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Bäcklund transformations for fifth-order Cosgrove's equations

A. H. Sakka [1] and S. R. Elshamy
Department of Mathematics, Islamic University of Gaza
P.O.Box 108, Rimal, Gaza, Palestine

[1] e-mail: asakka@mail.iugaza.edu
Fax Number: (+972)(7)2863552

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Abstract

In this article we use a generalization of Fokas and Ablowitz algorithm to obtain Bäcklund transformations (BTs) for the fifth-order ordinary differential equations of Cosgrove, Fif-I, Fif-II, Fif-III, and Fif-IV. We derive BTs between the equations Fif-I,..., Fif-IV and new Painlevé-type equations of the same order and higher degree. The form of some of the new equations are also given.

Keywords: *Bäcklund transformations, Cosgrove's equations, Painlevé equations.*

MSC2000 classification scheme numbers: *34M55, 37K35*

1 Introduction

The derivation of Bäcklund transformations (BTs) is an important problem in the theory of integrable equations. It is well known that six Painlevé equations $PI, PII, \ldots, PVI$ [1] admit BTs that relate a given equation to itself or to other ordinary differential equation (ODE) of the same order but perhaps of different degree.

Fokas and Ablowitz [2] used BT of the form

$$(a_1v^2 + b_1v + c_1)u - (v' + a_0v^2 + b_0v + c_0) = 0, \qquad (1)$$

where $a_i, b_i, c_i,\ i = 0, 1$, are all functions of z only, and they applied it to the equations $PI, \ldots, PVI$ to obtain BTs between these equations and second-order Painlevé-type equations of first and second degree. The same BT was used in [3, 4] to obtain all second-order second-degree equations of Painlevé-type related to the equations $PI, \ldots, PVI$ and in [5] the same method was used to obtain BTs between the equations PIII and PIV and Painlevé-type equations of second-order and higher-degree.

Gordoa and Pickering [6] generalize the method of Fokas and Ablowitz so that it can be applied to ODEs of order greater than two. The generalized method was applied to fourth-order Cosgrove's equations F-XVII, F-VI, and F-V in [6, 7, 8] respectively. In [9] we applied it to Cosgrove's equation F-XVIII.

In this paper, we will apply the same algorithm to fifth-order ODEs

$$v^{(5)} = f(z, v, v', \ldots, v^{(4)}). \qquad (2)$$

We can rewrite (1) as

$$v' = Av^2 + Bv + C, \qquad (3)$$

where $A = a_1u + a_0,\ B = b_1u + b_0$, and $C = c_1u + c_0$. Differentiating equation (3) four times, using (2) to replace $v^{(5)}$ and the j^{th} derivative of (3) to replace $v^{(j+1)},\ \ j = 0, 1, 2, 3$, we get an equation of the form

$$\sum_{i=0}^{n} \phi_i v^i = 0, \qquad (4)$$

where $\phi_i, i = 0, 1, \ldots, n$, are functions of $z, u, u', \ldots, u^{(4)}$. Eliminating v between equations (3) and (4), we can find a fifth-order ODE for u .

As an application of this method we will apply it to the fifth-order Cosgrove's equations Fif-I, Fif-II, Fif-III, and Fif-IV which was given in [10]. All of these equations admit first integrals, not necessarily of polynomial type, and it is believed that the members of this list define fourth-order Painlevé transcendents.

BTs for equations Fif-I,…, Fif-IV of the form

$$H(z, v, v', v'', v''')u - [v^{(4)} + G(z, v, v', v'', v''')] = 0, \qquad (5)$$

were considered in [11].

