Journal of The Islamic University of Gaza,(Series of Natural Studies &Engineering) Vol.13, No.2, P155-161,2005
ISSN 1726-6807, http://www.iugaza.edu/ara/research/

ON PROPERTIES OF GEOMETRY OF TYPE $D_{n,k}$

Mohammed Suliman El-Atrash
Professor of mathematics Science College, Mathematics Department
Islamic University of Gaza, Gaza, Palestine
E-mail: matrash@mail.iugaza.edu

Abdelsalam Abou zayda
Assistant professor of mathematics
Math. Department, Al-Aqsa University, Gaza, Palestine
E-mail : zayda8@hotmail.com

Abstract: In this paper we present a general case of the class of geometries $D_{n,2}$, $D_{n,3}$ and $D_{n,4}$ that is a point-line geometry of type $D_{n,k}(F)$ where $k \geq 2$, $n \geq k+3$ and F is a field. We define the varieties of the geometry and we prove that the geometry is a parapolar with diameter k+1. The properties related to the relations between the varieties will be investigated.

1- Introduction

In [5] and [7] it has been proved that the class of geomtries $D_{n,2}$, $D_{n,3}$ and $D_{n,4}$ are weak parapolar geometries with diameters 3, 4 and 5 respectively. We prove in this paper that the general case $D_{n,k}(F)$ is a parapolar geometry with diameter k+1 for $k \geq 2$ and $n \geq k+3$ and we get the same results with respect to the relations between symplecta themselves and the relations between points and symplecta.

$x^{\perp}$ means the set of all points in P collinear with x, including x itself. ***A subspace*** of a point-line geometry $\Gamma=(P, L)$ is a subset $X \subseteq P$ such that any line which has at least two of its incident points in X has all of its incident points in X. $\langle X \rangle$ means the intersection over all subspaces containing X, where $X \subseteq P$. Lines incident with more than two points are called *thick* lines, those incident with exactly two points are called *thin* lines.

The singular rank of a space Γ is the maximal number n (possibly ∞) for which there exist a chain of distinct subspaces $\emptyset \neq X_0 \subset X_1 \subset ... \subset X_n$ such that X_i is singular for each i, $X_i \neq X_j$, $i \neq j$, for example rank($\emptyset$)=-1, rank($\{p\}$)=0 where p is a point and rank(L)=1 where L a line.

In a point-line geometry $\Gamma=(P, L)$, a path of length n is a sequence of n+1 $(x_0,x_1,..,x_n)$ where, (x_i,x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. ***A geodesic*** from a point x to a point y is a path of minimal possible length with initial point *x* and end point *y*. We denote this length by $d_{\Gamma}(x, y)$.

A geometry Γ is called ***connected*** if and only if for any two of its points there is a path connecting them. A subset X of P is said to be ***convex*** if X contains all points of all geodesics connecting two points of X.

A polar space is a point-line geometry Γ=(P, L) satisfying the Buekenhout-Shult axiom [1] :
For each point-line pair (p, l) with p not incident with l; p is collinear with one or all points of l, that is $|p^{\perp}\cap l|=1$ or else $p^{\perp}\supset l$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of Γ for every point $p\in$P.

A point-line geometry Γ=(P, L) is called ***a projective plane*** if and only if it satisfies the following conditions [2]:

(i) Γ is a linear space; every two distinct points x, y in P lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of them are on a line.

A point-line geometry Γ=(P, L) is called ***a projective space*** if the following conditions are satisfied:

(i) every two points lie exactly on one line ,

(ii) if l_1, l_2 are two lines $l_1\cap l_2\neq\emptyset$, then $\langle l_1, l_2\rangle$ is a projective plane. ($\langle l_1, l_2\rangle$ means the smallest subspace of Γ containing l_1 and l_2.)

