ON SOME INTEGRAL INEQUALITIES OF OU-IANG’S TYPE

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Abstract: In this article, we establish some nonlinear integral inequalities of Iang’s type for functions of one or more than one independent variable. Also, we give some applications of these inequalities to study boundedness of solutions of nonlinear partial differential equations.

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1. Introduction

Integral inequalities play an important role in the study of existence, uniqueness, continuous dependence, perturbation, boundedness and stability of solutions of differential and integral equations [2, 3, 4].

The integral inequalities involving functions of more than one independent variables play a fundamental role in the study of the solutions of partial differential equations [1]. One of the most useful inequalities in the development of the theory of partial differential equations is Iang’s inequality which was first given by Ou-Iang [2] while studying the boundedness of solutions of certain second order differential equations.

**Theorem 1.1** [2] Let $u$ and $f$ be nonnegative continuous functions defined for $t \in R_+$. If

$$u^2(t) \leq c^2 + 2 \int_0^t f(s)u(s)ds,$$

for $t \in R_+$, where $c \geq 0$ is a constant, then

$$u(t) \leq c + \int_0^t f(s)ds, \quad \text{for} \quad t \in R_+.$$

In the past few years, Iang’s inequality has been applied with considerable success to study the global existence, uniqueness, stability and other properties of the solutions of various nonlinear differential equations. Generalizations of the inequality given in **Theorem 1.1** are given in [5, 6]. In the following theorem, we present one of those generalizations due to Pachpatte [5].
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Theorem 1.2 Let $u$, $f$ and $g$ be nonnegative continuous functions defined for $t \in R_+$ and $c$ be a nonnegative constant. If

$$u^2(t) \leq c^2 + 2 \int_0^t [f(s)u^2(s) + g(s)u(s)]ds,$$

for $t \in R_+$, then

$$u(t) \leq p(t)\exp\left(\int_0^t f(s)ds\right), \quad \text{for} \quad t \in R_+$$

where

$$p(t) = c + \int_0^t g(s)ds, \quad \text{for} \quad t \in R_+.$$ (1.5)

The purpose of this paper, is to establish some new integral inequalities of Lang’s type [2] for functions of one and more than one independent variables which can be used in the analysis of some classes of partial differential equations.

2. Main Results

In what follows, $R$ denotes the set of real numbers and $R_+ = [0, \infty)$. The first order partial derivatives of a function $Z(x, y)$ defined for $x, y \in R$ with respect to $x$ and $y$ are denoted by $Z_x(x, y)$ and $Z_y(x, y)$ respectively.

Throughout this paper, all the functions and their partial derivatives which appear in the inequalities are assumed to be real-valued and all the integrals involved are exist and of positive values on the respective domains of their definitions. Our main results are given in the following three theorems:

Theorem 2.1 Let $u(x)$, $h(x)$ be nonnegative continuous functions and $k(x) > 1$, defined for $x \in R_+$. Suppose that $k_i(x)$ be nonnegative continuous function defined for $x \in R_+$, $p_1$, $p_2$, $p_3$, and $p_4 \in R_+$. If

$$u^{p_1}(x) \leq k^{p_2}(x) + p_3 \int_0^x h(s)u^{p_4}(s)ds,$$ (2.1)

then

$$u(x) \leq \left[ k^{\frac{p_2}{p_1}}(0) + \left( \frac{p_4 - p_3}{p_1} \right) \left( k^{p_2}(x) - k^{p_2}(0) + p_3 \int_0^x h(s)ds \right) \right]^{\frac{1}{p_4 - p_3}},$$

$p_1 \neq p_4$. (2.2)
Proof: Let \( V^{\rho_1}(x) \) be the right hand side of (2.1), then

\[
V^{\rho_1}(x) = k^{\rho_2}(x) + p_3 \int_0^x h(s)u^{\rho_4}(s)ds. \tag{2.3}
\]

From (2.1), (2.3), we have \( V(x) > 0 \) and

\[
u(x) \leq V(x). \tag{2.4}
\]

Differentiating (2.3) with respect to \( x \), we get

\[
p_1V^{\rho_{p_1}}(x)V_x(x) = p_2k^{\rho_{p_2}}(x)k_x(x) + p_3h(x)u^{\rho_4}(x). \tag{2.5}
\]

From (2.4) and (2.5), we have

\[
p_1V^{\rho_{p_1}-1}(x)V_x(x) \leq p_2k^{\rho_{p_2}}(x)k_x(x) + p_3h(x)V^{\rho_4}(x),
\]

from which and by using (2.4), we get

\[
p_1V^{\rho_{p_1}-p_4-1}(x)V_x(x) \leq \frac{p_2k^{\rho_{p_2}}(x)k_x(x)}{V^{\rho_4}(x)} + p_3h(x). \tag{2.6}
\]

Since \( k(x) > 1 \), from (2.3) and (2.6), we obtain

\[
p_1V^{\rho_{p_1}-p_4-1}(x)V_x(x) \leq p_2k^{\rho_{p_2}}(x)k_x(x) + p_3h(x). \tag{2.7}
\]

Integrating (2.7) with respect to \( x \) from 0 to \( x \), we get

\[
\left( \frac{p_1}{p_1 - p_4} \right)\left( V^{\rho_{p_1}}(x) - V^{\rho_{p_1}}(0) \right) \leq k^{\rho_2}(x) - k^{\rho_2}(0) + p_3\int_0^x h(s)ds. \tag{2.8}
\]

From (2.3) and (2.8), we have

\[
V(x) \leq \left( k^{\rho_2}(x) - k^{\rho_2}(0) + p_3\int_0^x h(s)ds \right)^{\frac{1}{p_1-p_4}}(0), \quad p_1 \neq p_4. \tag{2.9}
\]

From (2.4) and (2.9) we have the required result. Hence the theorem is proved.

In the following, we obtain some special cases of the results in Theorem 2.1 by suitable substitutions given for each case.
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Special Cases I.