2 The Cosgrove's Fif-I equation

Consider the Cosgrove's Fif-I equation

$$v^{(5)} = 15vv''' + \frac{75}{2}v'v'' - 45v^2v' + zv' + 2v. \tag{6}$$

Equation (6) admits the first integral

$$\begin{array}{l} 2[v'' - 6v^2 + \frac{2}{3}z][v^{(4)} - 12vv'' - 12(v')^2] - [v''' - 12vv' + \frac{2}{3}]^2 \\ -3v[v'' - 6v^2 + \frac{2}{3}z]^2 + K_1 = 0, \end{array} \tag{7}$$

where K_1 is a constant of integration. Equation (6) has special solutions in term of the solutions of the first Painlevé equation

$$v'' = 6v^2 - \frac{2}{3}z. \tag{8}$$

Applying what we have introduced in the introduction to Cosgrove's Fif-I equation (6), we find that equation (4) reads

$$\sum_{i=0}^{6} \phi_i v^i = 0, \tag{9}$$

where

$$
\begin{aligned}
\phi_6 &= 120A^5,\\
\phi_5 &= 240A^3A' + 360A^4B - 165A^3,\\
\phi_4 &= 60A^2A'' + 90A(A')^2 + AA'(480AB - \tfrac{255}{2}) + 240A^4C\\
&\quad + 390A^3B^2 + 120A^3B' - \tfrac{735}{2}A^2B + 45A,\\
\phi_3 &= 10AA''' + 5(18AB + 4A' - 3)A'' + 70B(A')^2 + (340A^2C\\
&\quad + 290AB^2 + 110AB' - \tfrac{225}{2}B)A' + (480BC + 60C')A^3\\
&\quad + (180B^3 + 210BB' + 30B'' - 270C)A^2 - 255AB^2\\
&\quad - \tfrac{195}{2}AB' + 45B,\\
\phi_2 &= A^{(4)} + 9BA''' + (16B' + 31B^2 + 68AC)A'' + 56C(A')^2\\
&\quad + (374ABC + 49B^3 + 77BB' + 64AC' + 14B'' - \tfrac{195}{2}C)A'\\
&\quad + 6AB''' + (39AB - 15)B'' + 28A(B')^2 + (136A^2C + 101AB^2\\
&\quad - \tfrac{165}{2}B)B' + 12A^2C'' + (90A^2B - \tfrac{135}{2}A)C'\\
&\quad + 136A^3C^2 + 292A^2B^2C - 345ABC - \tfrac{105}{2}B^3 + 45C - Az,\\
\phi_1 &= B^{(4)} + 8CA''' + 5BB''' + 2AC''' + (12C' + 42BC)A'' + (26AC\\
&\quad + 10B^2 + 10B')B'' + (12AB + 8A' - 15)C'' + (108AC^2 + 56B^2C\\
&\quad + 56B'C + 22B^2C + 38BC')A' + 15B(B')^2 + (110ABC + 16B^3\\
&\quad + 26AC' - 30C)B' + 120A^2BC^2 + 30AB^3C + 52A^2CC'\\
&\quad + 22AB^3C + B^5 + 32AB^2C' + 16A^2BC^2 - 105AC^2\\
&\quad - 105B^2C - 105BC' - Bz - 2,\\
\phi_0 &= C^{(4)} + BC''' + 4B'''C + (8AC + B^2 + 4B')C''\\
&\quad + 6A(C')^2 + (26ABC + 24A'C + B^3 + 7BB' + 6B''\\
&\quad - \tfrac{75}{2}C)C' + 9BCB'' + 8(B')^2C + 12A''C^2\\
&\quad + 30A'BC^2 + (9B^2 + 24AC)B'C + 22AB^2C^2 + B^4C\\
&\quad + 16A^2C^3 - \tfrac{75}{2}BC^2 - zC.
\end{aligned}
\tag{10}
$$

The more interesting cases of this method result from reducing the degree of the equation (9) either by setting some of its coefficients to zero or by making a possible factorization. We will consider the following two cases.

