ON PRIMARY COMPACTLY PACKED MODULES

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Abstract. A proper submodule N of a module M over a ring R is compactly packed if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of prime submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, $N \subseteq P_\beta$ for some $\beta \in \lambda$. A module M is called compactly packed if every proper submodule is compactly packed. This concept was introduced in [17]. In this paper, we generalize this concept to primary submodules and introduce the concept of primary compactly packed modules. We also generalize the Prime Avoidance Theorem for modules that was proved in [13] to the Primary Avoidance Theorem for modules. In addition, we study various properties of primary compactly packed modules.

Contents

1. Introduction

Let M be a unitary R-module, where R is a commutative rings with identity. A proper submodule N of M is primary if $rm \in N$, for $r \in R$ and $m \in M$ implies that either $m \in N$, or $r^n M \subseteq N$ for some positive integer n.

In [17], the concept of compactly packed modules was introduced. We generalize this concept to the concept of primary compactly packed modules. A proper submodule N of M is primary compactly packed (*pcp*) if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$, $N \subseteq P_\beta$ for some $\beta \in \lambda$. A module M is called *pcp* if every submodule is *pcp*.

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In Section 2 of this paper we give general properties of compactly packed modules.

In [13], Chin Pi Lu proved the Prime Avoidance Theorem for modules. In Section 3 we introduce and prove the Primary Avoidance Theorem for modules.

In [10] and [11], Chin Pi Lu proved some results on minimal prime submodules. We introduce and prove some results concerning minimal primary submodules in Section 4.

In Section 5, we investigate some basic properties of primary radical submodules.

Throughout this paper, all rings are assumed to be commutative rings with identity and all modules will be unitary.

2. Primary Compactly Packed Modules

Definition 2.1. Let R be a commutative ring with identity and B an R-module. A proper submodule A of B is a primary submodule provided that if $r \in R, b \notin A$ and $rb \in A$, then $r^n B \subseteq A$ for some positive integer n.

Definition 2.2. Let M be a unitary R-module. A proper submodule N of M is primary compactly packed (*pcp*) if for each family $\{P_\alpha\}_{\alpha\in\lambda}$ of primary submodules of M with $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha, N \subseteq P_\beta$ for some $\beta \in \lambda$. A module M is called *pcp* if every submodule is *pcp*.

Definition 2.3. Let N be a submodule of an R-module M. If there exist primary submodules that contain N, then the intersection of all primary submodules containing N is called the primary radical of N and is denoted by $prad\ (N)$. If there is no primary submodule containing N, then $prad\ (N) = M$.
In a special case $prad(M) = M$.
We say that a submodule N is a primary radical submodule if $prad(N) = N$.
We denote that we will use the symbol $prad(\{0\})$ to represent the intersection of all primary submodules of the R-module M.

Example 2.4. *Let $R = \mathbb{Z}$. Since every ideal of R is a submodule of R, primary ideals of R are primary submodule of R. So for*
$n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$*, where p_i 's are prime numbers,*
$(n) = \bigcap_{i=1}^{k} (p_i^{\alpha_i})$*. Therefore in $\mathbb{Z}$ every ideal is a primary radical submodule of R.*

Proposition 2.5. *Let N and L be submodules of an R-module M. Then*
1) $N \subseteq prad(N)$.
2) If $N \subseteq L$, then $prad(N) \subseteq prad(L)$.
3) $prad(prad(N)) = prad(N)$, i.e., the primary radical of N is a primary radical submodule.
4) $prad(N \cap L) \subseteq prad(N) \cap prad(L)$.
5) $prad(N + L) = prad(prad(N) + prad(L))$.

Proof. 1) Trivial, since $N \subseteq Q$ for every primary submodule Q.
2) Let $N \subseteq L$ and let Qbe primary with $L \subseteq Q$. Then $N \subseteq Q$. Hence
$prad(N) \subseteq prad(L)$.
3) Since $N \subseteq prad(N)$, by 1, $prad(N) \subseteq prad(prad(N))$. Now let Q be primary such that $N \subseteq Q$. Then by the definition of $prad(N)$, $prad(N) \subseteq Q$. Hence
$prad(prad(N)) \subseteq prad(N)$ and $prad(N) = prad(prad(N))$.
4) Since $N \cap L \subseteq N$ and $N \cap L \subseteq L$, by 2

$prad(N \cap L) \subseteq prad(N)$ and $prad(N \cap L) \subseteq prad(L)$.
Thus $prad(N \cap L) \subseteq prad(N) \cap prad(L)$.