1) Let \( k(x) = c \), \( c \) is a positive constant and \( p_4 = 1 \). From Theorem 2.1, we have, if

\[
u^{p_1}(x) \leq c^{p_2} + p_3 \int_0^x h(s)u(s)ds,
\]

then

\[
u(x) \leq \left( \frac{p_2}{c^{p_1(p_1-1)}} + \frac{p_3}{p_1} (p_1 - 1) \int_0^x h(s)ds \right)^{\frac{1}{p_1-2}}, \quad p_1 \neq 1. \tag{2.10}
\]

2) Let \( k(x) = c \geq 0 \). From Theorem 2.1, we have, if

\[
u^{p_1}(x) \leq c^{p_2} + p_3 \int_0^x h(s)u^{p_1}(s)ds,
\]

then

\[
u(x) \leq \left( \frac{p_2}{c^{p_1(p_1-1)}} + \frac{p_3}{p_1} (p_1 - 1) \int_0^x h(s)ds \right)^{\frac{1}{p_1-p_4}}, \quad p_1 \neq p_4. \tag{2.11}
\]

3) If \( p_1 = p_2 = p_3 = 2 \) in (2.10), we have, if

\[
u^2(x) \leq c^2 + 2 \int_0^x h(s)u(s)ds,
\]

then

\[
u(x) \leq c + \int_0^x h(s)ds, \tag{2.12}
\]

which is the same result of Iang [2].

4) If \( p_2 = p_3 = 1 \) and \( p_1 = 2 \) in case 1, we have, if

\[
u^2(x) \leq c + \int_0^x h(s)u(s)ds,
\]

then

\[
u(x) \leq \sqrt{c} + \frac{1}{2} \int_0^x h(s)ds. \tag{2.13}
\]
5) If \( p_1 = p_2 = p_3 = 2 \) and \( p_4 = 1 \) in case 2, we have, if
\[
u^2(x) \leq c^2 + 2 \int_0^x h(s)u(s)ds,
\]
then
\[
u(x) \leq c + \int_0^x h(s)ds,
\]
which is the same result of \textbf{Iang}.

6) If \( p_2 = p_3 = p_4 = 1 \), \( p_1 = 2 \) in case 2, we have, if
\[
u^2(x) \leq c + \int_0^x h(s)u(s)ds,
\]
then
\[
u(x) \leq \sqrt{c + \frac{1}{2} \int_0^x h(s)ds}.
\]
which is the same result (2.13).

7) If \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = 2 \) in case 2, we have, if
\[
u(x) \leq c + \int_0^x h(s)u^2(s)ds,
\]
then
\[
u(x) \leq \left[ c^{-1} - \int_0^x h(s)ds \right]^{-1}.
\]

8) If \( p_1 = p_2 = p_3 = 2 \) and \( p_4 = 1 \) in \textbf{Theorem 2.1}, we have, if
\[
u^2(x) \leq k^2(x) + 2 \int_0^x h(s)u(s)ds,
\]
then
\[
u(x) \leq k(0) + \frac{1}{2} k^2(x) - \frac{1}{2} k^2(0) + \int_0^x h(s)ds.
\]

9) If \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = 2 \) in \textbf{Theorem 2.1}, we have, if
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\[ u(x) \leq k(x) + \int_{0}^{x} h(s)u^2(s) \, ds, \]

then

\[ u(x) \leq \left[ k^{-1}(0) - k(x) + k(0) - \int_{0}^{x} h(s) \, ds \right]^{-1}. \tag{2.18} \]

10) If \( p_2 = p_3 = p_4 = 1 \) and \( p_1 = 2 \) in Theorem 2.1, we have, if

\[ u^2(x) \leq k(x) + \int_{0}^{x} h(s)u(s) \, ds, \]

then

\[ u(x) \leq \frac{1}{2} \left[ 2k^{\frac{1}{2}}(0) + k(x) - k(0) + \int_{0}^{x} h(s) \, ds \right]. \tag{2.19} \]

11) If \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = \frac{1}{2} \) in Theorem 2.1, we have, if

\[ u(x) \leq k(x) + \int_{0}^{x} h(s)u^\frac{1}{2}(s) \, ds, \]

then

\[ u(x) \leq \left[ \frac{1}{2} k^{\frac{1}{2}}(0) + \frac{1}{2} k(x) - \frac{1}{2} k(0) + \frac{1}{2} \int_{0}^{x} h(s) \, ds \right]^2. \tag{2.20} \]

In the following theorem, we consider functions of two independent variables.

**Theorem 2.2** Let \( u(x, y), h(x, y) \) be nonnegative continuous functions and \( k(x, y) > 1 \), defined for \( x, y \in R_+ \). Suppose that \( k_x(x, y), k_y(x, y) \), and \( k_{xy}(x, y) \) be nonnegative and continuous functions defined for \( x, y \in R_+ \), and \( p_1, p_2, p_3, p_4 \in R_+ \). If

\[ u^{p_1}(x, y) \leq k^{p_2}(x, y) + p_3 \int_{0}^{x} \int_{0}^{y} h(s, t)u^{p_4}(s, t) \, dtds, \tag{2.21} \]

for \( x, y \in R_+ \), then
\[
\begin{align*}
 u(x, y) &\leq \left[ k^{p_4} p_4 (p_1 - p_3) (0, y) + k^{p_4} p_4 (p_1 - p_3) (x, 0) - k^{p_4} p_4 (0, 0) + \\
 &\left( \frac{p_4 - p_3}{p_4} \right) \left( k^{p_4} p_4 (0, 0) - k^{p_4} (x, 0) + k^{p_4} (x, y) - k^{p_4} (0, y) \right) + \\
 p_3 \int_0^x \int_0^y h(s, t) dt ds \right]^{\frac{1}{p_4 - p_4}}, \quad p_1 \neq p_4. 
\end{align*}
\]

**Proof:** Let \( V^{p_4} (x, y) \) be the right hand side of (2.21), then

\[
V^{p_4} (x, y) = k^{p_4} (x, y) + p_3 \int_0^x \int_0^y h(s, t) u^{p_4} (s, t) dt ds. 
\]