Case I: $\phi_j = 0,\ j = 3,4,5,6$ and $\phi_2 \neq 0$

Setting $\phi_6 = 120A^5$ to zero yields $A = 0$. As a result we find that $\phi_5 = \phi_4 = 0$ identically and $\phi_3 = 45B$. Now setting ϕ_3 to zero, we get $B = 0$. Without loss of generality we can assume that $C = u$. Thus the transformation (3) becomes

$$v' = u \tag{11}$$

and the equation (9) reads

$$45uv^2 - (15u'' + 2)v + (u^{(4)} - \frac{75}{2}uu' - zu) = 0. \tag{12}$$

Eliminating v between equations (11) and (12) yields the following fifth-order

ODE for u of degree two

$$\begin{aligned}&[u^2u^{(5)} + (u' - 2)uu^{(4)} - (\tfrac{5}{2}u'' + \tfrac{1}{3})uu''' \\ &\quad + \tfrac{5}{2}(u'')^2 + (\tfrac{2}{3} - \tfrac{75}{2}u^3)u'' \\ &\quad - 75u^2(u')^2 + (75 - 2z)u^2u' - (u^3 - 2zu^2 - \tfrac{2}{45})]^2 \\ &\quad = (\tfrac{1}{6}uu''' - \tfrac{1}{6}u'' - u^3 - \tfrac{1}{45})^2\times \\ &\qquad [(15u'' + 2)^2 - 180u(u^{(4)} - \tfrac{75}{2}uu' - zu)].\end{aligned} \tag{13}$$

Therefore we have obtained the BT (11) and (12) between the Fif-I equation (6) and equation (13).

Using the BT (11) and (12), we can find the following BT for the first integral (7)

$$v' = u \tag{14}$$

and

$$\begin{aligned}&2[u' - 6v^2 + \tfrac{2}{3}z][u''' - 12vu' - 12u^2] - [u'' - 12vu + \tfrac{2}{3}]^2 \\ &-3v[u' - 6v^2 + \tfrac{2}{3}z]^2 + K_1 = 0.\end{aligned} \tag{15}$$

Case II: $\phi_j = 0, j = 4, 5, 6$ and $\phi_3 \neq 0$
Following what we did in case I, we have $A = 0$ and $B \neq 0$, so we have two cases : $b_1 = 0$ and $b_1 \neq 0$.

Case II:a) $b_1 = 0$
Without loss of generality we can assume that $C = u$ and in this case no possible factorization can be done. Thus we have obtained the following BT for Fif-I

$$v' = b_0v + u \tag{16}$$

and

$$\begin{aligned}&(45b_0)v^3 - 15(\tfrac{7}{2}b_0^3 + \tfrac{11}{2}b_0b_0' + b_0'' - 3u)v^2 \\ &+[b_0^{(}4) + 5b_0b_0''' + 10(b_0' + b_0^2)b_0'' + 15(b_0')^2 + 10(b_0^3 - 3u)b_0' \\ &+b_0^5 - 105b_0^2u - 105b_0u' - 15u'' - b_0z - 2]v + [u^{(4)} + b_0u''' \\ &+(b_0^2 + 4b_0')u'' + (6b_0'' + 7b_0b_0' + b_0^3 - \tfrac{75}{2}u)u' \\ &+(b_0^+ 9b_0^2b_0' + 9b_0b_0'' + 8(b_0')^2 + 4b_0''' - z)u - \tfrac{75}{2}b_0u^2] = 0.\end{aligned} \tag{17}$$

Eliminating v between (16) and (17), we can obtain a fifth-order ODE for u of degree three.

Case II:b) $b_1 \neq 0$
Without loss of generality we can assume that $B = u$ and we can look for a possible factorization of (9)

$$\sum_{i=0}^{3}\phi_iv^i = (v - g)(45uv^2 + Dv + E), \tag{18}$$

where g is a function of z only. We can find D and E as a function of $u, u', \ldots, u^{(4)}, z$ by equating the coefficients of v^2 and v and then we are left with the coefficients of v^0 which we ask to be equal identically. In this way we find that $c_1 = -g, \quad c_0 = -c_1'$ and g is a second solution of equation (6). Our BT in this case reads

$$u = \frac{v' - g'}{v - g} \tag{19}$$

and

$$45uv^2 + Dv + E = 0, \tag{20}$$

where

$$\begin{aligned} D &= -15u'' - \tfrac{165}{2}uu' - \tfrac{105}{2}u^3 - 90gu + 45g', \\ E &= u^{(4)} + 5uu''' + 10(u' + u^2 + 3g)u'' + 15(u')^2 + (\tfrac{315}{2}gu + 16u^3)u' \\ &\quad + u^5 + \tfrac{315}{2}gu^3 + (90g^2 - 90g'' - z)u - (45gg' + 15g''' + 2). \end{aligned} \tag{21}$$

Eliminating v between equations (19) and (20), we can get a fifth-order ODE for u of degree two.

