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ON SOME INTEGRAL INEQUALITIES OF OU- IANG'S TYPE

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Abstract: In this article, we establish some nonlinear integral inequalities of Iang's type for functions of one or more than one independent variable . Also, we give some applications of these inequalities to study boundedness of solutions of nonlinear partial differential equations.

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1. Introduction

Integral inequalities play an important role in the study of existence, uniqueness, continuous dependence, perturbation, boundedness and stability of solutions of differential and integral equations [2, 3, 4].

The integral inequalities involving functions of more than one independent variables play a fundamental role in the study of the solutions of partial differential equations[1]. One of the most useful inequalities in the development of the theory of partial differential equations is ***Iang's* inequality** which was first given by **Ou-Iang** [2] while studying the boundedness of solutions of certain second order differential equations .

Theorem 1.1 [2] Let *u* and *f* be nonnegative continuous functions defined for $t \in R_+$. If

$$u^2(t) \le c^2 + 2\int_0^t f(s)u(s)ds, \tag{1.1}$$

for $t \in R_+$, where $c \geq 0$ is a constant , then

$$u(t) \le c + \int_0^t f(s)ds, \quad for \quad t \in R_+ . \tag{1.2}$$

In the past few years, ***Iang's* inequality** has been applied with considerable success to study the global existence , uniqueness, stability and other properties of the solutions of various nonlinear differential equations. Generalizations of the inequality given in **Theorem 1.1** are given in **[5, 6]**. In the following theorem, we present one of those generalizations due to **Pachpatte [5].**

Theorem 1.2 Let u, f and g be nonnegative continuous functions defined for $t \in R_+$ and c be a nonnegative constant . If

$$u^2(t) \le c^2 + 2\int_0^t [f(s)u^2(s) + g(s)u(s)]ds, \tag{1.3}$$

for $t \in R_+$, then

$$u(t) \le p(t)\exp(\int_0^t f(s)ds), \quad for \quad t \in R_+ \tag{1.4}$$

where

$$p(t) = c + \int_0^t g(s)ds, \quad for \quad t \in R_+ . \tag{1.5}$$

The purpose of this paper, is to establish some new integral inequalities of Iang's type [2] for functions of one and more than one independent variables which can be used in the analysis of some classes of partial differential equations.

2. Main Results

In what follows, R denotes the set of real numbers and $R_+ = [0,\infty)$. The first order partial derivatives of a function $Z(x,y)$ defined for $x, y \in R$ with respect to *x* and *y* are denoted by $Z_x(x,y)$ and $Z_y(x,y)$ respectively.

Throughout this paper , all the functions and their partial derivatives which appear in the inequalities are assumed to be real – valued and all the integrals involved are exist and of positive values on the respective domains of their definitions . Our main results are given in the following three theorems :

Theorem 2.1 Let $u(x)$, $h(x)$ be nonnegative continuous functions and k(x) > 1, defined for $x \in R_+$. Suppose that $k_x(x)$ be nonnegative continuous function defined for $x \in R_+$, p_1, p_2, p_3, and $p_4 \in R_+$. If

$$u^{p_1}(x) \le k^{p_2}(x) + p_3\int_0^x h(s)u^{p_4}(s)ds, \tag{2.1}$$

then

$$u(x) \le \left[k^{\frac{p_2}{p_1}(p_1-p_4)}(0) + \left(\frac{p_1-p_4}{p_1}\right)\left(k^{p_2}(x) - k^{p_2}(0) + p_3\int_0^x h(s)ds\right)\right]^{\frac{1}{p_1-p_4}},$$

$p_1 \ne p_4$. (2.2)

Proof : Let $V^{p_1}(x)$ be the right hand side of (2.1), then

$$V^{p_1}(x) = k^{p_2}(x) + p_3\int_0^x h(s)u^{p_4}(s)ds. \qquad (2.3)$$

From (2.1), (2.3), we have $V(x) > 0$ and

$$u(x) \le V(x). \qquad (2.4)$$

Differentiating (2.3) with respect to x, we get

$$p_1V^{p_1-1}(x)V_x(x) = p_2k^{p_2-1}(x)k_x(x) + p_3h(x)u^{p_4}(x). \qquad (2.5)$$

From (2.4) and (2.5), we have

$$p_1V^{p_1-1}(x)V_x(x) \le p_2k^{p_2-1}(x)k_x(x) + p_3h(x)V^{p_4}(x),$$

from which and by using (2.4), we get

$$p_1V^{p_1-p_4-1}(x)V_x(x) \le \frac{p_2k^{p_2-1}(x)k_x(x)}{V^{p_4}(x)} + p_3h(x). \qquad (2.6)$$

Since $k(x) > 1$, from (2.3) and (2.6), we obtain

$$p_1V^{p_1-p_4-1}(x)V_x(x) \le p_2k^{p_2-1}(x)k_x(x) + p_3h(x). \qquad (2.7)$$

Integrating (2.7) with respect to x from 0 to x , we get

$$\left(\frac{p_1}{p_1 - p_4}\right)\left(V^{p_1-p_4}(x) - V^{p_1-p_4}(0)\right) \le k^{p_2}(x) - k^{p_2}(0) + p_3\int_0^x h(s)ds. \qquad (2.8)$$

From (2.3) and (2.8), we have

$$V(x) \le \left(k^{\frac{p_2}{p_1}(p_1-p_4)}(0) + \frac{p_1 - p_4}{p_1}\left[k^{p_2}(x) - k^{p_2}(0) + p_3\int_0^x h(s)ds\right]\right)^{\frac{1}{p_1-p_4}}, p_1 \ne p_4. \qquad (2.9)$$

From (2.4) and (2.9) we have the required result. Hence the theorem is proved .