From (2.21), (2.23), we have \( V(x, y) > 0 \), and

\[
\begin{align*}
 u(x, y) &\leq V(x, y). 
\end{align*}
\]

Differentiating (2.23) with respect to \( x \), and then with respect to \( y \), we get

\[
p_1 V^{p_4} (x, y) V_x (x, y) = p_2 k^{p_4} (x, y) k_x (x, y) + p_3 \int_0^y h(x, t) u^{p_4} (x, t) dt ,
\]

and

\[
p_1 V^{p_4} (x, y) V_{xy} (x, y) + p_1 (p_1 - 1) V^{p_4} (x, y) V_y (x, y) V_y (x, y) = \\
p_2 k^{p_4} (x, y) k_{xy} (x, y) + p_2 (p_2 - 1) k^{p_4} (x, y) k_x (x, y) k_y (x, y) + \\
p_3 h(x, y) u^{p_4} (x, y).
\]

From (2.24) and (2.25), we have

\[
p_1 V^{p_4} (x, y) V_{xy} (x, y) + p_1 (p_1 - 1) V^{p_4} (x, y) V_y (x, y) V_y (x, y) \leq \\
p_2 k^{p_4} (x, y) k_{xy} (x, y) + p_2 (p_2 - 1) k^{p_4} (x, y) k_x (x, y) k_y (x, y) + \\
p_3 h(x, y) V^{p_4} (x, y).
\]
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From (2.26), we obtain

\[ p_1 V^{p_1 - p_4 - 1} V_{xy} + p_1 (p_1 - 1) V^{p_1 - p_4 - 2} V_x V_y \leq \]

\[ \frac{p_2 k^{p_2 - 1} k_{xy}}{V^{p_4}} + \frac{p_2 (p_2 - 1) k_{p_2 - 2} k_y}{V^{p_4}} + p_3 h(x,y). \]  

(2.27)

Since \( k(x, y) > 1 \), then from (2.23) and (2.27), we have

\[ p_1 V^{p_1 - p_4 - 1} V_{xy} + p_1 (p_1 - 1) V^{p_1 - p_4 - 2} V_x V_y \leq \]

\[ p_2 k^{p_2 - 1} k_{xy} + p_2 (p_2 - 1) k_{p_2 - 2} k_y + p_3 h(x,y). \]

(2.28)

Interchanging \( y \) by \( t \) in (2.28), and then integrating (2.28) with respect to \( t \) from 0 to \( y \), we get

\[ p_1 V^{p_1 - p_4 - 1} (x, y) V_x (x, y) - p_1 V^{p_1 - p_4 - 1} (x, 0) V_x (x, 0) \]

\[ + p_1 p_4 \int_0^y V^{p_1 - p_4 - 2} (x, t) V_x (x, t) V_t (x, t) dt \leq \]

\[ p_2 k^{p_2 - 1} (x, y) k_x (x, y) - p_2 k^{p_2 - 1} (x, 0) k_x (x, 0) + p_3 \int_0^y h(x, t) dt. \]

(2.29)

From (2.23), we obtain

\[ V(x, 0) = k^{p_4} (x, 0) \quad \text{and} \quad V_x (x, 0) = \frac{p_2 k^{p_2 - 1}}{p_1} (x, 0) k_y (x, 0). \]

(2.30)

Using (2.29) and (2.30), we get
\[ p_1 V^{p_1-p_4-1}(x,y)V_x(x,y) \leq p_2 k_{p_1}^{1/(p_1 p_2-p_3-p_4-p_1)} (x,0)k_x(x,0) - p_2 k_{p_1}^{p_1-1}(x,0)k_x(x,0) \]

\[ + p_2 k_{p_1-1}(x,y)k_x(x,y) + p_3 \int_0^y h(x,t)dt - p_1 p_4 \int_0^y V^{p_1-p_2-1}(x,t)V_x(x,t)V_t(x,t)dt, \]

from which, we have

\[ p_1 V^{p_1-p_4-1}(x,y)V_x(x,y) \leq p_2 k_{p_1}^{1/(p_1 p_2-p_3-p_4-p_1)} (x,0)k_x(x,0) - p_2 k_{p_1-1}(x,0)k_x(x,0) \]

\[ + p_2 k_{p_1-1}(x,y)k_x(x,y) + p_3 \int_0^y h(x,t)dt. \]

Integrating (2.31) with respect to \(x\) from 0 to \(y\), we have

\[ V^{p_1-p_4}(x,y) - V^{p_1-p_4}(0,y) \leq k_{p_1}^{p_3(p_1-p_4)}(x,0) - k_{p_1}^{p_3(0)}(0,0) \]

\[ + \left( \frac{p_1 - p_4}{p_1} \right) \left[ k_{p_3}^{p_3}(0,0) - k_{p_3}^{p_3}(x,0) + k_{p_3}^{p_3}(x,y) - k_{p_3}^{p_3}(0,y) \right] \]

\[ + \frac{p_3(p_1 - p_4)}{p_1} \int_0^y \int_0^s h(s,t)dtds, \quad p_1 \neq p_4. \]

(2.32)

Again from (2.23), we have

\[ V(0,y) = k^{p_3}(0,y). \]

(2.33)

Hence from (2.32) and (2.33), we obtain
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\[ V(x, y) \leq \left[ k \frac{p_2}{p_1} (p_1 - p_2) (0, y) + k \frac{p_2}{p_1} (x, 0) - k \frac{p_2}{p_1} (0, 0) \right. \]

\[ + \left( \frac{p_1 - p_4}{p_1} \right) \left( k^{p_2} (0, 0) - k^{p_2} (x, 0) + k^{p_2} (x, y) - k^{p_2} (0, y) \right) \]

\[ + p_3 \int_0^x \int_0^y h(s, t) dt ds \frac{1}{p_1 - p_4} , \quad p_1 \neq p_4. \]

(2.34)

From (2.34) and (2.24) we have the required result, and the theorem is proved.