A point-line geometry Γ=(P, L) is called ***a parapolar*** *space* if and only if it satisfies the following properties:

(i) Γ is a connected gamma space,

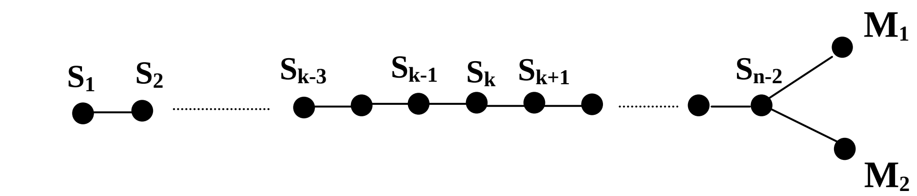
(ii) for every line l; $l^{\perp}$ s not a singular subspace,

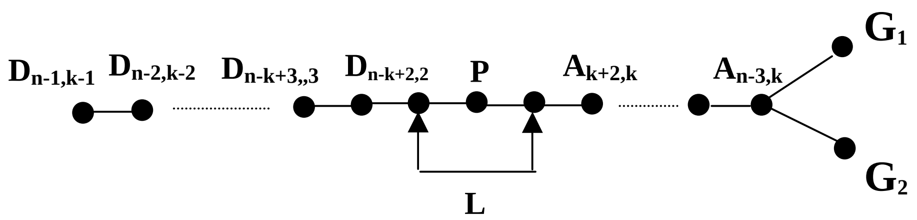
(iii) for every pair of non-collinear points x, y; $x^{\perp}\cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If x, y are distinct points in P, and if $|x^{\perp}\cap y^{\perp}|=1$, then (x, y) is called ***a special pair***, and if $x^{\perp}\cap y^{\perp}$ is a polar space, then (x, y) is called ***a polar pair*** (or ***a symplectic pair***). A parapolar space is called ***a strong parapolar*** space if it has no special pairs [4]**.**

2. Construction of $D_{n,k}$ (F)

Consider the polar space $\Delta=\Omega^{+}(2n, F)$ that comes from a vector space V of dimension 2n over a finite field F=GF(k) with a symmetric hyperbolic bilinear form B. $\mathbf{S_I}$ is the set of all totally isotropic i-dimensional subspaces of V*;* $1\leq i\leq n-2$. The two classes $\mathbf{M_1}$, $\mathbf{M_2}$ consist of maximal totally isotropic n-dimensional subspaces. Two n-subspaces fall in the same class if their intersection is of odd dimension.





$D_{n,k}(F)$

The geometry of type **$D_{n,k}(F)$** is the point-line geometry (P, L), whose set of points P is the collection of all k-dimensional subspaces of the vector space V, and whose lines are the pairs (A, B) where A is k-1-dimensional subspace of k+1-subspace B – that is, the set of (k-1, k+1)-subspace of V. A point C is incident with a line (A, B) if and only if $A \subset C \subset B$ as a subspaces of V.

To define the collinearity, let C_1 and C_2 be two point (the points are the T.I k-spaces), then C_1 is collinear to C_2 if and only if the intersection of $C_1 \cap C_2$ =k-1-space and $\langle C_1, C_2 \rangle$ =k+1-space.

The elements of the classes $\mathbf{G_1}$ and $\mathbf{G_2}$ are Grassmannian geometries of type $\mathbf{A_{n-1,k}}$ for more details about these kinds of geometries see [6] and [3]

There are two kinds of symlecta (1) The first kind is the convex polar spaces $A_{3,2}$ that represent the (k-2, k+2) subspaces of V. Then symplecton S of kind $A_{3,2}$ is the set of TI k-dimensional spaces that contain the TI k-2-dimensional space and contained in the TI k+2-dimensional space. (2) The second kind of symplecta is the convex polar spaces of type $D_{n-k+1,1}$, that represent the collection of all TI k-1-subspaces of V. Thus this kind of symplecta is defined as the collection of all TI k-subspaces of V that contain such TI k-1-subspaces.

Notation. Let the map Ψ:P$\to$V defined above, i.e., Ψ(p) is the T.I. k-dimensional subspace corresponding to the point p. We will use Ψ for the rest of the varieties of the geometry; for example $\Psi(D_{n-1,k-1})$ is the T.I. 1-dimensional subspace corresponding to a geometry of type $D_{n-1,k-1}(F)$. The inverse map Ψ^{-1} will be used for the inverse.

3-The main result.

The following theorem represents the first part of the main result of this paper that is: $D_{n,k}$ (F) *is weak parapolar geometry of diameter* k+1.