3 The Cosgrove's Fif-II equation

Consider the Cosgrove's Fif-II equation

$$v^{(5)} = 30vv''' + 30v'v'' - 180v^2v' + zv' + 2v. \tag{22}$$

Equation (22) admits the first integral

$$\begin{aligned} &2[v'' - 3v^2 + \tfrac{1}{12}z][v^{(4)} - 6vv'' - 6(v')^2] - [v''' - 6vv' + \tfrac{1}{12}]^2 \\ &-24v[v'' - 3v^2 + \tfrac{1}{12}z]^2 + K_2 = 0, \end{aligned} \tag{23}$$

where K_2 is a constant of integration, and it has special solutions satisfying the first Pianlevé equation

$$v'' = 3v^2 - \frac{1}{12}z. \tag{24}$$

Proceeding as in the previous example, we find that equation (4) reads

$$\sum_{i=0}^{6} \phi_i v^i = 0, \tag{25}$$

where $\phi_6 = 120A^5$. To reduce the degree of equation (25) we have two cases:
Case I: $\phi_j = 0, \quad j = 3, 4, 5, 6$ and $\phi_2 \neq 0$
Since $\phi_6 = 120A^5$, we have to choose $A = 0$ in order to make $\phi_6 = 0$ identically. This implies that $\phi_4 = \phi_5 = 0$ and $\phi_3 = 180B$. Now to make $\phi_3 = 0$ identically

we have to take $B = 0$. Without loss of generality we can assume that $C = u$ and our BT becomes

$$v' = u \tag{26}$$

and

$$180uv^2 - (30u'' + 2)v + (u^{(4)} - 30uu' - zu) = 0. \tag{27}$$

Eliminating v between equations (26) and (27), we get the following fifth-order ODE for u of degree two

$$\begin{aligned}&[-30uu''' + 30uu'' + 2u' + 360u^3]^2[-720uu^{(4)} + 900(u'')^2 + 120u'' \\ &\quad + 21600u^2u' + 720zu^2] = [-720u^2u^{(5)} + 2uu'u^{(4)} \\ &\quad + 60uu'''(15u'' + 1) - 90u'(u'')^2 + 120u''(180u^3 - u') \\ &\quad + 21540u^2(u')^2 + (720zu - 2zu^2 - 4)u']^2.\end{aligned} \tag{28}$$

Thus we have obtained the BT (26) and (27) between Fif-II equation (22) and equation (28).

Using the BT (26) and (27), we can find the following BT for the first integral (23)

$$v' = u \tag{29}$$

and

$$\begin{aligned}&2[u' - 3v^2 + \tfrac{1}{12}z][u''' - 6vu' - 6u^2] - [u'' - 6vu + \tfrac{1}{12}]^2 \\ &-24v[u' - 3v^2 + \tfrac{1}{12}z]^2 + K_2 = 0.\end{aligned} \tag{30}$$

Case II: $\phi_j = 0,\ j = 4, 5, 6$ and $\phi_3 \neq 0$

So we have $B \neq 0$ and we have two cases $b_1 = 0$ and $b_1 \neq 0$.

CaseII:a) $b_1 = 0$

Without loss of generality we can assume that $C = u$ and no possible factorization can be done in this case. Thus we obtain the following BT for (22)

$$v' = b_0v + u \tag{31}$$

and

$$\begin{aligned}&(180b_0)v^3 - 30(b_0'' + 4b_0b_0' + 2b_0^3 - 6u)v^2 + [b_0^{(4)} \\ &+5b_0b_0''' + 10(b_0' + b_0^2)b_0'' + 15b_0(b_0')^2 + 10b_0^3b_0' \\ &+b_0^5 - 90(b_0' + b_0^2)u - 60b_0u' - 30u'' - zb_0 - 2]v + [u^{(4)} \\ &+b_0u''' + (b_0^2 + 4b_0')u'' + (6b_0'' + 7b_0b_0' + b_0^3 - 30u)u' \\ &-30b_0u^2 + (4b_0''' + 9b_0b_0'' + 8(b_0')^2 \\ &+9b_0^2b_0' + b_0^4 - z)u] = 0.\end{aligned} \tag{32}$$

The elimination of v between (31) and (32) gives a fifth-order ODE for u of degree three.