In the following, we obtain some special cases of the results in Theorem 2.1 by suitable substitutions given for each case :

Special Cases I.

1) Let $k(x) = c$, c is a positive constant and $p_4 = 1$. From **Theorem 2.1**, we have, if

$$u^{p_1}(x) \le c^{p_2} + p_3 \int_0^x h(s)u(s)ds,$$

then

$$u(x) \le \left(c^{\frac{p_2}{p_1}(p_1-1)} + \frac{p_3}{p_1}(p_1-1) \int_0^x h(s)ds \right)^{\frac{1}{p_1-1}}, \quad p_1 \ne 1. \tag{2.10}$$

2) Let $k(x) = c \ge 0$. From **Theorem 2.1**, we have, if

$$u^{p_1}(x) \le c^{p_2} + p_3 \int_0^x h(s)u^{p_4}(s)ds,$$

then

$$u(x) \le \left(c^{\frac{p_2}{p_1}(p_1-p_4)} + \frac{p_3}{p_1}(p_1 - p_4) \int_0^x h(s)ds \right)^{\frac{1}{p_1-p_4}}, \quad p_1 \ne p_4. \tag{2.11}$$

3) If $p_1 = p_2 = p_3 = 2$ in (2.10), we have , if

$$u^2(x) \le c^2 + 2\int_0^x h(s)u(s)ds,$$

then

$$u(x) \le c + \int_0^x h(s)ds, \tag{2.12}$$

which is the same result of **Iang** [2].

4) If $p_2 = p_3 = 1$, *and* $p_1 = 2$ in case 1, we have, if

$$u^2(x) \le c + \int_0^x h(s)u(s)ds,$$

then

$$u(x) \le \sqrt{c} + \frac{1}{2}\int_0^x h(s)ds. \tag{2.13}$$

5) If $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$ in case 2, we have, if

$$u^2(x) \le c^2 + 2\int_0^x h(s)u(s)ds,$$

then

$$u(x) \le c + \int_0^x h(s)ds, \tag{2.14}$$

which is the same result of **Iang** .

6) If $p_2 = p_3 = p_4 = 1$ *, and* $p_1 = 2$ in case 2, we have, if

$$u^2(x) \le c + \int_0^x h(s)u(s)ds,$$

then

$$u(x) \le \sqrt{c} + \frac{1}{2}\int_0^x h(s)ds. \tag{2.15}$$

which is the same result (2.13).

7) If $p_1 = p_2 = p_3 = 1$ and $p_4 = 2$ in case 2, we have, if

$$u(x) \le c + \int_0^x h(s)u^2(s)ds,$$

then

$$u(x) \le \left[c^{-1} - \int_0^x h(s)ds \right]^{-1}. \tag{2.16}$$

8) If $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$ in **Theorem 2.1**, we have, if

$$u^2(x) \le k^2(x) + 2\int_0^x h(s)u(s)ds,$$

then

$$u(x) \le k(0) + \frac{1}{2}k^2(x) - \frac{1}{2}k^2(0) + \int_0^x h(s)ds. \tag{2.17}$$

9) If $p_1 = p_2 = p_3 = 1$ and $p_4 = 2$ in **Theorem 2.1**, we have, if

$$u(x) \le k(x) + \int_0^x h(s)u^2(s)ds,$$

then

$$u(x) \le \left[k^{-1}(0) - k(x) + k(0) - \int_0^x h(s)ds \right]^{-1}. \qquad (2.18)$$

10) If $p_2 = p_3 = p_4 = 1$ and $p_1 = 2$ in **Theorem 2.1**, we have , if

$$u^2(x) \le k(x) + \int_0^x h(s)u(s)ds,$$

then

$$u(x) \le \frac{1}{2}\left[2k^{\frac{1}{2}}(0) + k(x) - k(0) + \int_0^x h(s)ds \right]. \qquad (2.19)$$

11) If $p_1 = p_2 = p_3 = 1$ and $p_4 = \frac{1}{2}$ in **Theorem 2.1** , we have , if

$$u(x) \le k(x) + \int_0^x h(s)u^{\frac{1}{2}}(s)ds,$$

then

$$u(x) \le \left[k^{\frac{1}{2}}(0) + \frac{1}{2}k(x) - \frac{1}{2}k(0) + \frac{1}{2}\int_0^x h(s)ds \right]^2. \qquad (2.20)$$

In the following theorem, we consider functions of two independent variables .

Theorem 2.2 Let $u(x, y)$, $h(x, y)$ be nonnegative continuous functions and $k(x, y) > 1$, defined for $x, y \in R_+$. Suppose that $k_x(x, y)$, $k_y(x, y)$, and $k_{xy}(x, y)$ be nonnegative and continuous functions defined for $x, y \in R_+$, and $p_1, p_2, p_3, \ p_4 \in R_+$. If

$$u^{p_1}(x, y) \le k^{p_2}(x, y) + p_3 \int_0^x \int_0^y h(s,t)u^{p_4}(s,t)dtds, \qquad (2.21)$$

for $x, y \in R_+$, then

$$u(x,y) \le [\ k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y) + k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0) + \left(\frac{p_1-p_4}{p_1}\right)\left(k^{p_2}(0,0) - k^{p_2}(x,0) + k^{p_2}(x,y) - k^{p_2}(0,y)\right) + p_3\int_0^x\int_0^y h(s,t)dtds\ ]^{\frac{1}{p_1-p_4}}, \quad p_1 \ne p_4. \tag{2.22}$$