In the following, we obtain some special cases of the results in Theorem 2.2 by suitable substitutions given for each case:

**Special Cases II.**

1) Let \( k(x, y) = c \geq 0 \) and \( p_4 = 1 \). From Theorem 2.2, we have, if

\[ u^{p_1} (x, y) \leq c^{p_2} + p_3 \int_0^x \int_0^y h(s, t) dt ds , \]

then

\[ u(x, y) \leq \left( c^{\frac{p_2}{p_1}} + p_3 (p_1 - 1) \int_0^x \int_0^y h(s, t) dt ds \right) \frac{1}{p_1 - p_4} , \quad p_1 \neq 1. \]  

(2.35)

2) Let \( k(x, y) = c \geq 0 \). From Theorem 2.2, we have, if

\[ u^{p_1} (x, y) \leq c^{p_2} + p_3 \int_0^x \int_0^y h(s, t) dt ds , \]

then

\[ u(x, y) \leq \left( c^{\frac{p_2}{p_1}} + p_3 (p_1 - p_4) \int_0^x \int_0^y h(s, t) dt ds \right) \frac{1}{p_1 - p_4} , \quad p_1 \neq p_4. \]  

(2.36)

3) If \( p_1 = p_2 = p_3 = 2 \) in case 1, we have, if
\[ u^2(x, y) \leq c^2 + 2 \iiint_{0}^{x} h(s, t)u(s, t)\,dtds, \]

then

\[ u(x, y) \leq c + \iiint_{0}^{x} h(s, t)\,dtds, \quad (2.37) \]

which is the Iang's inequality for functions of two independent variables.

4) If \( p_2 = p_3 = 1 \), and \( p_1 = 2 \) in case 1, we have, if

\[ u^2(x, y) \leq c^2 + 2 \iiint_{0}^{x} h(s, t)u(s, t)\,dtds, \]

then

\[ u(x, y) \leq \sqrt{c} + \frac{1}{2} \iiint_{0}^{x} h(s, t)\,dtds. \quad (2.38) \]

5) If \( p_1 = p_2 = p_3 = 2 \) and \( p_4 = 1 \) in case 2, we have, if

\[ u^2(x, y) \leq c^2 + 2 \iiint_{0}^{x} h(s, t)u(s, t)\,dtds, \]

then

\[ u(x, y) \leq c + \iiint_{0}^{x} h(s, t)\,dtds, \quad (2.39) \]

which is the same result (2.37).

6) If \( p_2 = p_3 = p_4 = 1 \), and \( p_1 = 2 \) in (2.36), we have, if

\[ u^2(x, y) \leq c^2 + 2 \iiint_{0}^{x} h(s, t)u(s, t)\,dtds, \]

then

\[ u(x, y) \leq \sqrt{c} + \frac{1}{2} \iiint_{0}^{x} h(s, t)\,dtds. \quad (2.40) \]

which is the same result (2.38).

7) If \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = 2 \) in (2.36), we have, if
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\[ u(x, y) \leq c + \int_0^y \int_0^x h(s, t) u^2(s, t) dt ds, \]

then

\[ u(x, y) \leq \left[ c^{-1} - \int_0^y \int_0^x h(s, t) dt ds \right]^{-1}. \] (2.41)

8) If \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = \frac{1}{2} \) in (2.36), we have, if

\[ u(x, y) \leq c + \int_0^y \int_0^x h(s, t) u^2(s, t) dt ds, \]

then

\[ u(x, y) \leq \left[ \frac{1}{c^2} + \frac{1}{2} \int_0^y \int_0^x h(s, t) dt ds \right]^2. \] (2.42)

In the following theorem, we consider functions of three independent variables.

**Theorem 2.3** Let \( u(x, y, z), h(x, y, z) \) be nonnegative continuous functions and \( k(x, y, z) > 1 \), defined for \( x, y, z \in \mathbb{R} \). Suppose that the derivatives \( k_1(x, y, z), k_2(x, y, z), k_3(x, y, z), k_{xy}(x, y, z), \) and \( k_{xyz}(x, y, z) \) be nonnegative and continuous functions defined for \( x, y, z \in \mathbb{R} \), and \( p_1, p_2, p_3, p_4 \in R_+ \). If

\[ u^{p_1}(x, y, z) \leq k^{p_2}(x, y, z) + p_3 \int_0^y \int_0^x \int_0^z h(r, s, t) u^{p_4}(r, s, t) dt ds dr, \] (2.43)

for \( x, y, z \in \mathbb{R} \) and \( 0 \leq r \leq x, 0 \leq s \leq y, 0 \leq t \leq z \), then
\[ u(x, y, z) \leq \left[ \frac{p_2}{k^{p_1}} (0, y, z) + k^{p_1} (0, 0, z) \right] + \frac{p_2}{k^{p_1}} (x, 0, z) - k^{p_1} (x, 0, 0) \]

\[ + k^{p_1} (x, y, 0) - k^{p_1} (0, y, 0) - k^{p_1} (x, 0, 0) + k^{p_1} (0, 0, 0) \]

\[ + \left( \frac{p_1 - p_4}{p_1} \right) \left[ k^{p_1} (x, y, z) - k^{p_2} (0, y, z) - k^{p_1} (x, 0, z) + k^{p_1} (0, 0, z) - k^{p_1} (x, y, 0) \right] \]

\[ + k^{p_1} (0, y, 0) + k^{p_1} (x, 0, 0) - k^{p_1} (0, 0, 0) \]

\[ + p_3 \int_0^x \int_0^y \int_0^z h(r, s, t) dtdsdr \frac{1}{k^{p_1}} , p_1 \neq p_4. \]

(2.44)

**Proof:** Let \( W^{p_1}(x, y, z) \) be the right hand side of (2.43), then

\[ W^{p_1}(x, y, z) = k^{p_1} (x, y, z) + p_3 \int_0^x \int_0^y \int_0^z h(r, s, t) u^{p_1} (r, s, t) dtdsdr, \]

