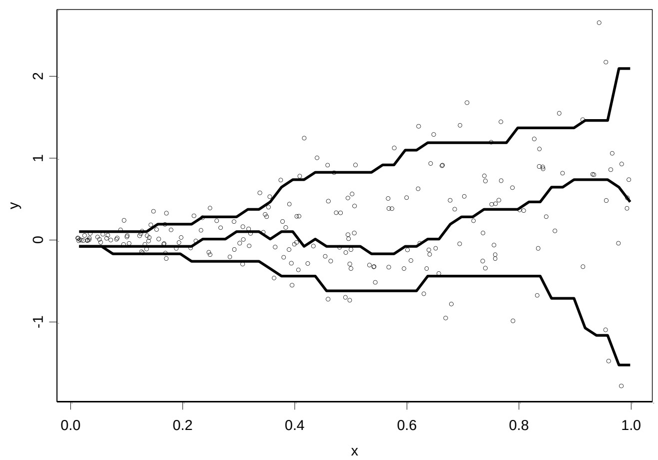

Figure 3: SNN kernel estimation of the conditional quantiles for the model $y = x^3 + xe$,

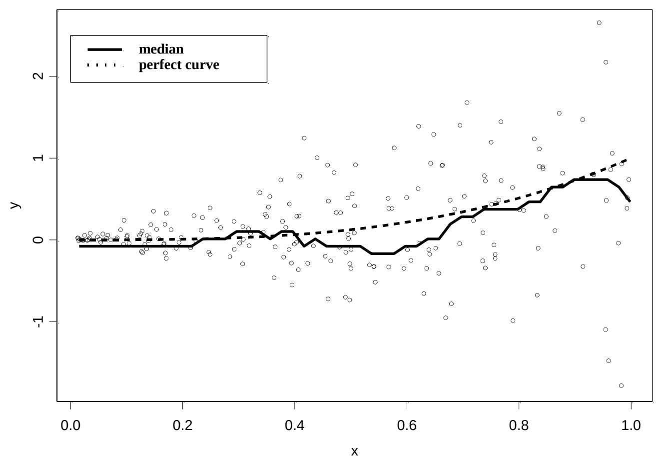

Figure 4: The true perfect curve and its median estimation for the model $y = x^3 + xe$,

For the third sample, Figure 5 presents the estimated conditional median, %10 quantile and %90 quantile. Also, for a direct comparison of the perfect curve and the conditional median estimator, a scatter plot of the original data, the perfect curve and the estimated conditional median curve are plotted in Figure 6.

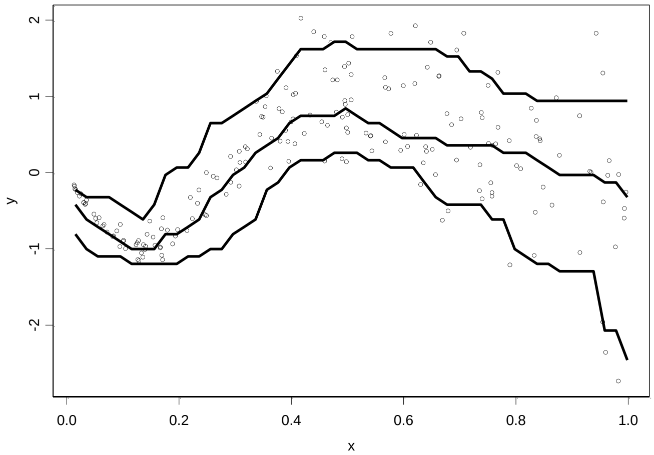

Figure 5: SNN kernel estimation of the conditional quantiles for the model $y = \sin(2\pi(1-x)^2) + xe$,

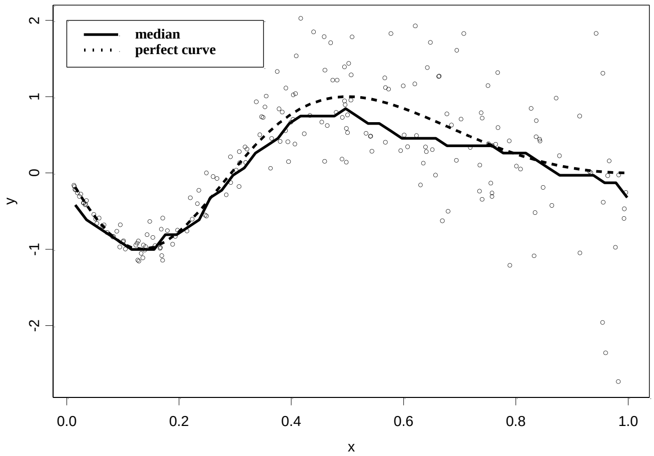

Figure 6: The true perfect curve and its median estimation for the model $y = \sin(2\pi(1-x)^2) + xe$,

Table 1 ***Performance measurement of SNN kernel estimators.***

	Model 1	Model 2	Model 3
$R^2_{y,\hat{y}}$	0.9126	0.6667	0.8147
MSE	0.0115	0.0282	0.0297

Source: (Simulation results).

We conclude from the figures and the results in Table 1 that the performance of the SNN kernel estimator of the conditional quantile is reasonably good.

6. Conclusion

Although numerous studies on the conditional distribution function $F(y|x)$ were made based on the conditional mean (or median) function, more investigations of other aspects of $F(y|x)$ are still needed. A deeper insight into the response variable Y given values x of a predictor variable X can be gained by considering the conditional quantiles functions. In literature, individual quantiles such as conditional median are commonly used to estimate specific threshold values associated with $F(y|x)$. However, in practice, we often wish to obtain a collection of conditional quantiles that can be used to characterize the entire conditional distribution $F(y|x)$. In this paper, we considered the SNN kernel estimation of the conditional quantile. The results from Stute (1986) were extended to the multivariate case by considering two approaches. In the first one, we derived the asymptotic normality of the joint distribution of two different quantiles estimated at the same conditional point. In the other one, we derived the asymptotic normality of the conditional quantile estimated at $k>1$ different conditional points. The constructing of confidence bands as well as the problem of bandwidth selection to avoid the boundary effects were discussed. Three different empirical simulation studies were conducted and the results indicated that the performance of the SNN kernel estimator is reasonably good.

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