5) Since $N + L \subseteq prad(N) + prad(L)$, by 2
$prad(N + L) \subseteq prad(prad(N) + prad(L))$.
Now let Q be primary such that $N + L \subseteq Q$, we want to prove that $prad(N) + prad(L) \subseteq Q$. Since $N + L \subseteq Q$, $N \subseteq Q$ and $L \subseteq Q$. Thus $prad(N) \subseteq Q$ and $prad(L) \subseteq Q$. Hence $prad(N) + prad(L) \subseteq Q$ and $prad(prad(N) + prad(L)) \subseteq Q$. Therefore,
$prad(prad(N) + prad(L)) \subseteq prad(N + L)$ and we have
$prad(prad(N) + prad(L)) = prad(N + L)$. □

Theorem 2.6. *Let M be an R-module. The following statements are equivalent:*

a) M is a pcp module.
b) For each proper submodule N of M there exists $a \in N$ such that $prad(N) = prad(Ra)$
c) For each proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\lambda}$ is a family of submodules of M and $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha$, then $N \subseteq prad(N_\beta)$ for some $\beta \in \lambda$.
d) For each proper submodule N of M, if $\{N_\alpha\}_{\alpha\in\lambda}$ is a family of primary radical submodules of M and $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha$, then $N \subseteq N_\beta$ for some $\beta \in \lambda$.

Proof. $(a \to b)$ Let N be a proper submodule of M, it is clear that $prad(Ra) \subseteq prad(N)$ for each $a \in N$. suppose that $prad(N) \nsubseteq prad(Ra)$ for each $a \in N$, then for each $a \in N$ there exists a primary submodule P_a for which $Ra \subseteq P_a$ and $N \nsubseteq P_a$. But $N = \bigcup_{a\in N} Ra \subseteq \bigcup_{a\in N} P_a$, i.e. M is not *pcp* which contradicts (a).
$(b \to c)$ Let N be a proper submodule of M and let $\{N_\alpha\}_{\alpha\in\lambda}$ be a family of submodules of M such that $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha$ by (b) there exists $a \in N$ such that $prad(N) = prad(Ra)$. Then $a \in \bigcup_{\alpha\in\lambda} N_\alpha$ and hence $a \in N_\beta$ for some $\beta \in \lambda$, so that $Ra \subseteq N_\beta$ and $N \subseteq prad(N) = prad(Ra) \subseteq prad(N_\beta)$.
$(c \to d)$ Let N be a proper submodule of M and let $\{N_\alpha\}_{\alpha\in\lambda}$ be a family of primary radical submodules of M such that $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha$, then by (c) there exists $\beta \in \lambda$ such that $N \subseteq prad(N_\beta) = N_\beta$ since N_β is primary radical.
$(d \to a)$ Let N be a proper submodule of M and suppose that $\{N_\alpha\}_{\alpha\in\lambda}$ is a family of primary submodules of M such that $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha$. Since N_α is primary for each $\alpha \in \lambda, N_\alpha = prad(N_\alpha)$. Thus $N \subseteq \bigcup_{\alpha\in\lambda} N_\alpha = \bigcup_{\alpha\in\lambda} prad(N_\alpha)$. By (d) there exists $\beta \in \lambda$ such that $N \subseteq prad(N_\beta) = N_\beta$. Thus M is *pcp*. □

Definitions 2.7. • An R-module M is called a multiplication module if every submodule N of M is of the form IM, for some ideal I of R, [5].

• A proper submodule N of an R-module M is said to be semiprime if $r^2x \in N$ for $r \in R$ and $x \in M$ implies that $rx \in N$, [3].

• A submodule N of an R-module M is said to be pure in M if for every ideal A of R, $N \cap AM = AN$. The module M is called regular if each submodule of M is pure, [4].

Proposition 2.8. *Let M be an R-module. If M is regular, then $prad(N) = N$ for all submodules N of M.*

Proof. Suppose that M is a regular R-module. Let N be a proper submodule of M and let $r^2x \in N$ for $r \in R$ and $x \in M$. Then as it is proved in [1],
$rx \in (r)M \cap (rx) = (r)(rx)$, because M is regular. Thus $rx \in N$ and N is a semiprime submodule. By [3], N is an intersection of prime submodules. Hence $N = \bigcap_{\alpha\in\lambda} P_\alpha$ where P_α is a prime submodule of M for each α. Therefore
$\bigcap_{\alpha\in\lambda} K_\alpha \subseteq N$, where K_α is a prime submodule of M such that $N \subseteq K_\alpha$. Since $prad(N) \subseteq \bigcap_{\alpha\in\lambda} K_\alpha$ because every prime submodule of M is primary,
$prad(N) \subseteq N$. But $N \subseteq prad(N)$. Thus $prad(N) = N$. □

Corollary 2.9. *Let M be a regular R-module, then M is a pcp module if and only each proper submodule of M is cyclic.*

Proof. $(\rightarrow)$ Let N be a proper submodule of M, then by Theorem 2.6 since M is *pcp* there exists $a \in N$ such that $prad(N) = prad(Ra)$ but M is regular module, then by the previous Proposition $N = Ra$, thus N is cyclic.
$(\leftarrow)$ Let N be a proper submodule of M. N is cyclic, thus there exists $a \in N$ such that $N = Ra$, thus $prad(N) = prad(Ra)$. By Theorem 2.6 M is *pcp*. □

Theorem 2.10. *If M is pcp module which has at least one maximal submodule, then M satisfies the ACC on primary radical submodules.*