Case II:b) $b_1 \neq 0$

Without loss of generality we can assume that $B = u$ and hence $\phi_3 = 180u$. In this case we can make a possible factorization

$$\sum_{j=0}^{3} \phi_j v^j = (v - g)(180uv^2 + Dv + E). \tag{33}$$

We can easily find that D and E are given by

$$\begin{aligned} D &= -30u'' - 120uu' - 60u^3 + 180g', \\ E &= u^{(4)} + 5uu''' + 10(u' + u^2)u'' + 15u(u')^2 \\ &\quad + 10(u^3 - 3g' + 60gu)u' + u^5 + 30gu^3 - 30g'u^2 \\ &\quad - (30g'' + z)u - 30g''' + 180gg' - 2. \end{aligned} \tag{34}$$

Moreover we have $c_1 = -g,\ c_0 = -c_1'$ and g is a second solution of equation (22). Our BT in this case reads

$$u = \frac{v' - g'}{v - g}, \tag{35}$$

and

$$180uv^2 + Dv + E = 0. \tag{36}$$

When we eliminate v between equations (35) and (36), we will get a fifth-order ODE for u of degree two.

4 The Cosgrove's Fif-III equation

Consider the Cosgrove's Fif-III equation

$$v^{(5)} = 20vv''' + 40v'v'' - 120v^2v' + zv' + 2v + \alpha. \tag{37}$$

where α is constant parameter. Equation (37) admits the first integral

$$\begin{aligned} &2[v'' - 6v^2 + 4\alpha v + \tfrac{1}{4}z - 4\alpha^2][v^{(4)} - 12vv'' - 12(v')^2 + 4\alpha v''] \\ &-[v''' - 12vv' + 4\alpha v' + \tfrac{1}{4}]^2 - 4(2v + \alpha)[v'' - 6v^2 \\ &+4\alpha v + \tfrac{1}{4}z - 4\alpha^2]^2 + K_3 = 0, \end{aligned} \tag{38}$$

where K_3 is a constant of integration. When $K_3 = 0$, a particular solution of Fif-III equation can be obtained satisfying the first Painlevé equation

$$v'' = 6v^2 - 4\alpha v - \frac{1}{4}z + 4\alpha^2. \tag{39}$$

For Fif-III equation (4) reads

$$\sum_{i=0}^{6} \phi_i v^i = 0, \tag{40}$$

where $\phi_6 = 120A^5$. Again we have two cases:
Case I: $\phi_j = 0, \quad j = 3,4,5,6$ and $\phi_2 \neq 0$
In order to make $\phi_6 = 0$, we take $A = 0$ and this implies that $\phi_4 = \phi_5 = 0$ identically. Now $\phi_3 = 120B$ and it will be identically zero if $B = 0$. Without loss of generality we can assume that $C = u$ and hence we obtain the following BT for Fif-III

$$v' = u \tag{41}$$

and

$$(120u)v^2 - 2(10u'' + 1)v + (u^{(4)} - 40uu' - zu - \alpha) = 0. \tag{42}$$

Eliminating v between equations (41) and (42), we get the following fifth-order ODE for u of degree two

$$\begin{aligned}&[-60u^2u^{(5)} + 60uu'u^{(4)} + 10uu'''(10u'' + 1) - 100u'(u'')^2\\ &\quad + 20u''(-u' + 120u^3) - (1 + 120zu^2 + 120\alpha u - 120u^2 - 60\alpha u)u'\\ &\quad + 60u^3]^2 = [-10uu''' + 10u'u'' + u' + 120u^3]^2[(10u'' + 1)^2\\ &\quad - 120u(u(4) - 40uu' - zu - \alpha)].\end{aligned} \tag{43}$$