Proof : Let $V^{p_1}(x,y)$ be the right hand side of (2.21), then

$$V^{p_1}(x,y) = k^{p_2}(x,y) + p_3\int_0^x\int_0^y h(s,t)u^{p_4}(s,t)dtds\,. \tag{2.23}$$

From (2.21) , (2.23), we have $V(x, y) > 0$, and

$$u(x,y) \le V(x,y)\,. \tag{2.24}$$

Differentiating (2.23) with respect to x, and then with respect to y, we get

$$p_1V^{p_1-1}(x,y)V_x(x,y) = p_2k^{p_2-1}(x,y)k_x(x,y) + p_3\int_0^y h(x,t)u^{p_4}(x,t)dt\,,$$

and

$$p_1V^{p_1-1}(x,y)V_{xy}(x,y) + p_1(p_1-1)V^{p_1-2}(x,y)V_x(x,y)V_y(x,y) = p_2k^{p_2-1}(x,y)k_{xy}(x,y) + p_2(p_2-1)k^{p_1-2}(x,y)k_x(x,y)k_y(x,y) + p_3h(x,y)u^{p_4}(x,y). \tag{2.25}$$

From (2.24) and (2.25), we have

$$p_1V^{p_1-1}(x,y)V_{xy}(x,y) + p_1(p_1-1)V^{p_1-2}(x,y)V_x(x,y)V_y(x,y) \le p_2k^{p_2-1}(x,y)k_{xy}(x,y) + p_2(p_2-1)k^{p_1-2}(x,y)k_x(x,y)k_y(x,y) + p_3h(x,y)V^{p_4}(x,y).$$

(2.26)

From (2.26), we obtain

$$p_1 V^{p_1-p_4-1} V_{xy} + p_1(p_1-1) V^{p_1-p_4-2} V_x V_y \leq$$

$$\frac{p_2 k^{p_2-1} k_{xy}}{V^{p_4}} + \frac{p_2(p_2-1) k^{p_2-2} k_x k_y}{V^{p_4}} + p_3 h(x,y).$$

(2.27)

Since $k(x, y) > 1$, then from (2.23) and (2.27), we have

$$p_1 V^{p_1-p_4-1} V_{xy} + p_1(p_1-1) V^{p_1-p_4-2} V_x V_y \leq$$

$$p_2 k^{p_2-1} k_{xy} + p_2(p_2-1) k^{p_2-2} k_x k_y + p_3 h(x,y).$$

(2.28)

Interchanging y by t in (2.28), and then integrating (2.28) with respect to t from 0 to y, we get

$$p_1 V^{p_1-p_4-1}(x,y) V_x(x,y) - p_1 V^{p_1-p_4-1}(x,0) V_x(x,0)$$

$$+ p_1 p_4 \int_0^y V^{p_1-p_4-2}(x,t) V_x(x,t) V_t(x,t)\,dt \leq$$

$$p_2 k^{p_2-1}(x,y) k_x(x,y) - p_2 k^{p_2-1}(x,0) k_x(x,0) + p_3 \int_0^y h(x,t)\,dt.$$

(2.29)

From (2.23), we obtain

$$V(x,0) = k^{\frac{p_2}{p_1}}(x,0) \quad \text{and} \quad V_x(x,0) = \frac{p_2}{p_1} k^{\frac{p_2}{p_1}-1}(x,0) k_x(x,0). \tag{2.30}$$

Using (2.29) and (2.30), we get

$$p_1 V^{p_1-p_4-1}(x,y)V_x(x,y) \le p_2 k^{\frac{1}{p_1}(p_1p_2-p_2p_4-p_1)}(x,0)k_x(x,0) - p_2 k^{p_2-1}(x,0)k_x(x,0)$$

$$+ p_2 k^{p_2-1}(x,y)k_x(x,y) + p_3\int_0^y h(x,t)dt - p_1p_4\int_0^y V^{p_1-p_4-2}(x,t)V_x(x,t)V_t(x,t)dt,$$

from which , we have

$$p_1 V^{p_1-p_4-1}(x,y)V_x(x,y) \le p_2 k^{\frac{1}{p_1}(p_1p_2-p_2p_4-p_1)}(x,0)k_x(x,0) - p_2 k^{p_2-1}(x,0)k_x(x,0)$$

$$+ p_2 k^{p_2-1}(x,y)k_x(x,y) + p_3\int_0^y h(x,t)dt. \tag{2.31}$$

Integrating (2.31) with respect to x from 0 to x, we have

$$V^{p_1-p_4}(x,y) - V^{p_1-p_4}(0,y) \le k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0)$$

$$+\left(\frac{p_1-p_4}{p_1}\right)\left(k^{p_2}(0,0) - k^{p_2}(x,0) + k^{p_2}(x,y) - k^{p_2}(0,y)\right)$$

$$+\frac{p_3(p_1-p_4)}{p_1}\int_0^x\int_0^y h(s,t)dtds, \qquad p_1 \ne p_4. \tag{2.32}$$

Again from (2.23), we have

$$V(0,y) = k^{\frac{p_2}{p_1}}(0,y). \tag{2.33}$$

Hence from (2.32) and (2.33), we obtain

$$V(x,y) \le [\ k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y) + k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0)$$

$$+\left(\frac{p_1-p_4}{p_1}\right)\left(k^{p_2}(0,0) - k^{p_2}(x,0) + k^{p_2}(x,y) - k^{p_2}(0,y)\right)$$

$$+\, p_3\int_0^x\int_0^y h(s,t)dtds\ ]^{\frac{1}{p_1-p_4}}, \qquad p_1 \ne p_4. \tag{2.34}$$

From (2.34) and (2.24) we have the required result , and the theorem is proved .