Proof. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of primary radical submodules of M and let $L = \bigcup_i N_i$. If $L = M$ and H is a maximal submodule of M, then $H \subsetneq \bigcup_i N_i$. Since M is *pcp*, by Theorem 2.6 $H \subseteq N_j$ for some j. Therefore $H = N_j$. Since $N_j \subseteq N_{j+n} \subseteq \bigcup_i N_i$ for $n = 1, 2, ...$ and N_j is maximal either $N_j = N_{j+n}$ for every $n = 1, 2, ...$, thus $N_j = \bigcup_i N_i = M$, which is impossible, or $N_{j+n} = \bigcup_i N_i = M$ which is also impossible. Thus L is a proper submodule of M.
Since M is *pcp* by Theorem 2.6 $L \subseteq N_j$ for some j and hence
$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_j = N_{j+1} = N_{j+2} = \cdots$ therefore the ACC is satisfied for primary radical submodules. □

Since every finitely generated module and every multiplication module has a proper maximal submodule[2], then we have the following Corollary

Corollary 2.11. *If M is a finitely generated or a multiplication pcp R-module, then M satisfies the ACC on primary radical submodules.*

Proposition 2.12. *Let $f : B \to D$ be an R-module epimorphism and C a proper submodule of D. Then C is a primary submodule of D if and only if $f^{-1}(C)$ is a primary submodule of B.*

Proof. $(\rightarrow)$ Let $rm \in f^{-1}(C)$ with $r \in R, m \in B$ and $m \notin f^{-1}(C)$, then
$f(rm) \in C$, thus $rf(m) \in C$. Since C is primary with $f(m) \notin C, r^nD \subseteq C$ for some $n \in \mathbb{Z}^+$. Thus $r^nf^{-1}(D) \subseteq f^{-1}(C)$ for some $n \in \mathbb{Z}^+$. Therefore $r^nB \subseteq f^{-1}(C)$ for some $n \in \mathbb{Z}^+$ hence $f^{-1}(C)$ is a primary submodule of B.
$(\leftarrow)$ Let $rd \in C$ with $r \in R$ and $d \in D - C$. Since f is epimorphism, there exists $b \in B$ such that $f(b) = d$. Thus $rf(b) \in C$, hence $f(rb) \in C$. Therefore $rb \in f^{-1}(C)$. But $f^{-1}(C)$ is a primary submodule of B with $r \in R$ and $b \in B - f^{-1}(C)$, thus $r^nB \subseteq f^{-1}(C)$ for some $n \in \mathbb{Z}^+$. Therefore $f(r^nB) = r^nf(B) = r^nD \subseteq C$ for some $n \in \mathbb{Z}^+$ and C is a primary submodule of D. □

Proposition 2.13. *Let $f : B \to D$ be an R-module epimorphism. If L is a primary submodule of B with $ker f \subseteq L$, then $f(L)$ is a primary submodule of D.*

Proof. Let $rs \in f(L)$ with $r \in R$ and $s \in D - f(L)$. Since $rs \in f(L)$, there exists $x \in L$ such that $f(x) = rs$. Since $s \in D$ and f is epimorphism, there exists $b \in B$ such that $f(b) = s$. Thus $f(x) = rf(b)$, and hence $f(x - rb) = 0$. Therefore $x - rb \in ker(f) \subseteq L$. Thus $x \in L$ and $x - rb \in L$. Hence $rb \in L$. Since L is a primary submodule of B with $r \in R$ and $b \in B - L, r^n B \subseteq L$ for some $n \in \mathbb{Z}^+$. Hence $r^n D \subseteq f(L)$ for some $n \in \mathbb{Z}^+$ and $f(L)$ is a primary submodule of D. □

Proposition 2.14. *Let $\varphi : M \to \bar{M}$ be an epimorphism. If M is pcp then so is $\bar{M}$. The converse is true if $ker(\varphi) \subseteq prad(\{0\})$.*