Moreover a BT for the first integral (38) of equations (37) is given by

$$v' = u \tag{44}$$

and

$$\begin{aligned}&2[u' - 6v^2 + 4\alpha v + \tfrac{1}{4}z - 4\alpha^2][u''' - 12vu' - 12u^2 + 4\alpha u']\\ &-[u'' - 12vu + 4\alpha u + \tfrac{1}{4}]^2 - 4(2v + \alpha)[u' - 6v^2\\ &+4\alpha v + \tfrac{1}{4}z - 4\alpha^2]^2 + K_3 = 0.\end{aligned} \tag{45}$$

Case II: $\phi_j = 0, \ j = 4,5,6$ and $\phi_3 \neq 0$
So we have $B \neq 0$ and we have two cases $b_1 = 0$ and $b_1 \neq 0$
CaseII:a) $b_1 = 0$
Without loss of generality we can assume that $C = u$ and no possible factorization can be found in this case. Our BT in this case reads

$$v' = b_0 v + u \tag{46}$$

and

$$\begin{aligned}&(120b_0)v^3 - 10(2b_0'' + b_0b_0' + 6b_0^3 - 12u)v^2\\ &+[b_0^{(4)} + 5b_0b_0''' + 10(b_0' + b_0^2)b_0'' + 15b_0(b_0')^2 + 10b_0^3b_0'\\ &+b_0^5 - 20(4b_0' + 5b_0^2)u - 60b_0u' - 20u'' - zb_0 - 2]v\\ &+[u^{(4)} + b_0u''' + (b_0^2 + 4b_0')u'' + (6b_0'' + 7b_0b_0' + b_0^3 - 40u)u'\\ &-40b_0u^2 + (4b_0''' + 9b_0b_0'' + 8(b_0')^2 + 9b_0^2b_0' + b_0^4 - z)u - \alpha] = 0.\end{aligned} \tag{47}$$

Eliminating v between equations (46) and (47), we get a fifth-order ODE for u of degree three.

Case II:b) $b_1 \neq 0$
Without loss of generality we can assume that $B = u$ and hence $\phi_3 = 120u$. We have the following factorization for (40)

$$\sum_{i=0}^{3} \phi_i v^i = (v - g)(120uv^2 + Dv + E), \tag{48}$$

where

$$\begin{aligned} D &= -20u'' - 100uu' - 60u^3 + 120g', \\ E &= u^{(4)} + 5uu''' + 10(u' + u^2)u'' + 15u(u')^2 + 10(u^3 - 4g' + 13gu)u' \\ &\quad + u^5 + 40gu^3 - 40g'u^2 - (40g'' + z)u - 20g''' + 120gg' - 2, \end{aligned} \tag{49}$$

$g = -c_1,\ c_0 = -c_1'$, and g is a second solution of equation (37). The BT in this case is given by

$$u = \frac{v' - g'}{v - g} \tag{50}$$

and

$$120uv^2 + Dv + E = 0. \tag{51}$$

The elimination of v between equations (50) and (51) yields a fifth-order ODE for u of degree two.

5 The Cosgrove's Fif-IV equation

Consider the Cosgrove's Fif-IV equation

$$\begin{aligned} v^{(5)} &= 18vv''' + 36v'v'' - 72v^2v' + 3\lambda v'' + \tfrac{1}{2}\lambda z(5v''' - 36vv') \\ &\quad - \tfrac{1}{2}\lambda^2 z(2zv' + v) + \tfrac{1}{z}[v^{(4)} - 18vv'' - 9(v')^2 + 24v^3 - 3\lambda v' + \kappa], \end{aligned} \tag{52}$$

where λ and κ are constant parameters.

When $\lambda = 0$ equation (52) admits F-VI as a first integral

$$v^{(4)} = 18vv'' + 9(v')^2 - 24v^3 - \alpha v + \kappa z + \beta, \tag{53}$$

where β is a constant of integration.

