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The Topology τ^* on Alexandroff Spaces

Hisham Mahdi[1,*]

Lubna Elostath

[1]Department of Mathematics, Faculty of Science, Islamic University of Gaza, Gaza Strip, Palestine

*** Corresponding author**
e-mail address: hmahdi@iugaza.edu.ps

Abstract

The generalized closure operator cl^* induces a topology τ^*. In this paper, we study the topology τ^* on lower bounded Alexandroff spaces. We prove that (X,τ^*) is a submaximal T_0-Alexandroff space. We get some new results about the relation between τ^* and τ_α. Then we prove that a subset A in a lower bounded space is $g-$closed set if and only if A is $\alpha-$open in the dual space.

Keywords:
Alexandroff spaces,
upper and lower bounded space,
$\alpha-$open sets, generalized closed sets, τ^* spaces.

MSC: 54A05, 54D10, 54F05, 54F65.

1. Introduction:

A topological space X is submaximal [6] if each dense subset is open. X is $T_{\frac{1}{2}}-$space if every singleton set is either open or closed [12]. A subset A of X is said to be $\alpha-$open [9] (resp. preopen [1], semi-open [8]) if $A\subseteq \overline{A^o}^o$ (resp. $A\subseteq \overline{A}^o$, $A\subseteq \overline{A^o}$). A set F is called $j-$closed for $j\in\{\alpha, semi, pre\}$ if $X\setminus F$ is $j-$open. The family of all $\alpha-$open (resp. preopen, preclosed, semi-open, semi-closed) is denoted by τ_α (resp. $PO(X)$, $PC(X)$, $SO(X)$, $SC(X)$). We have the following facts: The collection τ_α forms a topology on X [9]. $\tau_\alpha = PO(X)\cap SO(X)$ [11]. $\tau\subseteq\tau_\alpha$ (resp. $\tau\subseteq JO(X)$ for $J\in\{S,P\}$). For $j\in\{semi, pre\}$, the union (intersection) of any family of $j-$open ($j-$closed) sets is $j-$open ($j-$closed).

An Alexandroff space [10] (briefly $A-$space) X is a topological space in which an arbitrary intersection of open sets is open. In this space, each element x possesses a smallest open neighborhood $V(x)$ which is the intersection of all open sets containing x. For every T_0 $A-$space (X,τ), there is a corresponding poset $(X,\leq_\tau)$ in one to one and onto way, where each one of them is completely determined by the other. If (X,τ) is a T_0 $A-$space, we define the corresponding partial order $\leq_\tau$, called (Alexandroff) specialization order, by: $a\leq_\tau b$ iff $a\in\overline{\{b\}}$ iff $b\in V(a)$. On the other hand, if $(X,\leq)$ is a poset, then the collection $\mathcal{B}=\{\uparrow x : x\in P\}$ forms a base for a T_0 $A-space$ on X, denoted by $\tau_\leq$.

So throughout this paper, we consider $(X,\tau(\leq))$ to be a T_o $A-$space (X,τ) together with its corresponding poset $(X,\leq)$. A space is upper bounded T_0 $A-$space (UB T_0 $A-$space) [2] if every chain of points in the corresponding poset $(X,\leq)$ is bounded above [2]. A space is lower bounded (LB) [2] if every chain of points is bounded below. Given a poset $(X,\leq)$, the set of all maximal elements is denoted by $M(X)$ (or simply by M) and the set of all minimal elements is denoted by $m(X)$ (or simply by m). Moreover, for each $x\in X$, we define $\uparrow x=\{y\in X: x\leq y\}$, $\downarrow x=\{y\in X: x\geq y\}$, $\hat{x}$ to be the set of all maximal elements greater than or equal to x, and $\breve{x}$ the set of all minimal elements less than or equal to x [4]. If X is a UBT_0 $A-$space, then $M\neq\varnothing$ and $\hat{x}\neq\varnothing$ $\forall x\in X$. Similarly in an LBT_0 $A-$spaces, $m\neq\varnothing$ and $\breve{x}\neq\varnothing$. For a T0 $A-$space, $V(x)=\uparrow x$, and $\overline{x}=\downarrow x$ $\forall x\in X$. Moreover, if $A\subseteq X$, then $\overline{A}=\downarrow A$ $(=\downarrow A=\bigcup_{x\in A}\downarrow x)$, $A^o=\{x\in A:\uparrow x\in A\}$, and the smallest open set containing A equals $\uparrow A=\bigcup_{x\in A}\uparrow x$.

Theorem 1.1. [5] Let $(X,\tau(\leq))$ be a T_0 $A-$space, then all the following are equivalent:

a. X is $T_{\frac{1}{2}}-$space.

b. X is submaximal.

c. Each $\alpha-$open set is open; that is, $\tau_\alpha=\tau$.

d. Each element of X is either maximal or minimal.