Proof. Let M be *pcp* and suppose that $\bar{N} \subseteq \bigcup_{\alpha\in\lambda} K_\alpha$ where $\bar{N}$ is a proper submodule of $\bar{M}$ and K_α is a primary submodule of $\bar{M}$ for each $\alpha \in \lambda$. Since φ is epimorphism, $\varphi^{-1}(\bar{N}) \subseteq \varphi^{-1}(\bigcup_{\alpha\in\lambda} K_\alpha)$. Thus $\varphi^{-1}(\bar{N}) \subseteq \bigcup_{\alpha\in\lambda}(\varphi^{-1}(K_\alpha))$. Since K_α is primary for each $\alpha \in \lambda$, by Proposition 2.12 $\varphi^{-1}(K_\alpha)$ is a primary submodule of M for each $\alpha \in \lambda$. But M is *pcp*, thus there exists $\beta \in \lambda$ such that $\varphi^{-1}(\bar{N}) \subseteq \varphi^{-1}(K_\beta)$. Therefore $\bar{N} \subseteq K_\beta$ for some $\beta \in \lambda$ and hence $\bar{N}$ is *pcp*. Thus $\bar{M}$ is *pcp*.
Now suppose that $\bar{M}$ is *pcp* and $ker(\varphi) \subseteq prad(\{0\})$. Let $N \subseteq \bigcup_{\alpha\in\lambda} P_\alpha$ where N is a submodule of M and P_α is a primary submodule of M for each $\alpha \in \lambda$. Then $\varphi(N) \subseteq \varphi(\bigcup_{\alpha\in\lambda} P_\alpha)$. Thus $\varphi(N) \subseteq \bigcup_{\alpha\in\lambda} \varphi(P_\alpha)$. But $ker(\varphi) \subseteq P_\alpha$ for each $\alpha \in \lambda$. Therefore by Proposition 2.13 $\varphi(P_\alpha)$ is a primary submodule of $\bar{M}$ for each $\alpha \in \lambda$. Since $\bar{M}$ is *pcp*, $\varphi(N) \subseteq \varphi(P_\beta)$ for some $\beta \in \lambda$. Thus for every $x \in N, \varphi(x) \in \varphi(N) \subseteq \varphi(P_\beta)$. So $\varphi(x) \in \varphi(P_\beta)$. Therefore there exists $b \in P_\beta$ such that $\varphi(x) = \varphi(b)$. Thus $x - b \in ker(\varphi)$.
Since $ker(\varphi) \subseteq P_\beta$, and $b \in P_\beta$, then $x \in P_\beta$. Therefore $N \subseteq P_\beta$ and hence N is *pcp*. Thus M is *pcp*. □

Definition 2.15. Let M be an R-module, and let S be multiplicatively closed subset of R. An S-component of M is denoted by M_S and defined as
$M_S = \{a : a \in R \text{ and } as \in M \text{ for some } s \in S\}$.

Proposition 2.16. *Let M be an R-module, and S a multiplicatively closed subset of R. If M is pcp, then so is M_S.*

Proof. Suppose that $H \subseteq \bigcup_{\alpha\in\lambda} W_\alpha$ where H is a proper submodule of M_S and W_α is a primary submodule of M_S for each $\alpha \in \lambda$. Define $\varphi : M \to M_S$ as follow :
$\varphi(m) = \frac{m}{1}$ for every $m \in M$. Thus φ is epimorphism. Therefore $\varphi^{-1}(H) \subseteq \bigcup_{\alpha\in\lambda} \varphi^{-1}(W_\alpha)$ for each $\alpha \in \lambda$. Since W_α is a primary submodule of M_S and φ is epimorphism, $\varphi^{-1}(W_\alpha)$ is a primary submodule of M for each $\alpha \in \lambda$ (Proposition 2.12). But M is *pcp*, thus $\varphi^{-1}(H) \subseteq \varphi^{-1}(W_\beta)$ for some $\beta \in \lambda$.
Therefore $(\varphi^{-1}(H))_S \subseteq (\varphi^{-1}(W_\beta))_S$. Now we need only to prove that $(\varphi^{-1}(N))_S = N$ for any submodule N of M_S. Let $\frac{x}{s} \in (\varphi^{-1}(N))_S$, where $x \in \varphi^{-1}(N)$ and $s \in S$. Then $\varphi(x) \in N$. Therefore $\frac{x}{1} \in N$, hence $\frac{x}{s} = \frac{1}{s}\frac{x}{1} \in N$. Thus $(\varphi^{-1}(N))_S \subseteq N$.
Now let $\frac{x}{s} \in N$, then $\frac{1}{s}\frac{x}{1} \in N$ and hence $\frac{x}{1} \in N$. This implies that $\varphi(x) \in N$. Therefore $x \in \varphi^{-1}(N)$ and $\frac{x}{s} \in (\varphi^{-1}(N))_S$. Thus $N \subseteq (\varphi^{-1}(N))_S$. Therefore $N = (\varphi^{-1}(N))_S$ for any submodule N of M_S. Now since $(\varphi^{-1}(H))_S \subseteq (\varphi^{-1}(W_\beta))_S$ for some $\beta \in \lambda$, $H \subseteq W_\beta$ for some $\beta \in \lambda$. Thus H is *pcp*. Therefore M_S is *pcp*. □

3. Primary Avoidance Theorem for Modules.

Definitions 3.1. Let N be a submodule of an R-module M

1) $(N : M) = \{r|r \in R : rM \subseteq N\}$.
2) $\sqrt{(N : M)} = \{r|r \in R : r^n M \subseteq N$, for some $n \in \mathbb{Z}^+\}$

Proposition 3.2. *If N is a primary submodule of an R-module M, then $(N : M)$ is a primary ideal of R. The converse is not true.*

Proof. Let $rs \in (N : M)$ with $s \notin (N : M)$, then $sM \nsubseteq N$ that is there exists $m \in M$ such that $sm \notin N$. But $rsm \in N$ and N is a primary submodule of M. Thus $r^n M \subseteq N$ for some $n \in \mathbb{Z}^+$. Therefore $r^n \in (N : M)$ for some $n \in \mathbb{Z}^+$ and $(N : M)$ is a primary ideal of R.