3.1 Theorem. *Let* $\Gamma=(P, L)$ *be the geometry* of type $D_{n,k}(F)$. *Thus*:

i- Γ *is of diameter* k+1,.

ii- Γ *is parapolar geometry.*

Proof: i We shall use the mathematical induction for $k\geq2$. If k=2, then we have geometry of type $D_{n,2}(F)$ with diameter 3 [5]. Now for all the cases k-1, k-2, ….., 4, 3, 2 the geometries $D_{n,k-1}$, $D_{n,k-2}$,….., $D_{n,4}$, $D_{n,3}$, $D_{n,2}$ have the diameters k, k-1, ….., 5, 4, 3 respectively. Now we shall prove that the diameter of $D_{n,k}$ is k+1. Suppose that p and q are two arbitrary points of $D_{n,k}$,that represent TI k-spaces $\Psi(p)=C_1=\langle x_1,x_2,\dots,x_k\rangle$ and $\Psi(q)=C_2=\langle y_1,y_2,\dots,y_k\rangle$. If $\langle x_1,x_2,\dots,x_k\rangle\cap\langle y_1,y_2,\dots,y_k\rangle$.=1-space, 2-space,……or (k-2)-space, then the two points lie in the geometries $D_{n,k-1}$, $D_{n,k-2}$,……and $D_{n,2}$ respectively. Thus by the hypotheses the diameter are k, k-1,….,3 respetively.

If $\langle x_1,x_2,\dots,x_k\rangle\cap\langle y_1,y_2,\dots,y_k\rangle$.=(k-1)-space, without loss of generality we take $z_1=x_2=y_2$, $z_2=x_3=y_3$,…..,$z_{k-1}=x_k=y_k$, then $\langle x_1,x_2,\dots,x_k\rangle\cap\langle y_1,y_2,\dots,y_k\rangle=\langle z_1,z_2,\dots,z_{k-1}\rangle$. Thus we have two cases:

1- $x_1^{\perp}\cap C_2=C_2$

2- $x_1^{\perp}\cap C_2=\langle z_1,z_2,\dots,z_{k-1}\rangle$.

In case 1, $\langle x_1,y_1,z_1,z_2,\dots,z_{k-1}\rangle$ forms a (k+1)-space and $(\langle z_1,z_2,\dots,z_{k-1}\rangle,\langle x_1,y_1,z_1,z_2,...,z_{k-1}\rangle)$ is the line that contains the two points. Thus d(p,q)=1.

In case 2, $B(x_1,y_1)\neq0$ and C_2 is contained in a maximal TI n-space K, say $K=\langle y_1,z_1,z_2,...,z_{k-1},z_k,z_{k+1},\dots z_{n-1}\rangle$. Then $x_1^{\perp}\cap K$ is a hyperplane of K, say $H=\langle z_1,z_2,...,z_{k-1},z_k,z_{k+1},\dots z_{n-1}\rangle$. We have two (k+1)-spaces $\langle x_1,z_1,z_2,...,z_{k-1},z_k\rangle$ and $\langle y_1,z_1,z_2,...,z_{k-1},z_k\rangle$, then $\langle z_1,z_2,...,z_{k-1},z_k\rangle$ is a point that is collinear to each of C_1 and C_2. Thus d(p,q)=2.

If $\langle x_1,x_2,\dots,x_k\rangle\cap\langle y_1,y_2,\dots,y_k\rangle$=0-space, then we the following cases:

1- $x_i^{\perp}\cap C_2=C_2$, i=1, 2, …,k.

2- $x_i^{\perp}\cap C_2=\langle y_1,y_2,\dots,y_{i-1},y_{i+1},\dots,y_k\rangle$ and $x_j^{\perp}\cap C_2=C_2$ where $i\neq j$ and i, j=1, 2,…,k.

3- $x_i^{\perp}\cap C_2=\langle y_1,y_2,\dots,y_{i-1},y_{i+1},\dots,y_k\rangle$, , i=1, 2, …,k.