When $\lambda \neq 0$, equation (52) admits the first integral

$$\begin{aligned} &L^3 - \tfrac{3}{2}\lambda^2 z^2 L[(v + \tfrac{1}{4}\lambda z)H'' - v'H' + \tfrac{1}{2}H^2 + \tfrac{1}{4}\lambda^2 z^2 v] \\ &+ \tfrac{3}{16}\lambda^3 z^3[(H'')^2 - 4(v + \tfrac{1}{4}\lambda z)HH'' - 4v(H')^2 + 4v'HH' - \tfrac{4}{3}H^3] \\ &+ \tfrac{3}{8}\lambda^4 z^3[v'H'' - 4v(v + \tfrac{1}{4}\lambda z)H' + \lambda vH - \tfrac{1}{4}\lambda(v')^2 + \lambda v(v + \tfrac{1}{4}\lambda z)^2] \\ &+ \tfrac{3}{4}K_4 z^3 = 0, \end{aligned} \tag{54}$$

where

$$\begin{aligned} H &= v'' - 6v^2 - 2\lambda z v - \tfrac{1}{8}\lambda^2 z^2, \\ L &= H'' - 6(v + \tfrac{1}{12}\lambda z)H + 3(v')^2 + \lambda v' \\ &\quad - 12v^3 - 6\lambda z v^2 - \tfrac{3}{4}\lambda^2 z^2 v + \tfrac{1}{4}\lambda^2 + \kappa, \end{aligned} \tag{55}$$

and K_4 is a constant of integration.

Proceeding as in the three examples above, we find that for Fif-IV equation (4) has the form

$$\sum_{i=0}^{6} \phi_i v^i = 0. \tag{56}$$

Once again we have two cases:
Case I: $\phi_j = 0, \quad j = 3, 4, 5, 6$ and $\phi_2 \neq 0$
In this case $\phi_6 = 120A^5$ and hence we have to take $A = 0$ in order to make $\phi_6 = 0$. This implies that $\phi_4 = \phi_5 = 0$ and $\phi_3 = 72B - \frac{24}{z}$. Setting $\phi_3 = 0$, we get $B = \frac{1}{3z}$ and without loss of generality we can assume that $C = u$. Thus we obtain the following BT for (52)

$$v' = \frac{1}{3z}v + u \tag{57}$$

and

$$(72u - \frac{7}{z^3} + 6\lambda)v^2 + \psi_1 v + \psi_0 = 0, \tag{58}$$

where

$$\begin{aligned} \psi_1 &= \tfrac{2390}{243}(\tfrac{1}{z^5}) + (\tfrac{26}{z^2} + 18\lambda z)u - 18u'' \\ &\quad + \tfrac{40}{54}(\tfrac{\lambda}{z^2}) + \tfrac{5}{6}\lambda^2 z, \\ \psi_0 &= u^{(4)} - \tfrac{2}{3z}u''' - (\tfrac{14}{9z^2} + \tfrac{5}{2}\lambda z)u'' - (\tfrac{110}{27z^3} + 36u \\ &\quad + \tfrac{23}{6}\lambda)u' - \tfrac{3}{z}u^2 - (\tfrac{560}{81z^4} + \tfrac{61}{18z}\lambda - \lambda^2 z^2)u - \tfrac{\kappa}{z}. \end{aligned} \tag{59}$$

Eliminating v between equations (57) and (58) gives a fifth-order ODE for u of degree two.
Case II: $\phi_j = 0, \ j = 4, 5, 6$ and $\phi_3 \neq 0$
We will consider here two cases $B = 0$ and $B \neq 0$.
CaseII:a) $B = 0$
When $B = 0$, we have $\phi_3 = -\frac{24}{z}$. Without loss of generality we can assume that $C = u$ and no possible factorization can be done in this case. Thus our BT is given by

$$v' = u \tag{60}$$

and

$$\begin{aligned} &-\tfrac{24}{z}v^3 - 72uv^2 + (-18u'' + \tfrac{18}{z}u' + 18\lambda z u + \tfrac{1}{2}\lambda^2 z)v \\ &+[u^{(4)} - \tfrac{1}{z}u''' - \tfrac{5}{2}\lambda z u'' - 36uu' + (\lambda^2 z^2 + \tfrac{3}{z}\lambda)u + \tfrac{9}{z}u^2 - \tfrac{\kappa}{z}] = 0. \end{aligned} \tag{61}$$

Elimination of v between equation (60) and (61) we get a fifth-order ODE for u of degree three.