To see that the converse is not true, consider the following example.
Let M be the free $\mathbb{Z}$ -module $\mathbb{Z} \times \mathbb{Z}$ and $N = (a, 0)\mathbb{Z}$ for any integer $a > 0$. Then $(N : M) = \{0\}$ is a primary ideal of $\mathbb{Z}$, while N is not a primary submodule of M. □

The radical of a primary ideal is always a prime ideal (see [8], p 41 Proposition 2.11). Therefore Corollary 3.3 follows immediately from Proposition 3.2.

Corollary 3.3. *If N is a primary submodule of an R-module M, then $\sqrt{(N : M)}$ is a prime ideal of R*

Definition 3.4. Let $L, L_1, L_2, ..., L_n$ be submodules of an R-module M. An efficient covering of L is a covering $L \subseteq L_1 \cup L_2 \cup ... \cup L_n$ in which no L_k , $k \in \{1, 2, ..., n\}$ satisfies $L \subseteq L_k$.

The following result was proved for ideals in [16] and Lu in [13] pointed out that the same result also remains valid if ideals are replaced with subgroups of any group as in the following Lemma.

Lemma 3.5. *Let $L = L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient union of submodules of an R-module M for $n > 1$. Then $\bigcap_{j \neq k} L_j = \bigcap_{j=1}^{n} L_j$ for all k.*

Proposition 3.6. *Let $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient covering of submodules of an R-module M where $n > 1$. If $(L_j : M) \nsubseteq \sqrt{(L_k : M)}$ for every $j \neq k$, then no L_k for $k \in \{1, 2, \cdots , n\}$ is a primary submodule of M.*

Proof. Since $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ is an efficient covering,
$L = (L \cap L_1) \cup (L \cap L_2) \cup \cdots \cup (L \cap L_n)$ is an efficient covering. Hence for every $k \in \{1, 2, \cdots , n\}$, there exists an element $e_k \in L - L_k$. Moreover

$\bigcap_{j \neq k}(L \cap L_j) \subseteq L \cap L_k$, by the previous Lemma. If $j \neq k$, then
$(L_j : M) \nsubseteq \sqrt{(L_k : M)}$. So that there exists an element $s_j \in (L_j : M)$,
but $s_j \notin \sqrt{(L_k : M)}$. Now suppose that some L_k is a primary submodule, then $\sqrt{(L_k : M)}$ is a prime ideal by Corollary 3.3. Therefore $s = \prod_{j \neq k} s_j \in (L_j : M)$ but $s \notin \sqrt{(L_k : M)}$ Consequently, $se_k \in L \cap L_j$ for each $j \neq k$, but $se_k \notin L \cap L_k$, because if $se_k \in L \cap L_k$, then $se_k \in L_k$. Since $e_k \notin L_k$ and L_k is a primary submodule, $s^r M \subseteq L_k$ for some $r \in \mathbb{Z}^+$. Thus $s \in \sqrt{(L_k : M)}$ contradiction. Therefore $se_k \notin L \cap L_k$, which contradicts to $\bigcap_{i \neq k}(L \cap L_j) \subseteq L \cap L_k$. Therefore no L_k is primary. □

It is well-known that if I, A_1 and A_2 are ideals of a ring such that $I \subseteq A_1 \cup A_2$, then $I \subseteq A_1$ or $I \subseteq A_2$. Hence a covering of an ideal by two ideals is never efficient. As McCoy remarked in [16], this result remains valid if I, A_1 and A_2 are subgroups of any arbitrary group. Consequently, a covering of a submodule by two submodules of a module is never efficient. Thus $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_m$ can be an efficient cover only when $m > 2$ or $m = 1$.

Before we go on, it should be noted that any covering of a union of submodules can be reduced to an efficient one, simply by the deletion of any unnecessary terms. We call this an efficient reduction of the cover.

In [13], Chin pi Lu introduced and proved the Prime Avoidance Theorem for modules. We now introduce and prove the Primary Avoidance Theorem for modules.

Theorem 3.7. *(Primary Avoidance Theorem for Modules).*
Let M be an R-module, $L_1, L_2, \cdots, L_n$ a finite number of submodules of M and let L be a submodule of M such that $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$. Assume that at most two of the L_i 's, i = 1, 2, ..., n are not primary and that $(L_j : M) \nsubseteq \sqrt{(L_k : M)}$ whenever $j \neq k$. Then $L \subseteq L_k$ for some $k \in \{1, 2, \cdots, n\}$.