(2.45)

From (2.43) and (2.45), we have \( W(x, y, z) > 0 \) and

\[ u(x, y, z) \leq W(x, y, z). \]

(2.46)

Differentiating (2.45) with respect to \( x \), and then with respect to \( y \), and then with respect to \( z \), we get

\[ p_1 W^{p_1-1} W_x = p_2 k^{p_2-1} k_x + p_3 \int_0^x \int_0^y \int_0^z h(x, s, t) u^{p_1} (x, s, t) dtds, \]

\[ p_1 (p_1 - 1) W^{p_1-2} W_{1x} + p_1 W^{p_1-1} W_y = p_2 k^{p_2-1} k_y + p_3 (p_2 - 1) k^{p_2-2} k_x k_y \]

\[ + p_3 \int_0^x \int_0^y \int_0^z h(x, y, t) u^{p_1} (x, y, t) dt, \]

and
ON SOME INTEGRAL INEQUALITIES...

\[ p_1W^{p-1}W_{xyz} + p_1(p_1 - 1)W^{p-2}W_{xy} + p_1(p_1 - 1)W^{p-2}(W_yW_z) + \]

\[ p_1(p_1 - 1)(p_1 - 2)W^{p-3}W_xW_yW_z = p_2k^{p-1}k_2 + p_2(p_2 - 1)k^{p-2}k_2k_3 + \]

\[ + p_2(p_2 - 1)k^{p-2}(k_3k_2) + p_2(p_2 - 1)(p_2 - 2)k^{p-3}k_3k_2k_3 + \]

\[ + p_3h(x, y, z)u^{p_3} (x, y, z). \]  

(2.47)

From (2.46) and (2.47), we have

\[ p_1W^{p-1}W_{xyz} + p_1(p_1 - 1)W^{p-2}W_xW_y + p_1(p_1 - 1)W^{p-2}(W_yW_z) + \]

\[ p_1(p_1 - 1)(p_1 - 2)W^{p-3}W_xW_yW_z \leq p_2k^{p-1}k_2 + p_2(p_2 - 1)k^{p-2}k_2k_3 + \]

\[ p_2(p_2 - 1)k^{p-2}(k_3k_2) + p_2(p_2 - 1)(p_2 - 2)k^{p-3}k_3k_2k_3 + \]

\[ + p_3h(x, y, z)W^{p_3} (x, y, z). \]  

(2.48)

From (2.48), we obtain

\[ p_1W^{p-p_{x-1}}W_{xyz} + p_1(p_1 - 1)W^{p-p_{x-2}}W_xW_y + p_1(p_1 - 1)W^{p-p_{x-2}}(W_yW_z) + \]

\[ p_1(p_1 - 1)(p_1 - 2)W^{p-p_{x-3}}W_xW_yW_z \leq \frac{p_2k^{p-1}k_2}{W^{p_3}(x, y, z)} + \frac{p_2(p_2 - 1)k^{p-2}k_2k_3}{W^{p_3}(x, y, z)} + \]

\[ \frac{p_2(p_2 - 1)k^{p-2}(k_3k_2)}{W^{p_3}(x, y, z)} + \frac{p_2(p_2 - 1)(p_2 - 2)k^{p-3}k_3k_2k_3}{W^{p_3}(x, y, z)} + p_3h(x, y, z). \]  

(2.49)

Since \( k(x, y, z) > 1 \), then from (2.45) and (2.49), we get
\[ p_1 W^{p_1-p_2-1} W_{xy} + p_1 (p_1 - 1) W^{p_1-p_2-2} W_x W_y + p_1 (p_1 - 1) W^{p_1-p_2-2} (W_x W_y)_z + \]
\[ p_1 (p_1 - 1)(p_1 - 2) W^{p_1-p_2-3} W_x W_y W_z \leq p_2 k^{p_2-1} k_{xyz} + p_2 (p_2 - 1) k^{p_2-2} k_x k_y + \]
\[ p_2 (p_2 - 1) k^{p_2-2} (k_x k_y)_z + p_2 (p_2 - 1)(p_2 - 2) k^{p_2-3} k_x k_y k_z + p_3 h(x, y, z). \]

(2.50)

Integrating (2.50) with respect to \( z \) from 0 to \( z \), we have
\[ p_1 W^{p_1-p_2-1} W_x - p_1 W^{p_1-p_2-1} (x, y, 0) W_y (x, y, 0) + p_1 (p_1 - 1) W^{p_1-p_2-2} W_y - \]
\[ p_1 (p_1 - 1) W^{p_1-p_2-2} (x, y, 0) W_x (x, y, 0) W_y (x, y, 0) \leq p_2 k^{p_2-1} k_{xy} - p_2 k^{p_2-1} (x, y, 0) k_{xy} (x, y, 0) \]
\[ + p_2 (p_2 - 1) k^{p_2-2} (x, y, z) k_x k_y - p_2 (p_2 - 1) k^{p_2-2} (x, y, 0) k_x (x, y, 0) k_y (x, y, 0) \]
\[ - p_2 \int_0^z W^{p_1-p_2-2} (x, y, t) W_y (x, y, t) W_y (x, y, t) dt \]
\[ - p_2 (p_2 - 1) \int_0^z W^{p_1-p_2-3} (x, y, t) W_x (x, y, t) W_y (x, y, t) W_y (x, y, t) dt + p_3 \int_0^z h(x, y, t) dt. \]

(2.51)

From (2.51), we have
\[ p_1 W^{p_1-p_2-1} W_y - p_1 W^{p_1-p_2-1} (x, y, 0) W_y (x, y, 0) + p_1 (p_1 - 1) W^{p_1-p_2-2} W_y - \]
\[ p_1 (p_1 - 1) W^{p_1-p_2-2} (x, y, 0) W_y (x, y, 0) W_y (x, y, 0) \leq p_2 k^{p_2-1} k_{xy} - \]
\[ p_2 k^{p_2-1} (x, y, 0) k_{xy} (x, y, 0) + p_2 (p_2 - 1) k^{p_2-2} (x, y, z) k_x k_y - \]
\[ p_2 (p_2 - 1) k^{p_2-2} (x, y, 0) k_x (x, y, 0) k_y (x, y, 0) + p_3 \int_0^z h(x, y, t) dt. \]