In the following, we obtain some special cases of the results in Theorem 2.2 by suitable substitutions given for each case :

Special Cases II.

1) Let $k(x, y) = c \ge 0$ and $p_4 = 1$. From **Theorem 2.2**, we have, if

$$u^{p_1}(x,y) \le c^{p_2} + p_3\int_0^x\int_0^y h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le \left(c^{\frac{p_2}{p_1}(p_1-1)} + \frac{p_3}{p_1}(p_1-1)\int_0^x\int_0^y h(s,t)dtds\right)^{\frac{1}{p_1-1}}, \quad p_1 \ne 1. \tag{2.35}$$

2) Let $k(x, y) = c \ge 0$. From **Theorem 2.2**, we have, if

$$u^{p_1}(x,y) \le c^{p_2} + p_3\int_0^x\int_0^y h(s,y)u^{p_4}(s,t)dtds,$$

then

$$u(x,y) \le \left(c^{\frac{p_2}{p_1}(p_1-p_4)} + \frac{p_3}{p_1}(p_1-p_4)\int_0^x\int_0^y h(s,t)dtds\right)^{\frac{1}{p_1-p_4}}, \quad p_1 \ne p_4. \tag{2.36}$$

3) If $p_1 = p_2 = p_3 = 2$ in case 1, we have, if

$$u^2(x,y) \le c^2 + 2\int_0^x\int_0^y h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le c + \int_0^x\int_0^y h(s,t)dtds, \tag{2.37}$$

which is the **Iang's** inequality for functions of two independent variables .

4) If $p_2 = p_3 = 1$ $, and$ $p_1 = 2$ in case 1, we have, if

$$u^2(x,y) \le c + \int_0^x\int_0^y h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le \sqrt{c} + \frac{1}{2}\int_0^x\int_0^y h(s,t)dtds. \tag{2.38}$$

5) If $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$ in case 2, we have , if

$$u^2(x,y) \le c^2 + 2\int_0^x\int_0^y h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le c + \int_0^x\int_0^y h(s,t)dtds, \tag{2.39}$$

which is the same result (2.37).

6) If $p_2 = p_3 = p_4 = 1$ $, and$ $p_1 = 2$ in (2.36), we have, if

$$u^2(x,y) \le c + \int_0^x\int_0^y h(s,t)u(s,t)dtds,$$

then

$$u(x,y) \le \sqrt{c} + \frac{1}{2}\int_0^x\int_0^y h(s,t)dtds. \tag{2.40}$$

which is the same result (2.38).

7) If $p_1 = p_2 = p_3 = 1$ and $p_4 = 2$ in (2.36), we have , if

$$u(x,y) \le c + \int_0^x \int_0^y h(s,t)u^2(s,t)dtds,$$

then

$$u(x,y) \le \left[c^{-1} - \int_0^x \int_0^y h(s,t)dtds \right]^{-1}. \qquad (2.41)$$

8) If $p_1 = p_2 = p_3 = 1$ and $p_4 = \frac{1}{2}$ in (2.36), we have, if

$$u(x,y) \le c + \int_0^x \int_0^y h(s,t)u^{\frac{1}{2}}(s,t)dtds,$$

then

$$u(x,y) \le \left[c^{\frac{1}{2}} + \frac{1}{2}\int_0^x \int_0^y h(s,t)dtds \right]^2. \qquad (2.42)$$

In the following theorem , we consider functions of three independent variables .

Theorem 2.3 Let $u(x, y, z)$, $h(x, y, z)$ be nonnegative continuous functions and $k(x, y, z)>1$, defined for $x,y,z \in R_+$. Suppose that the derivatives $k_x(x,y,z)$, $k_y(x,y,z), k_z(x,y,z)$, $k_{xy}(x,y,z),$ *and* $k_{xyz}(x,y,z)$ be nonnegative and continuous functions defined for $x,y,z \in R_+$, and $p_1, p_2, p_3, p_4 \in R_+$. If

$$u^{p_1}(x,y,z) \le k^{p_2}(x,y,z) + p_3 \int_0^x \int_0^y \int_0^z h(r,s,t,)u^{p_4}(r,s,t)dtdsdr, \qquad (2.43)$$

for $x,y,z \in R_+$ and $0 \le r \le x, 0 \le s \le y$, $0 \le t \le z$, then

$$u(x,y,z) \le \Bigg[k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y,z) + k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,z) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,z)$$

$$+ k^{\frac{p_2}{p_1}(p_1-p_4)}(x,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,0) + k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,0)$$

$$+ \left(\frac{p_1-p_4}{p_1}\right)\Big(k^{p_2}(x,y,z) - k^{p_2}(0,y,z) - k^{p_2}(x,0,z) + k^{p_2}(0,0,z) - k^{p_2}(x,y,0)$$

$$+ k^{p_2}(0,y,0) + k^{p_2}(x,0,0) - k^{p_2}(0,0,0)\Big) + p_3 \int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr \Bigg]^{\frac{1}{p_1-p_4}}, \; p_1 \ne p_4. \tag{2.44}$$

Proof : Let $W^{p_1}(x,y,z)$ be the right hand side of (2.43), then

$$W^{p_1}(x,y,z) = k^{p_2}(x,y,z) + p_3 \int_0^x\int_0^y\int_0^z h(r,s,t)u^{p_4}(r,s,t)dtdsdr, \tag{2.45}$$