Proof. For the given covering $L \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$, let $L \subseteq L_{i1} \cup L_{i2} \cup \cdots \cup L_{im}$ be its efficient reduction. Then $1 \leq m \leq n$ and $m \neq 2$. If $m > 2$, then there exists at least one L_{ij} to be primary. In view of Proposition 3.6 this is impossible as $(L_j : M) \nsubseteq \sqrt{(L_k : M)}$ if $j \neq k$. Hence $m = 1$, namely $L \subseteq L_k$ for some $k \in \{1, 2, \cdots, n\}$. □

Let $L_1, L_2, \cdots, L_n$ be submodules of an R-module M and let $L_1 + e_1, L_2 + e_2, \cdots, L_n + e_n$ be cosets in M. We call a covering $L \subseteq (L_1 + e_1) \cup (L_2 + e_2) \cup \cdots \cup (L_n + e_n)$ efficient if no coset is superfluous (i.e, $\nexists k$ s.t. $L \subseteq L_k + e_k$, $k \in \{1, 2, \cdots, n\}$). If $e_k = e$ for every $k \in \{1, 2, \cdots, n\}$, then the above covering is equivalent to $L - e \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ and this is a coset efficiently covered by a union of submodules.

The following Lemma was proved by C. Gottlieb in 1994 [6].

Lemma 3.8. *Let $L \subseteq (L_1 + e_1) \cup (L_2 + e_2) \cup \cdots \cup (L_n + e_n)$ be an efficient covering of a submodule L by cosets, where $n \geq 2$. Then $L \cap (\bigcap_{j \neq k} L_j) \subseteq L_k$, but $L \nsubseteq L_k$ for all k.*

Proposition 3.9. *Let $L + e \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ be an efficient covering with $n \geqslant 2$. If $(L_i : M) \nsubseteq \sqrt{(L_k : M)}$ for all $j \neq k$, then no L_k is primary.*

Proof. By the previous Lemma $L \cap (\bigcap_{j \neq k} L_j) \subseteq L_k$ and $L \nsubseteq L_k$.
Put $I = ((\bigcap_{j \neq k} L_j) : M)$. Then $IL \subseteq L \cap (\bigcap_{j \neq k} L_j) \subseteq L_k$. Suppose L_k is primary for some k, then either $L \subseteq L_k$ or $I = ((\bigcap_{j \neq k} L_j) : M) = \bigcap_{j \neq k} (L_j : M) \subseteq \sqrt{(L_k : M)}$. So that $(L_j : M) \subseteq \sqrt{(L_k : M)}$ for some $j \neq k$. However, both cases are impossible, hence no L_k is primary.

□

Theorem 3.10. *Let $L + e \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$ be a covering such that at most one submodule L_i is not primary and that $(L_j : M) \nsubseteq \sqrt{(L_k : M)}$ if $j \neq k$. Then the submodule $L + eR \subseteq L_k$ for some k.*

Proof. For the given covering $L+e \subseteq L_1 \cup L_2 \cup \cdots \cup L_n$, let $L+e \subseteq L_{i1} \cup L_{i2} \cup \cdots \cup L_{im}$ be its efficient reduction. Then $1 \leq m \leq n$. It is immediate from the proof of Theorem 3.7 that $m = 1$. Hence $L+e \subseteq L_k$ for some k. This implies that $L+eR \subseteq L_k$ as $e = 0 + e \in L + e \subseteq L_k$. □

4. Minimal Primary Submodules

Definition 4.1. A set Ω on which a partial order is given is called an inductive system if every totally ordered subset is bounded above.

Lemma 4.2. *(Zorn's lemma)*
Every non-empty inductive system possesses at least one maximal element.

Definition 4.3. A primary submodule Q of an R-module M is called a minimal primary submodule over a submodule N if $N \subseteq Q$ and if there is no smaller primary submodule with this property.
A minimal primary submodule of $\{0\}$ is known as a minimal primary submodule of M. Thus a primary submodule Q is a minimal primary submodule of an R-module M if it does not strictly contain any other primary submodule.

The following Theorem was proved in [8],p40 Theorem 2.7.

Theorem 4.4. *If the R-module M satisfies the ACC on submodules, then every submodule of M can be written as an intersection of a finite number of primary submodules.*

Now we can prove the following Lemma.

Lemma 4.5. *Let $\{q_i\}_{i\in I}$ be a non-empty family of primary submodules of an R-module M that satisfies the ACC and suppose that the family is totally ordered by inclusion. Then $\bigcap_{i\in I} q_i$ and $\bigcup_{i\in I} q_i$ are primary submodules of the R-module M.*