(2.52)

From (2.45), we have
ON SOME INTEGRAL INEQUALITIES…

\[ W(x, y; 0) = k^p \ (x, y; 0) \]

and

\[ W_x(x, y; 0) = \frac{p_2}{p_1} k^{p_2-1} (x, y; 0)k_x(x, y; 0), \]

\[ W_y(x, y; 0) = \frac{p_2}{p_1} k^{p_3-1} (x, y; 0)k_y(x, y; 0). \]

Hence

\[ W_{xy}(x, y; 0) = \frac{p_2}{p_1} (\frac{p_2}{p_1} - 1) k^{p_2-2} (x, y; 0)k_x(x, y; 0)k_y(x, y; 0) + \frac{p_2}{p_1} k^{p_2-1} (x, y; 0)k_{xy}(x, y; 0), \]

from which (2.52) becomes

\[ p_1W^{p_1}W_x + p_1(p_1 - 1)W^{p_2}W_y \leq \]

\[ p_2(p_2 - 1)k^{p_2-2} (x, y; 0)k_x(x, y; 0)k_y(x, y; 0) + p_2 k^{p_2-1} (x, y; 0)k_{xy}(x, y; 0) + \]

\[ p_2k^{p_1-1}(x, y; z)k_{xy} - p_2k^{p_1-1}(x, y; 0)k_{xy}(x, y; 0) + p_2(p_2 - 1)k^{p_2-2}(x, y; z)k_{xy} - \]

\[ p_2(p_2 - 1)k^{p_2-2}(x, y; 0)k_x(x, y; 0)k_y(x, y; 0) + p_2 \int_0^z h(x, y; t)dt. \]  

(2.53)

Interchanging \( y \) by \( t \) in (2.53), and then integrating with respect to \( t \) from 0 to \( y \), we have
\begin{align*}
& p_1 W^{n-p_1} W - p_1 W^{p_1} (x,0,z) W (x,0,z) \leq p_2 k^{p_2/p_1} (x,y,0) k^* (x,y,0) - \\
& p_2 k^{p_2/p_1} (x,0,0) k^* (x,0,0) + p_2 k^{p_1} (x,y,z) k^* - p_2 k^{p_2-1} (x,0,z) k^* (x,0,z) \\
& \quad - p_2 k^{p_2-1} (x,y,0) k^* (x,y,0) + p_2 k^{p_2-1} (x,0,0) k^* (x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t) dt ds \\
& - \frac{p_2^2 P_4}{p_1} \int_0^y \int_0^{p_2-p_1-2} k^* (x,s,0) k^* (x,s,0) ds, \\
& \text{from which we have} \\
& p_1 W^{n-p_1} W - p_1 W^{p_1} (x,0,z) W (x,0,z) \leq p_2 k^{p_2/p_1} (x,y,0) k^* (x,y,0) - \\
& p_2 k^{p_2/p_1} (x,0,0) k^* (x,0,0) + p_2 k^{p_1} (x,y,z) k^* - p_2 k^{p_2-1} (x,0,z) k^* (x,0,z) - \\
& p_2 k^{p_2-1} (x,y,0) k^* (x,y,0) + p_2 k^{p_2-1} (x,0,0) k^* (x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t) dt ds
\end{align*}

From (2.45), we have

\[ W(x,0,z) = k^* (x,0,z), \]

and

\[ W_x (x,0,z) = \frac{p_2}{p_1} k^{p_2-1} (x,0,z) k^* (x,0,z). \] (2.55)

From (2.54) and (2.55), we have
ON SOME INTEGRAL INEQUALITIES...

\[ p_1 W^{\eta_1 - \eta_2} W_\gamma \leq p_2 k^{\frac{\eta_1 - \eta_2}{\eta_1}} (x,0,z) k_\gamma (x,0,z) + p_2 k^{\frac{\eta_2 - \eta_1}{\eta_2}} (x,y,0) k_\gamma (x,y,0) \]

\[- p_2 k^{\frac{\eta_1 - \eta_2}{\eta_1}} (x,0,0) k_\gamma (x,0,0) + p_2 k^{\eta_2 - \eta_1} k_\gamma (x,0,z) + p_2 k^{\frac{\eta_2 - \eta_1}{\eta_2}} (x,0,0) k_\gamma (x,0,z) \]

\[- p_2 k^{\eta_1 - \eta_2} (x,y,0) k_\gamma (x,y,0) + p_2 k^{\eta_1 - \eta_2} (x,0,0) k_\gamma (x,0,0) + p_3 \int_0^y \int_0^z h(x,s,t) dt ds \]

Again integrating (2.56) with respect to \( x \), we get

\[ \left( \frac{p_1}{p_1 - p_2} \right) \left[ W^{\eta_1 - \eta_2} (x,y,z) - W^{\eta_1 - \eta_2} (0,y,z) \right] \leq \left( \frac{p_1}{p_1 - p_2} \right) \left( \frac{p_2 (\eta_1 - \eta_2)}{k^{\eta_1 - \eta_2} (x,0,z)} \right) (x,0,z) \]

\[- k^{\frac{\eta_2 (\eta_1 - \eta_2)}{\eta_1}} (0,0,z) + k^{\frac{\eta_2 (\eta_1 - \eta_2)}{\eta_1}} (x,y,0) - k^{\frac{\eta_2 (\eta_1 - \eta_2)}{\eta_1}} (0,y,0) - k^{\frac{\eta_2 (\eta_1 - \eta_2)}{\eta_1}} (x,0,0) \]

\[ + k^{\frac{\eta_2 (\eta_1 - \eta_2)}{\eta_1}} (0,0,0) \]

\[- k^{\eta_1} (x,y,0) + k^{\eta_2} (0,y,0) - k^{\eta_1} (x,0,0) - k^{\eta_2} (0,0,0) + p_3 \int_0^y \int_0^z h(r,s,t) dt ds dr \]

(2.57)

From (2.45), we obtain

\[ W(0,y,z) = k^{\eta_1} (0,y,z). \]

So from (2.57) and (2.58), we get
Hence, from (2.46) and (2.59) we get the required result, and the theorem is proved.