CaseII:b) $B \neq 0$.

We have two cases $b_1 = 0$ and $b_1 \neq 0$.

CaseII:b1) $b_1 = 0$

Without loss of generality we can assume that $C = u$. No possible factorization can be done in this case and our BT is given by

$$v' = b_0 v + u \tag{62}$$

and

$$(72b_0 - \frac{24}{z})v^3 + \psi_2 v^2 + \psi_1 v + \psi_0 = 0, \tag{63}$$

where

$$\begin{aligned}
\psi_2 &= -18b_0'' + (\tfrac{18}{z} - 90b_0)b_0' - 54b_0^3 + \tfrac{27}{z}b_0^2 + 18\lambda z b_0 + 72u, \\
\psi_1 &= b_0^{(4)} + (5b_0 - \tfrac{1}{z})b_0''' + (10b_0' + 10b_0^2 - \tfrac{4}{z}b_0 \\
&\quad - \tfrac{5}{2}\lambda z)b_0'' + (15b_0 - \tfrac{3}{2})(b_0')^2 \\
&\quad + (10b_0^3 - 72u - 3\lambda - \tfrac{15}{2}\lambda z b_0 - \tfrac{6}{z}b_0^2)b_0' \\
&\quad + b_0^5 - \tfrac{1}{z}b_0^4 - \tfrac{5}{2}\lambda z b_0^3 - 3(30u + \lambda)b_0^2 \\
&\quad + (-54u' + \lambda^2 z^2 + \tfrac{3\lambda}{z} + \tfrac{36}{z}u)b_0 + (-18u'' \\
&\quad + \tfrac{18}{z}u' + 18\lambda z u + \tfrac{1}{2}\lambda^2 z, \\
\psi_0 &= 4ub_0''' + (6u' + 9b_0 u - \tfrac{3}{z}u)b_0'' + 8u(b_0')^2 \\
&\quad + (9ub_0^2 + 7b_0 u' - \tfrac{5}{z}ub_0 + 4u'' - \tfrac{3}{z}u' - 5\lambda z u)b_0' \\
&\quad + ub_0^4 + (u' - \tfrac{u}{z})b_0^3 + (u'' - \tfrac{1}{z}u' - \tfrac{5}{2}\lambda z u)b_0^2 \\
&\quad + (u''' - \tfrac{1}{z}u'' - \tfrac{5}{2}\lambda z u' - 36u^2 - 3\lambda u)b_0 \\
&\quad + u^{(4)} - \tfrac{1}{z}u''' - \tfrac{5}{2}\lambda z u'' - 3(12u + \lambda)u' \\
&\quad + \tfrac{9}{z}u^2 + (\lambda^2 z^2 + \tfrac{3}{z}\lambda)u - \tfrac{\kappa}{z}.
\end{aligned} \tag{64}$$

Eliminating v between (62) and (63), we obtain a a fifth-order third-degree ODE for u.

Case II:b2) $b_1 \neq 0$

Without loss of generality we can assume that $B = u$ and hence $\phi_3 = 72u - \frac{24}{z}$. As a result we have the factorization

$$\sum_{i=0}^{3} \phi_i v^i = (v - g)(\phi_3 v^2 + Dv + E), \tag{65}$$

where D and E can be obtained as function of $u, u', \ldots, u^{(4)}, z$ that we will not give here. Therefore we get the following BT for Fif-IV

$$u = \frac{v' - g'}{v - g} \tag{66}$$

and

$$\phi_3 v^2 + Dv + E = 0, \quad (67)$$

where g is a second solution of equation (52). The elimination of v between (66) and (67) yields fifth-order ODE for u of degree two.

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Bäcklund Transformations

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