Proof. It is easy to show that $\bigcup_{i\in I} q_i$ and $\bigcap_{i\in I} q_i$ are submodules of M, we shall prove that they are primary.
Put $Q^* = \bigcap_{i\in I} q_i$. By the previous Theorem $Q^* = \bigcap_{i\in I^*} q_i^*$ where I^* is finite and q_i^* is primary for every $i \in I^*$. Also the family $\{q_i^*\}_{i\in I^*}$ is still totally ordered since it is a subset of the totally ordered family $\{q_i\}_{i\in I}$. Let $rs \in Q^*$ with $s \in M - Q^*$ and $r \in R$, then there exists $j \in I^*$ such that $rs \in q_j^*$ and $s \notin q_j^*$. Since q_j^* is primary, there exists $n_j \in \mathbb{Z}^+$ such that $r^{n_j} M \subseteq q_j^*$. Let k be an arbitrary element of I^*. Since $\{q_i^*\}_{i\in I^*}$ is totally ordered, either $q_j^* \subseteq q_k^*$ or $q_k^* \subseteq q_j^*$. If $q_j^* \subseteq q_k^*$,then $r^{n_j} M \subseteq q_k^*$. If $q_k^* \subseteq q_j^*$, then $rs \in q_k^*$ and $s \notin q_k^*$. But q_k^* is primary, thus $r^{n_k} M \subseteq q_k^*$ for some $n_k \in \mathbb{Z}^+$. Let $J = \{i \in I^* : r^{n_i} M \subseteq q_i\}$, and let $n = \sum_{i\in J} n_i$. Then $r^n M \subseteq \bigcap_{i\in I^*} q_i^* = Q^*$. Thus Q^* is primary. In a similar way we can prove that $\bigcup_{i\in I} q_i$ is also primary since it is a submodule of M. □

Theorem 4.6. *If an R-module M satisfies the ACC on submodules, and A is a submodule of M that is contained in a primary submodule Q of M, then Q contains a minimal primary submodule over A.*

Proof. Denote by Ω the set of all primary submodules which contain A, and are contained in Q. Then $Q \in \Omega$ and therefore Ω is not empty. If $\bar{Q}$, and $\bar{\bar{Q}}$ belongs to Ω, then we shall write $\bar{Q} \leq \bar{\bar{Q}}$ if $\bar{\bar{Q}} \subseteq \bar{Q}$. (Note the change in the order of $\bar{Q}$ and $\bar{\bar{Q}}$). This gives a partial order on Ω. We shall prove that Ω is an inductive system. Let Σ be a non-empty totally ordered subset of Ω. Let $\bar{Q}$ be the intersection of all

the members of Σ. By Lemma 4.5, $\bar{Q}$ is primary submodule of M. And $A \subseteq \bar{Q} \subseteq Q$. Consequently $\bar{Q} \in \Omega$. Also since $\bar{Q} \subseteq B$ for every $B \in \Sigma$,we have $B \leq \bar{Q}$ for every $B \in \Sigma$. Thus $\bar{Q}$ is an upper bound for Σ. Therefore Ω is an inductive system.
By Zorn's lemma. Ω contains a maximal element Q^*. Since $Q^* \in \Omega$, it is primary submodule with $A \subseteq Q^* \subseteq Q$. Suppose now that Q_1 is a primary submodule satisfying $A \subseteq Q_1 \subseteq Q$. Then $Q_1 \in \Omega$ and $Q^* \leq Q_1$. Consequently, since Q^* is maximal in $\Omega, Q_1 = Q^*$. This shows that Q^* is a minimal primary submodule of A and completes the proof.

□

Since every proper submodule of a finitely generated module is contained in a prime submodule [9] and every prime submodule is primary, we can conclude by applying Theorem 4.6 the following Corollary.

Corollary 4.7. *Every proper submodule of a finitely generated R-module M that satisfies the ACC on submodules possesses at least one minimal primary submodule.*

Since it is known that an R-module M is finitely generated if and only if M satisfies the ACC on submodules (see[8], p8 Theorem 1.8.), We can conclude the following Corollary.

Corollary 4.8. *If an R-module M satisfies the ACC on submodules, then every proper submodule of M possesses at least one minimal primary submodule.*

Corollary 4.9. *Every primary submodule of an R-module M that satisfies the ACC on submodules contains at least one minimal primary submodule of M.*

Proof. Let $A = \{0\}$ in Theorem 4.6 . □

Theorem 4.10. *If an R-module M satisfies the ACC on submodules, then the primary radical of a proper submodule N of M is the intersection of its minimal primary submodules.*

Proof. N has at least one minimal primary submodule due to Corollary 4.8. Hence the intersection L of all minimal primary submodules of N contains $prad(N)$. On the other hand, let Q be any primary submodule containing N. Then Q contains some minimal primary submodule Q_i of N by Theorem 4.6. Hence
$L = \bigcap_{i \in I} Q_i \subseteq pradN \subseteq L$ and the proof is complete. □

5. Basic Properties of Primary Radical Submodules

The following Theorem follows immediately from Theorem 4.4.

Theorem 5.1. *Let M be an R-module. If M satisfies the ACC for primary radical submodules, then M satisfies the ACC for primary submodules and each primary radical submodule is an intersection of a finite number of primary submodules.*

Theorem 5.2. *For any R-module M, if M satisfies the ACC for primary radical submodules, then every primary radical submodule is the primary radical of a finitely generated submodule.*

Proof. Assume that there exists a primary radical submodule N which is not primary radical of a finitely generated submodule. Let $e_1 \in N$ and let $N_1 = prade_1R$. Then $N_1 \subsetneq N$ so that there exists $e_2 \in N - N_1$. Let $N_2 = prad(e_1R + e_2R)$. Then $N_1 \subsetneq N_2 \subsetneq N$. so that there exists $e_3 \in N - N_2$ etc. This gives an ascending chain of primary radical submodules $N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \cdots$ which is a contradiction. □

The following Theorem is trivial.