In the following, we obtain some special cases of the results in Theorem 2.3 by suitable substitutions given for each case:

**Special Cases III.**

1) Let \( k(x, y, z) = c \geq 0 \). From Theorem 2.3, we have, if

\[
W(x, y, z) \leq \frac{p_2}{k^{p_1}} (0, y, z) + k^{p_1} (x,0,z) - k^{p_1} (0,0,z)
\]

\[
+ \frac{p_2}{k^{p_1}} (x, y, 0) - k^{p_1} (0, y, 0) - k^{p_1} (x,0,0) + k^{p_1} (0,0,0)
\]

\[
+ \left( \frac{p_1 - p_4}{p_1} \right) \left[ k^{p_2} (x, y, z) - k^{p_2} (0, y, z) - k^{p_2} (x,0,z) + k^{p_2} (0,0,z) - k^{p_2} (x, y, 0) \right]
\]

\[
+ k^{p_2} (0, y, 0) + k^{p_2} (0,0,0) - k^{p_2} (0,0,0) + p_3 \int_0^x \int_0^y \int_0^z h(r,s,t)dt ds dr \right] \frac{1}{k^{p_1 - p_4}},
\]

then

\[
u_{p_1} (x, y, z) \leq c^{p_2} + p_3 \int_0^x \int_0^y \int_0^z h(r,s,t)dt ds dr,
\]

then

\[
u(x, y, z) \leq \left( \frac{p_2}{c^{p_1}} + \frac{p_2}{p_1} (p_1 - p_4) \int_0^x \int_0^y \int_0^z h(r,s,t)dt ds dr \right) \frac{1}{k^{p_1 - p_4}}, \quad p_1 \neq p_4. \quad (2.59)
\]

2) If \( p_1 = p_2 = p_3 = 2 \) and \( p_4 = 1 \) in case 1, we have

\[
u^2 (x, y, z) \leq c^2 + 2 \int_0^x \int_0^y \int_0^z h(r,s,t)dt ds dr.
\]

then

\[
u(x, y, z) \leq c + \int_0^x \int_0^y \int_0^z h(r,s,t)dt ds dr. \quad (2.61)
\]
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3) If \( p = p = p = 1 \), and \( p = 2 \) in case 1, we have

\[
\int_0^1 \int_0^1 \int_0^1 h(r,s,t)u(r,s,t)dt ds dr,
\]

then

\[
u(x,y,z) \leq \sqrt{c} + \frac{1}{2} \int_0^1 \int_0^1 \int_0^1 h(r,s,t)dt ds dr. \tag{2.62}
\]

4) If \( p = p = p = 1 \) and \( p = 2 \) in case 1, we have

\[
u(x,y,z) \leq c + \int_0^1 \int_0^1 \int_0^1 h(r,s,t)u^2(r,s,t)dt ds dr,
\]

then

\[
u(x,y,z) \leq \left[ c^{-1} - \int_0^1 \int_0^1 \int_0^1 h(r,s,t)dt ds dr \right]^{-1}. \tag{2.63}
\]

6) If \( p = p = p = 2 \) and \( p = 1 \) in Theorem 2.3, we have, if

\[
u^2(x,y,z) \leq k^2(x,y,z) + \int_0^1 \int_0^1 \int_0^1 h(r,s,t)u(r,s,t)dt ds dr,
\]

then

\[
u(x,y) \leq k(0,y,z) + k(x,0,z) - k(0,0,z) + k(x,y,0) - k(0,y,0) - k(x,0,0)
\]

\[
+ k(0,0,0) + \frac{1}{2}(k^2(x,y,z) - k^2(0,y,z) - k^2(x,0,z) + k^2(0,0,z) - k^2(x,y,0)
\]

\[
+ k^2(0,y,0) + k^2(x,0,0) - k^2(0,0,0) + 2 \int_0^1 \int_0^1 \int_0^1 h(r,s,t)dt ds dr \right\}. \tag{2.64}
\]

6) If \( p = p = p = 1 \), and \( p = 2 \) in Theorem 2.3, we have, if
\[ u(x, y, z) \leq k(x, y, z) + \int \int \int h(r, s, t)u^2(r, s, t)dt ds dr, \]
then
\[ u(x, y) \leq \left[ k^{-1}(0, y, z) + k^{-1}(x, 0, z) - k^{-1}(0, 0, z) + k^{-1}(x, y, 0) \right. \]
\[ - k^{-1}(0, y, 0) - k^{-1}(x, 0, 0) + k^{-1}(0, 0, 0) - k(x, y, z) + k(0, y, z) \]
\[ + k(x, 0, z) - k(0, 0, z) + k(x, y, 0) - k(0, y, 0) - k(x, 0, 0) + k(0, 0, 0) \]
\[ - \int \int \int h(r, s, t)dt ds dr \right]^{-1}. \]

(2.65)

7) If \( p_2 = p_3 = p_4 = 1 \) , and \( p_1 = 2 \) in Theorem 2.3, we have, if
\[ u^2(x, y, z) \leq k(x, y, z) + \int \int \int h(r, s, t)u(r, s, t)dt ds dr, \]
then
\[ u(x, y, z) \leq \frac{1}{2} k^2(0, y, z) + \frac{1}{2} k^2(x, 0, z) - \frac{1}{2} k^2(0, 0, z) + \frac{1}{2} k^2(x, y, 0) - \frac{1}{2} k^2(0, y, 0) \]
\[ - k^2(x, 0, 0) + \frac{1}{2} \left( k(x, y, z) - k(0, y, z) - k(x, 0, z) + k(0, 0, z) \right) \]
\[ - k(x, y, 0) + k(0, y, 0) + k(x, 0, 0) - k(0, 0, 0) + \int \int \int h(r, s, t)dt ds dr \right], \)