From (2.43) and (2.45) , we have $W(x, y, z) > 0$ and

$$u(x, y, z) \le W(x, y, z) . \tag{2.46}$$

Differentiating (2.45) with respect to *x*, and then with respect to *y*, and then with respect to *z*, we get

$$p_1 W^{p_1-1}W_x = p_2 k^{p_2-1}k_x + p_3 \int_0^y\int_0^z h(x,s,t)u^{p_4}(x,s,t)dtds,$$

$$p_1(p_1-1)W^{p_1-2}W_xW_y + p_1W^{p_1-1}W_{xy} = p_2k^{p_2-1}k_{xy} + p_2(p_2-1)k^{p_2-2}k_xk_y$$

$$+ p_3 \int_0^z h(x,y,t)u^{p_4}(x,y,t)dt,$$

and

$$p_1W^{p_1-1}W_{xyz} + p_1(p_1-1)W^{p_1-2}W_zW_{xy} + p_1(p_1-1)W^{p_1-2}(W_xW_y)_z +$$

$$p_1(p_1-1)(p_1-2)W^{p_1-3}W_xW_yW_Z = p_2k^{p_2-1}k_{xyz} + p_2(p_2-1)k^{p_2-2}k_zk_{xy}$$

$$+ p_2(p_2-1)k^{p_2-2}(k_xk_y)_z + p_2(p_2-1)(p_2-2)k^{p_2-3}k_xk_yk_z$$

$$+ p_3h(x,y,z)u^{p_4}(x,y,z). \tag{2.47}$$

From (2.46) and (2.47), we have

$$p_1W^{p_1-1}W_{xyz} + p_1(p_1-1)W^{p_1-2}W_zW_{xy} + p_1(p_1-1)W^{p_1-2}(W_xW_y)_z +$$

$$p_1(p_1-1)(p_1-2)W^{p_1-3}W_xW_yW_Z \le p_2k^{p_2-1}k_{xyz} + p_2(p_2-1)k^{p_2-2}k_zk_{xy} +$$

$$p_2(p_2-1)k^{p_2-2}(k_xk_y)_z + p_2(p_2-1)(p_2-2)k^{p_2-3}k_xk_yk_z$$

$$+ p_3h(x,y,z)W^{p_4}(x,y,z). \tag{2.48}$$

From (2.48), we obtain

$$p_1W^{p_1-p_4-1}W_{xyz} + p_1(p_1-1)W^{p_1-p_4-2}W_zW_{xy} + p_1(p_1-1)W^{p_1-p_4-2}(W_xW_y)_z +$$

$$p_1(p_1-1)(p_1-2)W^{p_1-p_4-3}W_xW_yW_Z \le \frac{p_2k^{p_2-1}k_{xyz}}{W^{p_4}(x,y,z)} + \frac{p_2(p_2-1)k^{p_2-2}k_zk_{xy}}{W^{p_4}(x,y,z)} +$$

$$\frac{p_2(p_2-1)k^{p_2-2}(k_xk_y)_z}{W^{p_4}(x,y,z)} + \frac{p_2(p_2-1)(p_2-2)k^{p_2-3}k_xk_yk_z}{W^{p_4}(x,y,z)} + p_3h(x,y,z). \tag{2.49}$$

Since $k(x, y, z) > 1$, then from (2.45) and (2.49), we get

$$p_1W^{p_1-p_4-1}W_{xyz}+p_1(p_1-1)W^{p_1-p_4-2}W_zW_{xy}+p_1(p_1-1)W^{p_1-p_4-2}(W_xW_y)_z+$$

$$p_1(p_1-1)(p_1-2)W^{p_1-p_4-3}W_xW_yW_Z\le p_2k^{p_2-1}k_{xyz}+p_2(p_2-1)k^{p_2-2}k_zk_{xy}+$$

$$p_2(p_2-1)k^{p_2-2}(k_xk_y)_z+p_2(p_2-1)(p_2-2)k^{p_2-3}k_xk_yk_z+p_3h(x,y,z). \tag{2.50}$$

Integrating (2.50) with respect to z from 0 to z, we have

$$p_1W^{p_1-p_4-1}W_{xy}-p_1W^{p_1-p_4-1}(x,y,0)W_{xy}(x,y,0)+p_1(p_1-1)W^{p_1-p_4-2}W_xW_y-$$

$$p_1(p_1-1)W^{p_1-p_4-2}(x,y,0)W_x(x,y,0)W_y(x,y,0)\le p_2k^{p_2-1}k_{xy}-p_2k^{p_2-1}(x,y,0)k_{xy}(x,y,0)$$

$$+p_2(p_2-1)k^{p_2-2}(x,y,z)k_xk_y-p_2(p_2-1)k^{p_2-2}(x,y,0)k_x(x,y,0)k_y(x,y,0)$$

$$-p_1p_4\int_0^z W^{p_1-p_4-2}(x,y,t)W_{xy}(x,y,t)W_t(x,y,t)dt$$

$$-p_1p_4(p_1-1)\int_0^z W^{p_1-p_4-3}(x,y,t)W_x(x,y,t)W_y(x,y,t)W_t(x,y,t)dt+p_3\int_0^z h(x,y,t)dt. \tag{2.51}$$

From (2.51), we have

$$p_1W^{p_1-p_4-1}W_{xy}-p_1W^{p_1-p_4-1}(x,y,0)W_{xy}(x,y,0)+p_1(p_1-1)W^{p_1-p_4-2}W_xW_y-$$

$$p_1(p_1-1)W^{p_1-p_4-2}(x,y,0)W_x(x,y,0)W_y(x,y,0)\le p_2k^{p_2-1}k_{xy}-$$

$$p_2k^{p_2-1}(x,y,0)k_{xy}(x,y,0)+p_2(p_2-1)k^{p_2-2}(x,y,z)k_xk_y-$$

$$p_2(p_2-1)k^{p_2-2}(x,y,0)k_x(x,y,0)k_y(x,y,0)+p_3\int_0^z h(x,y,t)dt. \tag{2.52}$$