In the first two cases we have always $d(p,q) \leq k$ because at least one of the vectors x_i we have $x_i^{\perp} \cap C_2 = C_2$.
In case 3, the least number of vertices that can make a geodesic between p and q is equal k. Since C_2 is contained in a maximal TI n-space, then without loss of generality we can choose the points $r_1, r_2, \ldots, r_k$ such that $\Psi(r_1) = \langle z, x_2, x_3, \ldots, x_k \rangle$, $\Psi(r_2) = \langle z, y_1, x_3, \ldots, x_k \rangle$, $\Psi(r_3) = \langle z, y_1, y_2, x_4, \ldots, x_k \rangle, \ldots\ldots,$ $\Psi(r_k) = \langle z, y_1, y_2, \ldots, y_{k-1} \rangle$.where z is a vector in the n-space in which C_2 is contained. Thus $p \sim r_1$, $r_1 \sim r_2, \ldots, r_k \sim q$ i.e., $d(p, q) = k+1$ and diameter of $D_{n,k}$ is k+1.
ii- To prove that the geometry is parapolar, we first prove that $D_{n,k}$ is gamma space. Assume that *l* is a line containing two points p and q such that $\psi(p) = \langle x_1, x_2, \ldots, x_k \rangle$ and $\psi(p) = \langle y_1, y_2, \ldots, y_k \rangle$. Then by the definition of collinearity, the two points form a TI (k+1)-space $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle$, where $z_1 = x_2 = y_2$, $z_2 = x_3 = y_3, \ldots$, $z_{k-1} = x_k = y_k$. Let s be a point not in *l* such that s is collinear to p and q, we shall prove that s is collinear to every point incident to *l*. Since s is collinear to p and q, then s does not contain the TI (k-1)-space $\langle z_1, z_2, \ldots, z_{k-1} \rangle$ and form (k+1)-spaces with p and q. Now assume that r is arbitrary point in *l* such that $r \neq p \neq q$, then $\psi(r)$, $\psi(s)$ contain the same (k-1)-space $\langle z_1, z_2, \ldots, z_{k-1} \rangle$ and they are contained in (k+1)-space $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle$. Thus s is collinear to r.
Now for the completion of the proof we need to show that for any line *l* ; $l^{\perp}$ is non-singular. For this purpose we take the same above line i.,e., $l = (\langle z_1, z_2, \ldots, z_{k-1} \rangle, \langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle)$. Thus the points r, s can be chosen to be non-collinear and each of them is collinear to p and q. Since $\psi(s)$ is contained in a maximal n-space, then we take $\langle z_1, z_2, \ldots, z_{k-2}, x_1, y_1 \rangle$, $\langle z_1, z_2, \ldots, z_{k-1}, u \rangle$ to the points r, s where u is a vector in the n-space not in $\psi(s)$. Now $\langle z_1, z_2, \ldots, z_{k-2}, x_1, y_1 \rangle \cap \langle z_1, z_2, \ldots, z_{k-1}, u \rangle$ does not contain a (k-1)-space, this means that r is not collinear to s but $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle$ and $\langle z_1, z_2, \ldots, z_{k-1}, y_1, u \rangle$ form (k+1)-spaces, then r, s are collinear to each of p and q. Thus $l^{\perp}$ is non-singular. ♦

3.2 Corollary. $\mathbf{D_{n,k}}$ *is a non-degenerate weak geometry.*

Proof. By the definition of collinearity and By Theorem 3.1, $D_{n,k}$ is a non-degenerate. Since the geometry of type $D_{n,2}$ is a weak sub-geometry of $D_{n,k}$ [5], then $D_{n,k}$ is a weak. ♦

4- Properties of $\mathbf{D_{n,k}}$

It has been proved that for any pair of distinct sympleta (S_1, S_2) of the class of geometries $D_{n,2}$, $D_{n,3}$, $D_{n,4}$, $\mathrm{rank}(S_1 \cap S_2) = -1, 0, 2$, Moreover for any pair of non-incident point and symplecton (p, S), we have rank($p^{\perp}\cap S$)=-1,1,2 see [5] and [7]. In this Paper we prove the same result is satisfied for the general case $D_{n,k}$.

Remark. Each geometry of the class $\mathbf{G_1}$ or $\mathbf{G_2}$ of $D_{n,k}$ is denoted by A_T, where T is the TI n-space that corresponds to such geometry and A_D denotes to the symlecton of type $A_{3,2}$, where D is the TI (k+2)-space that corresponds to such symplecton.

4.1 Theorem. *Let* Γ *be a geometry of type* $D_{n,k}$, *then*:

1- Let A_{D1},A_{D2} be two distinct symplecta of type $A_{3,2}$ with then rank($A_{D1}\cap A_{D2}$)=-1,0,2.

2- If S_1 and S_2 are symplecta of type $D_{n-1,1}$, then rank($S_1\cap S_2$)=-1, 0.

3- If S is a symplecton of type $D_{n-1,1}$ and A_D is a symplecton of type $A_{3,2}$, then rank($S\cap A_D$)=-1, 2.