Theorem 5.3. *If every primary radical submodule is the primary radical of a finitely generated submodule, then every primary submodule is the primary radical of a finitely generated submodule.*

Proposition 5.4. *Let N be a submodule of an R-module M. If M satisfies the ACC on submodules, then $prad(N) = M$ if and only if $N = M$.*

Proof. It is clear that if $M = N$, then $prad(N) = M$. Now suppose that M satisfies the ACC on submodules. Then every proper submodule of M is contained in a primary submodule, see Corollary 4.8. Thus if N is proper then $prad(N) \neq M$. Hence if $prad(N) = M$ then $N = M$. Thus
$prad(N) = M$ if and only if $N = M$. □

Corollary 5.5. *If an R-module M satisfies the ACC on submodules, then*
$prad(N) + prad(L) = M$ if and only if $N + L = M$.

Proof. If $N + L = M$, then it is clear that
$prad(N) + prad(L) = prad(M) = M$. Now assume that $prad(N) + prad(L) = M$, then
$prad(prad(N) + prad(L)) = prad(M)$. Thus $prad(N + L) = M$. But M satisfies the ACC on submodules, then by the previous Proposition
$N + L = M$. □

Proposition 5.6. *Let N and L be submodules of an R-module M such that whenever $N \cap L \subseteq Q$ we have $N \subseteq Q$ or $L \subseteq Q$ for any primary submodule Q of M. Then $prad(N \cap L) = prad(N) \cap prad(L)$.*

Proof. From 4 of Proposition 2.5, $prad(N \cap L) \subseteq prad(N) \cap prad(L)$.
Now if $prad(N \cap L) = M$, then clearly $prad(N) = prad(L) = M$ and so
$prad(N \cap L) = prad(N) \cap prad(L)$.
If $prad(N \cap L) \neq M$, then then there exists a primary submodule Q such that
$N \cap L \subseteq Q$. By hypothesis, $N \subseteq Q$ or $L \subseteq Q$ so that $prad(N) \subseteq Q$ or $prad(L) \subseteq Q$.
Since this is true for all primary submodules Q containing $N \cap L$,
$prad(N) \cap prad(L) \subseteq prad(N \cap L)$ and therefore
$prad(N \cap L) = prad(N) \cap prad(L)$. □

We can generalize Proposition 5.6 as follows.

Proposition 5.7. *Let $N_1, N_2, ..., N_k$ be submodules of an R-module M such that whenever $N_1 \cap N_2 \cap ... \cap N_k \subseteq Q$, we have $N_i \subseteq Q$ for some $i = 1, 2, ..., k$, for any primary submodule Q of M. Then $prad(\bigcap_{i=1}^{k} N_i) = \bigcap_{i=1}^{k} prad(N_i.)$*

Proposition 5.8. *Let $N_1, N_2, ..., N_k$ be submodules of an R-module $M$and let N be a primary submodule of M. If $N_1 \cap N_2 \cap ... \cap N_k \subseteq N$, then there exists an i such that either $N_i \subseteq N$ or $(N_i : M) \subseteq \sqrt{(N : M)}$.*

Proof. Assume the contrary. Then there exists an $e \in N_1$ such that $e \notin N$ and an $r_i \in (N_i : M)$ such that $r_i \notin \sqrt{(N : M)}$, for every $i \neq 1$. Consequently, $r_i e \in N_1 \cap N_i$ for every $i \neq 1$. So that
$r_2 r_3 ... r_k e \in N_1 \cap N_2 \cap ... \cap N_k \subseteq N$. However, $e \notin N$ and $r_2 r_3 ... r_k \notin \sqrt{(N : M)}$, which contradicts that N is primary and also by Corollary 3.3 $\sqrt{(N : M)}$ is a prime ideal of R. □

By combining Propositions 5.7 and 5.8, we have the following Corollary.

Corollary 5.9. *Let* $N_1, N_2, ..., N_k$ *be submodules of an* R*-module* M *such that whenever* $N_1 \cap N_2 \cap ... \cap N_k \subseteq Q$*, we have* $(N_i : M) \nsubseteq \sqrt{(N : M)}$ *for every* $i = 1, 2, ..., k$*, for any primary submodule* Q *of* M*. Then* $prad(\bigcap_{i=1}^{k} N_i) = \bigcap_{i=1}^{k} prad(N_i.)$

Definition 5.10. A module M is called a Bezout module if every finitely generated submodule is cyclic.

Theorem 5.11. *Let* M *be a Bezout module. If* M *satisfies the ACC on primary radical submodules, then* M *is pcp.*

Proof. Let N be a proper submodule of M. By Theorem 5.2, there exists a finitely generated submodule L of M such that $prad(N) = prad(L)$ and hence L is cyclic submodule, because M is Bezout. By Theorem 2.6, M is *pcp*. □

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