From (2.45), we have

$$W(x,y,0) = k^{\frac{p_2}{p_1}}(x,y,0)$$

and

$$W_x(x,y,0) = \frac{p_2}{p_1} k^{\frac{p_2}{p_1}-1}(x,y,0)k_x(x,y,0),$$

$$W_y(x,y,0) = \frac{p_2}{p_1} k^{\frac{p_2}{p_1}-1}(x,y,0)k_y(x,y,0).$$

Hence

$$W_{xy}(x,y,0) = \frac{p_2}{p_1}\left(\frac{p_2}{p_1}-1\right)k^{\frac{p_2}{p_1}-2}(x,y,0)k_x(x,y,0)k_y(x,y,0) + \frac{p_2}{p_1}k^{\frac{p_2}{p_1}-1}(x,y,0)k_{xy}(x,y,0),$$

from which (2.52) becomes

$$\begin{aligned} &p_1 W^{p_1-p_4-1}W_{xy} + p_1(p_1-1)W^{p_1-p_4-2}W_xW_y \le \\ &p_2(p_2-1)k^{p_2-\frac{p_2p_4}{p_1}-2}(x,y,0)k_x(x,y,0)k_y(x,y,0) + p_2k^{p_2-\frac{p_2p_4}{p_1}-1}(x,y,0)k_{xy}(x,y,0) + \\ &p_2k^{p_2-1}(x,y,z)k_{xy} - p_2k^{p_2-1}(x,y,0)k_{xy}(x,y,0) + p_2(p_2-1)k^{p_2-2}(x,y,z)k_xk_y - \\ &p_2(p_2-1)k^{p_2-2}(x,y,0)k_x(x,y,0)k_y(x,y,0) + p_3\int_0^z h(x,y,t)dt. \end{aligned} \qquad (2.53)$$

Interchanging y by t in (2.53), and then integrating with respect to t from 0 to y, we have

$$p_1 W^{p_1-p_4-1} W_x - p_1 W^{p_1-p_4-1}(x,0,z) W_x(x,0,z) \le p_2 k^{p_2-\frac{p_2 p_4}{p_1}-1}(x,y,0) k_x(x,y,0) -$$

$$p_2 k^{p_2-\frac{p_2 p_4}{p_1}-1}(x,0,0) k_x(x,0,0) + p_2 k^{p_2-1}(x,y,z) k_x - p_2 k^{p_2-1}(x,0,z) k_x(x,0,z)$$

$$- p_2 k^{p_2-1}(x,y,0) k_x(x,y,0) + p_2 k^{p_2-1}(x,0,0) k_x(x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t) dt ds$$

$$- \frac{p_2^2 p_4}{p_1} \int_0^y k^{p_2-\frac{p_2 p_4}{p_1}-2}(x,s,0) k_x(x,s,0) k_s(x,s,0) ds,$$

from which , we have

$$p_1 W^{p_1-p_4-1} W_x - p_1 W^{p_1-p_4-1}(x,0,z) W_x(x,0,z) \le p_2 k^{p_2-\frac{p_2 p_4}{p_1}-1}(x,y,0) k_x(x,y,0) -$$

$$p_2 k^{p_2-\frac{p_2 p_4}{p_1}-1}(x,0,0) k_x(x,0,0) + p_2 k^{p_2-1}(x,y,z) k_x - p_2 k^{p_2-1}(x,0,z) k_x(x,0,z) -$$

$$p_2 k^{p_2-1}(x,y,0) k_x(x,y,0) + p_2 k^{p_2-1}(x,0,0) k_x(x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t) dt ds. \tag{2.54}$$

From (2.45), we have

$$W(x,0,z) = k^{\frac{p_2}{p_1}}(x,0,z),$$

and

$$W_x(x,0,z) = \frac{p_2}{p_1} k^{\frac{p_2}{p_1}-1}(x,0,z) k_x(x,0,z). \tag{2.55}$$

From (2.54) and (2.55), we have

$$p_1 W^{p_1-p_4-1} W_x \le p_2 k^{p_2-\frac{p_2p_4}{p_1}-1}(x,0,z)k_x(x,0,z) + p_2 k^{p_2-\frac{p_2p_4}{p_1}-2}(x,y,0)k_x(x,y,0)$$

$$-p_2 k^{p_2-\frac{p_2p_4}{p_1}-2}(x,0,0)k_x(x,0,0) + p_2 k^{p_2-1}k_x + p_2 k^{p_2-1}(x,0,z)k_x(x,0,z)$$

$$-p_2 k^{p_2-1}(x,y,0)k_x(x,y,0) + p_2 k^{p_2-1}(x,0,0)k_x(x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t)\,dtds. \tag{2.56}$$

Again integrating (2.56) with respect to x, we get

$$\left(\frac{p_1}{p_1-p_4}\right)\left[W^{p_1-p_4}(x,y,z) - W^{p_1-p_4}(0,y,z)\right] \le \left(\frac{p_1}{p_1-p_4}\right)\left[k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,z)\right.$$

$$-k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,z) + k^{\frac{p_2}{p_1}(p_1-p_4)}(x,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,0)$$

$$\left.+k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,0)\right] + k^{p_2}(x,y,z) - k^{p_2}(0,y,z) - k^{p_2}(x,0,z) + k^{p_2}(0,0,z)$$

$$-k^{p_2}(x,y,0) + k^{p_2}(0,y,0) + k^{p_2}(x,0,0) - k^{p_2}(0,0,0) + p_3 \int_0^x \int_0^y \int_0^z h(r,s,t)\,dtdsdr. \tag{2.57}$$