(2.66)

8) If \( p_1 = p_2 = p_3 = 1 \) , and \( p_4 = \frac{1}{2} \) in Theorem 2.3, we have, if
\[ u(x, y, z) \leq k(x, y, z) + \int \int \int h(r, s, t)u^2(r, s, t)dt ds dr, \]
then
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\[ u(x,y,z) \leq \left[ \frac{1}{k^2}(0,0,z) + k^2(x,0,z) - k^2(0,0,z) + k^2(x,y,0) - k^2(0,y,0) - k^2(x,0,0) \right. \]
\[ + \frac{1}{2} \left( k(x,y,z) - k(0,y,z) - k(x,0,z) + k(0,0,z) - k(x,y,0) \right) \]
\[ + k(0,0,0) + k(x,0,0) - k(0,0,0) + \int_0^t \int_0^s \int_0^r h(r,s,t)dt ds dr \right]^{\frac{1}{2}}. \]  
(2.67)

3. Applications

In this section, we present some applications of **Theorem 2.1** and **Theorem 2.2**. Our aim is to study the boundedness of the solutions of some hyperbolic partial differential equations.

In the following two examples, we use **Theorem 2.1** to obtain explicit bounds on the solutions of some first order partial differential equations.

**Example 1.** Let \( u_t(t) = h(t)u^2(t) \) where \( u(0) = c \) and \( h(t) \leq t \), then we have the partial differential inequality

\[ u_t(t) \leq tu^2(t). \]  
(3.1)

Interchanging \( t \) by \( s \) in (3.1), and then integrating (3.1) with respect to \( s \) from 0 to \( t \), we have the integral inequality

\[ u(t) \leq c + \int_0^t su^2(s)ds. \]

By (2.11) with \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = 2 \) we get

\[ u(t) \leq \left[ c^{-1} - \frac{1}{2} t^2 \right]^{-1}. \]

**Example 2.** Let \( u(x)u_x(x) = e^{2x} + h(x)u(x), \) \( u(0) = 0, \) and \( h(x) \leq x, \) then we have the partial differential inequality

\[ u(x)u_x(x) \leq e^{2x} + xu(x). \]  
(3.2)
Interchanging $x$ by $s$ in (3.2), and then integrating with respect to $s$ from 0 to $x$, we have the integral inequality

$$u^2(x) \leq \left(e^x\right)^2 + 2 \int_0^x su(s)ds.$$  

By Theorem 2.1 with $k(x) = e^x$, $p_1 = p_2 = p_3 = 2$, and $p_4 = 1$, we have

$$u(x) \leq \frac{1}{2} \left[1 + e^{2x} + x^2\right] \quad e^x > 1.$$  

We now present three examples showing how Theorem 2.2 can be used to obtain explicit bounds on the solutions of some second order partial differential equations

**Example 3.** Let $u(x, y)u_{xy}(x, y) + u_x(x, y)u_y(x, y) = h(x, y)u(x, y)$ where $u_x(x, 0) = 0$, $u(0, y) = c$, $h(x, y) \leq ye^x$, then we have the partial differential inequality

$$u(x, y)u_{xy}(x, y) + u_x(x, y)u_y(x, y) \leq ye^x u(x, y). \quad (3.3)$$  

Integrating (3.3) with respect to $y$, we get

$$u(x, y)u_x(x, y) \leq \int_0^y te^x u(x, t)dt. \quad (3.4)$$  

Again, integrating (3.4) with respect to $x$, we have the nonlinear integral inequality

$$u^2(x, y) \leq c^2 + 2 \int_0^x \int_0^y te^x u(s, t)dt ds.$$  

By Theorem 2.2 with $p_1 = p_2 = p_3 = 2$ and $p_4 = 1$, we get

$$u(x, y) \leq c + \int_0^x \int_0^y te^x dt ds = c + \frac{1}{2} y^2 (e^x - 1).$$  

**Example 4.** Let $u_{xy}(x, y) = h(x, y)u^2(x, y)$, where $u(0, y) = u(x, 0) = c$ and $u_x(x, 0) = 0$, $h(x, y) \leq x \sin y$, then we have the partial differential inequality

$$u_{xy} \leq x \sin y \ u^2(x, y). \quad (3.5)$$  

Integrating (3.5) with respect to $y$ and with respect to $x$, we get the nonlinear integral inequality
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\[ u(x, y) \leq c + \int_{0}^{x} \int_{0}^{y} s \sin t \ u^2(s, t) dt \, ds. \]

By Theorem 2.2 with \( p_1 = p_2 = p_3 = 1 \) and \( p_4 = 2 \), we have

\[ u(x, y) \leq c^{-1} - \int_{0}^{x} \int_{0}^{y} s \sin t \ dt \, ds \leq \left[ c^{-1} + \frac{1}{2} x^2 (\cos y - 1) \right]^{-1}. \]

Example 5.
Let \( u_{xy}(x, y) = 2xe^{y} + h(x, y)u^2(x, y) \), \( u_{x}(x, 0) = 2x \), \( u(0, y) = 0 \), \( h(x, y) \leq x + y \), then we have partial differential inequality

\[ u_{xy} \leq 2xe^{y} + (x + y)u^2(x, y). \] (3.6)

Integrating (3.6) with respect to \( y \) and then with respect to \( x \), we get

\[ u(x, y) \leq x^2 e^{y} + \int_{0}^{x} \int_{0}^{y} (s + t)u^2(s, t) dt \, ds. \]

From Theorem 2.2 with \( k(x, y) = x^2 e^{y} \), \( p_1 = p_2 = p_3 = 1 \), and \( p_4 = 2 \),
we have

\[ u(x, y) \leq \left[ x^{-2} + x^2 - x^2 e^{y} - \frac{1}{2} xy(x + y) \right]^{-1}, \]

where \( x^2 e^{y} > 1 \).

References