4- If (p, S) is a non-incidence pair of point and symplecton, then rank($p^{\perp}\cap S$)=-1, 1, 2.

Proof. 1- A_{T1} and A_{T2} are located in the same class ($\mathbf{G_1}$), so A_{D1} and A_{D2} are located in the located in $\mathbf{G_1}$**,** then we have two cases:

a- A_{D1}, A_{D2} are symplecta of the same geometry A_T, then it has three cases:

a1- If $D_1\cap D_2\leq$(k-1)-space, then rank($A_{D1}\cap A_{D2}$)=-1.

a2- if $D_1\cap D_2$=k-space, then rank($A_{D1}\cap A_{D2}$)=0.

a3- if $D_1\cap D_2$=(k+1)-space, then rank($A_{D1}\cap A_{D2}$)=2.

b- A_{D1}, A_{D2} are symplecta of different geometries A_{T1} and A_{T2} respectively, then $T_1\cap T_2$ is a space of odd dimension and we have three cases:

b1- For dim($T_1\cap T_2$)≤k-1, dim($D_1\cap D_2$)≤k-1, so rank($A_{D1}\cap A_{D2}$)=-1.

b2- For dim($T_1\cap T_2$)=k, dim($D_1\cap D_2$)≤k. Then rank($A_{D1}\cap A_{D2}$)=-1, 0.

b3- For dim($T_1\cap T_2$)>k, then either dim($D_1\cap D_2$)≤k and rank($A_{D1}\cap A_{D2}$)=-1, 0 or dim($D_1\cap D_2$)=k+1 and rank($A_{D1}\cap A_{D2}$)=2. Then rank($A_{D1}\cap A_{D2}$)=-1, 0, 2.

Now the case in which A_{T1} and A_{T2} are located in different classes is similar to that cases above i.e., rank($A_{D1}\cap A_{D2}$)=-1, 0, 2.

2- In this case $\psi(S_1)$ and $\psi(S_2)$ correspond to (k-1)-spaces. We have two cases:

i- If ($\psi(S_1)\cap\psi(S_2)$)<k-2, then we cannot find a k-space that contains $\langle\psi(S_1),\psi(S_2)\rangle$, i.e., $S_1\cap S_2=\varphi$. Thus rank($S_1\cap S_2$)=-1.

ii- If $\psi(S_1)\cap\psi(S_2)$=k-2 and $\langle\psi(S_1),\psi(S_2)\rangle$=k-space, then rank($S_1\cap S_2$)=0.

3- If S is a symlecton of type $D_{n-1,1}$ and A_D is a symplecton of type $A_{3,2}$, then we have two cases:

i- $\psi(S)\cap D\leq(k-2)$-space, then we there is no a k-space that contains $\psi(S)$ and contained in D, i.e., $rank(S\cap A_D)=-1$.

ii- $\psi(S)\subseteq D$, then the number of different k-spaces in D that contains $\psi(S)$ form a space of rank 3 i.e., $rank(S\cap A_D)=2$. ♦

4- a- For any pair (p, A_D) of a point p and a symplecton A_D of type $A_{3,2}$, there are two cases:

ai- If $dim(\psi(p)\cap D)<k-1$, then there is no any k-space contained in D and meets $\psi(p)$ in a (k-1)-space i.e., $\psi(p)\cap D=\varphi$ and $rank(p^{\perp}\cap S)=-1$.

aii- If $dim(\psi(p)\cap D)=k-1$, then set of k-spaces in D that contain the (k-1)-space $\psi(p)\cap D$ form a projective plane i.e., $rank(p^{\perp}\cap S)=2$.

b- For any pair (p, S) of a point p and a symplecton S of type $D_{n-k+1,1}$, there are two cases:

bi- If $dim((\psi(p)\cap\psi(S))<k-2$, then we cannot find any k-space contains $\psi(S)$ and meets $\psi(p)$ in a (k-1)-space i.e., $rank(p^{\perp}\cap S)=-1$.

bii- If $dim((\psi(p)\cap\psi(S))=k-2$, and $\langle\psi(p),\psi(S)\rangle=(k+1)$-space, then there are two k-spaces meet $\psi(p)$ in a (k-1)-space and contain $\psi(S)$. Since the two k-space form a (k+1)-space, then $p^{\perp}\cap S$ is a line i.e., $rank(p^{\perp}\cap S)=1$.

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