From (2.45), we obtain

$$W(0,y,z) = k^{\frac{p_2}{p_1}}(0,y,z). \tag{2.58}$$

So from (2.57) and (2.58), we get

$$W(x,y,z) \le \Bigg[k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y,z) + k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,z) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,z)$$

$$+k^{\frac{p_2}{p_1}(p_1-p_4)}(x,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(0,y,0) - k^{\frac{p_2}{p_1}(p_1-p_4)}(x,0,0) + k^{\frac{p_2}{p_1}(p_1-p_4)}(0,0,0)$$

$$+\left(\frac{p_1-p_4}{p_1}\right)\Bigg(k^{p_2}(x,y,z) - k^{p_2}(0,y,z) - k^{p_2}(x,0,z) + k^{p_2}(0,0,z) - k^{p_2}(x,y,0)$$

$$+k^{p_2}(0,y,0) + k^{p_2}(x,0,0) - k^{p_2}(0,0,0) + p_3\int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr \Bigg) \Bigg]^{\frac{1}{p_1-p_4}},$$

$p_1 \ne p_4$.

(2.59)

Hence, from (2.46) and (2.59) we get the required result , and the theorem is proved .

In the following, we obtain some special cases of the results in Theorem 2.3 by suitable substitutions given for each case :

Special Cases III.

1) Let $k(x, y, z) = c \ge 0$. From **Theorem 2.3**, we have, if

$$u^{p_1}(x,y,z) \le c^{p_2} + p_3\int_0^x\int_0^y\int_0^z h(r,s,t)u^{p_4}(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le \left(c^{\frac{p_2}{p_1}(p_1-p_4)} + \frac{p_3}{p_1}(p_1-p_4)\int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr \right)^{\frac{1}{p_1-p_4}}, \quad p_1 \ne p_4. \quad (2.60)$$

2) If $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$ in case 1, we have

$$u^2(x,y,z) \le c^2 + 2\int_0^x\int_0^y\int_0^z h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le c + \int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr. \quad (2.61)$$

3) If $p_2 = p_3 = p_4 = 1$ $, and$ $p_1 = 2$ in case 1, we have

$$u^2(x,y,z) \le c + \int_0^x\int_0^y\int_0^z h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le \sqrt{c} + \frac{1}{2}\int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr. \qquad (2.62)$$

4) If $p_1 = p_2 = p_3 = 1$ and $p_4 = 2$ in case 1, we have

$$u(x,y,z) \le c + \int_0^x\int_0^y\int_0^z h(r,s,t)u^2(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le \left[c^{-1} - \int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr\right]^{-1}. \qquad (2.63)$$

6) If $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$ in **Theorem 2.3**, we have, if

$$u^2(x,y,z) \le k^2(x,y,z) + 2\int_0^x\int_0^y\int_0^z h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y) \le k(0,y,z) + k(x,0,z) - k(0,0,z) + k(x,y,0) - k(0,y,0) - k(x,0,0)$$

$$+ k(0,0,0) + \frac{1}{2}\Big(k^2(x,y,z) - k^2(0,y,z) - k^2(x,0,z) + k^2(0,0,z) - k^2(x,y,0)$$

$$+ k^2(0,y,0) + k^2(x,0,0) - k^2(0,0,0) + 2\int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr\Big).$$

$$(2.64)$$

6) If $p_1 = p_2 = p_3 = 1$ $, and$ $p_4 = 2$ in **Theorem 2.3**, we have, if

$$u(x,y,z) \le k(x,y,z) + \int_0^x\int_0^y\int_0^z h(r,s,t)u^2(r,s,t)dtdsdr,$$

then

$$u(x,y) \le [\ k^{-1}(0,y,z) + k^{-1}(x,0,z) - k^{-1}(0,0,z) + k^{-1}(x,y,0)$$

$$- k^{-1}(0,y,0) - k^{-1}(x,0,0) + k^{-1}(0,0,0) - k(x,y,z) + k(0,y,z)$$

$$+ k(x,0,z) - k(0,0,z) + k(x,y,0) - k(0,y,0) - k(x,0,0) + k(0,0,0)$$

$$- \int_0^x \int_0^y \int_0^z h(r,s,t)dtdsdr\ ]^{-1}. \tag{2.65}$$

7) If $p_2 = p_3 = p_4 = 1$, *and* $p_1 = 2$ in **Theorem 2.3**, we have, if

$$u^2(x,y,z) \le k(x,y,z) + \int_0^x\int_0^y\int_0^z h(r,s,t)u(r,s,t)dtdsdr,$$

then

$$u(x,y,z) \le k^{\frac{1}{2}}(0,y,z) + k^{\frac{1}{2}}(x,0,z) - k^{\frac{1}{2}}(0,0,z) + k^{\frac{1}{2}}(x,y,0) - k^{\frac{1}{2}}(0,y,0)$$

$$- k^{\frac{1}{2}}(x,0,0) + k^{\frac{1}{2}}(0,0,0) + \frac{1}{2}\Bigg(k(x,y,z) - k(0,y,z) - k(x,0,z) + k(0,0,z)$$

$$- k(x,y,0) + k(0,y,0) + k(x,0,0) - k(0,0,0) + \int_0^x \int_0^y \int_0^z h(r,s,t)dtdsdr \Bigg). \tag{2.66}$$

8) If $p_1 = p_2 = p_3 = 1$, *and* $p_4 = \frac{1}{2}$ in **Theorem 2.3**, we have, if

$$u(x,y,z) \le k(x,y,z) + \int_0^x\int_0^y\int_0^z h(r,s,t)u^{\frac{1}{2}}(r,s,t)dtdsdr,$$

then

$$u(x,y,z)\le$$
$$\Big[\ k^{\frac{1}{2}}(0,y,z)+k^{\frac{1}{2}}(x,0,z)-k^{\frac{1}{2}}(0,0,z)+k^{\frac{1}{2}}(x,y,0)-k^{\frac{1}{2}}(0,y,0)-k^{\frac{1}{2}}(x,0,0)$$
$$+k^{\frac{1}{2}}(0,0,0)+\frac{1}{2}\Big(\ k(x,y,z)-k(0,y,z)-k(x,0,z)+k(0,0,z)-k(x,y,0)$$
$$+k(0,y,0)+k(x,0,0)-k(0,0,0)+\int_0^x\int_0^y\int_0^z h(r,s,t)dtdsdr\Big)\ \Big]^2. \tag{2.67}$$

3. Applications

In this section, we present some applications of **Theorem 2.1** and **Theorem 2.2**. Our aim is to study the boundedness of the solutions of some hyperbolic partial differential equations .

In the following two examples, we use **Theorem 2.1** to obtain explicit bounds on the solutions of some first order partial differential equations

Example 1. Let $u_t(t)=h(t)u^2(t)$ where $u(0)=c$ and $h(t)\le t,$ then we have the partial differential inequality

$$u_t(t)\le tu^2(t). \tag{3.1}$$

Interchanging t by s in (3.1), and then integrating (3.1) with respect to s from 0 to t , we have the integral inequality

$$u(t)\le c+\int_0^t su^2(s)ds.$$

By (2.11) with $p_1 = p_2 = p_3 = 1$,and $p_4 = 2$ we get

$$u(t)\le\Big[\ c^{-1}-\frac{1}{2}t^2\ \Big]^{-1}.$$

Example 2. Let $u(x)u_x(x)=e^{2x}+h(x)u(x),$ $u(0)=0,$ and $h(x)\le x,$ then we have the partial differential inequality

$$u(x)u_x(x)\le e^{2x}+xu(x). \tag{3. 2}$$

Interchanging *x* by *s* in (3.2), and then integrating with respect to *s* from 0 to *x*, we have the integral inequality

$$u^2(x) \le \left(e^x\right)^2 + 2\int_0^x su(s)ds.$$

By **Theorem 2.1** with $k(x) = e^x, \quad p_1 = p_2 = p_3 = 2 \ , and \quad p_4 = 1$, we have

$$u(x) \le \frac{1}{2}\left[1 + e^{2x} + x^2\right], \quad e^x > 1.$$

We now present three examples showing how **Theorem 2.2** can be used to obtain explicit bounds on the solutions of some second order partial differential equations

Example 3. Let $u(x,y)u_{xy}(x,y) + u_x(x,y)u_y(x,y) = h(x,y)u(x,y)$ where $u_x(x,0) = 0, \ u(0,y) = c, \ h(x,y) \le ye^x$, then we have the partial differential inequality

$$u(x,y)u_{xy}(x,y) + u_x(x,y)u_y(x,y) \le ye^x u(x,y). \tag{3. 3}$$

Integrating (3. 3) with respect to y, we get

$$u(x,y)u_x(x,y) \le \int_0^y te^x u(x,t)dt. \tag{3. 4}$$

Again, integrating (3. 4) with respect to *x* , we have the nonlinear integral inequality

$$u^2(x,y) \le c^2 + 2\int_0^x \int_0^y te^s u(s,t)dtds.$$

By **Theorem 2.2** with $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$, we get

$$u(x,y) \le c + \int_0^x \int_0^y te^s dtds = c + \frac{1}{2}y^2(e^x - 1).$$

Example 4. Let $u_{xy}(x,y) = h(x,y)u^2(x,y)$, where $u(0,y) = u(x,0) = c$ and $u_x(x,0) = 0, \quad h(x,y) \le x\sin y,$ then we have the partial differential inequality

$$u_{xy} \le x\sin y \ \ u^2(x,y). \tag{3. 5}$$

Integrating (3. 5) with respect to *y* and with respect to *x*, we get the nonlinear integral inequality

$$u(x,y) \le c + \int_0^x \int_0^y s \sin t \quad u^2(s,t)dtds.$$

By **Theorem 2.2** with $p_1 = p_2 = p_3 = 1$ *, and* $p_4 = 2$, we have

$$u(x,y) \le c^{-1} - \int_0^x \int_0^y s \sin t \quad dtds \le \left[c^{-1} + \frac{1}{2} x^2 (\cos y - 1) \right]^{-1}.$$

Example 5.

Let $u_{xy}(x,y) = 2xe^y + h(x,y)u^2(x,y),\ u_x(x,0) = 2x,\ u(0,y) = 0,$

and $h(x,y) \le x + y,$ then we have partial differential inequality

$$u_{xy} \le 2xe^y + (x+y)u^2(x,y). \tag{3. 6}$$

Integrating (3. 6) with respect to *y* and then with respect to *x*, we get

$$u(x,y) \le x^2 e^y + \int_0^x \int_0^y (s+t)u^2(s,t)dtds.$$

From **Theorem 2.2** with $k(x,y) = x^2 e^y$, $p_1 = p_2 = p_3 = 1$, and $p_4 = 2$, we have

$$u(x,y) \le \left[x^{-2} + x^2 - x^2 e^y - \frac{1}{2} xy(x+y) \right]^{-1},$$

where $x^2 e^y > 1.$

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