Theorem 1.2. [3] Let $(X,\tau(\leq))$ be a UB T_0 $A-$space. Then:

a. $PO(X)=\tau_\alpha$ and $PO(X)\subseteq SO(X)$.

b. a subset A is semi open set if and only if $\hat{x}\cap A\neq\varnothing$ $\forall x\in A$.

c. a subset A is $\alpha-$open set if and only if $\hat{x}\subseteq A$ $\forall x\in A$.

d. a subset A is clopen in τ if and only if A is clopen in τ_α.

e. (X,τ_α) is a submaximal T_0 $A-$space.

1. Generalized Closed Sets on LB T_0 $A-$Spaces:

Definition 2.1. [7] A subset A of a topological space (X,τ) is called a generalized closed set (briefly g-closed), if $\overline{A}\subseteq U$ whenever $A\subseteq U$, and U is open in X. A subset A of X is g-open if A^c is g-closed.

Remark 2.2. (a) In general, the collection of all $g-open$ sets need not form a topology on X.

(b) In a T_0 A-space, since $\uparrow A$ is the smallest open nhood of A, we can use $\uparrow A$ instead the open set U which contains A in the Definition 2.1.

(c) An alternating definition of $T_{\frac{1}{2}}-$space found in [7] as a space where each $g-$closed set is closed.

Proposition 2.3. If $(X,\tau(\leq))$ is a T_0 A-space and $A\subseteq X$, then A is g-closed set if and only if $\forall x\in A$, and $\forall a\in\overline{x}$ we have $\overline{a}\cap A\neq\varnothing$.

Proof. Let $x\in A$ and $a\in\overline{x}(=\downarrow x)$. As A $g-$closed, $\overline{A}\subseteq\uparrow A$, and hence $a\in\uparrow A$. Then there exists $y\in A$ such that $a\in\uparrow y$. Equivalently, $y\in\downarrow a(=\overline{a})$. Conversely, it suffices to prove that $\overline{A}\subseteq\uparrow A$, so let $r\in\overline{A}(=\downarrow A)$. Then there exists $x\in A$ such that $r\in\downarrow x$. By given, $\overline{r}\cap A\neq\varnothing$. Let $y\in\overline{r}\cap A$. Then $r\in\uparrow y\subseteq\uparrow A$. □

Corollary 2.4. Let $(X,\tau(\leq))$ be an LB T_0 A-space, and A a subset of X. Then A is g-closed set if and only if $\forall x\in A$, $\breve{x}\subseteq A$.

Proof. For $x\in A$, $\breve{x}\subseteq\downarrow x$. Let $a\in\breve{x}$, then $a=\overline{a}$ and $\overline{a}\cap A\neq\varnothing$. Hence $a\in A$. The converse is easy. □

Corollary 2.5. If $(X,\tau(\leq))$ is an LB T_0 A-space, and $A\subseteq X$, then A is g-open if and only if $\uparrow x\subseteq A$ whenever $x\in m\bigcap A$.

Proof. Suppose to contrary that $y\in m\cap A$ such that $\uparrow y\cap A^c\neq\varphi$. Then there exists $r\geq y$ such that $r\in A^c$. By Corollary 2.4, $\breve{r}\subseteq A^c$. Since $y\in m$, $y\in\breve{r}\subseteq A^c$ which is a contradiction. Conversely, Let

$r \in A^c$ and let $s \in \breve{r}$. Then $r \in \uparrow s$. This implies that $s \in A^c$, and hence $\breve{r} \subseteq A^c$. Therefore, A^c is g-closed. □

Example 2.6. Let $X = \{a, b, d, c, d, e, f, g, h, r\}$, equipped with the partial order as shown in Figure 1 below:

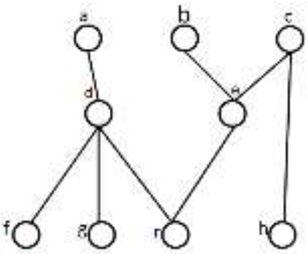

Figure 1: g-closed and g-open sets

For the set $A = \{a, e, r, g, f\}$, we have that $\breve{a} = \{f, g, r\}$, $\breve{e} = \{r\}$, $\breve{r} = \{r\}$, $\breve{f} = \{f\}$, and $\breve{g} = \{g\}$. That is, $\forall x \in A$, $\breve{x} \subseteq A$. Hence A is $g-$closed. Let $B = \{d, f, c, g, r\}$. Then B is not g-closed set, since $\breve{c} \not\subset B$. Consider the set $D = \{h, c, f, d, a\}$. We have that $D \cap m = \{h, f\}$, and $\uparrow h \subseteq D$, $\uparrow f \subseteq D$. Hence D is $g-$open.

2. The Topology τ^* :

A Kuratowski closure operator cl^* is defined in [13] as the intersection of all g-closed sets. The topology generated by cl^* is denoted as τ^*. For any topological space (X, τ), the space (X, τ^*) is always a $T_{\frac{1}{2}}$-space. It is worth mentioned that if A is a $g-$closed, then $cl^*(A) = A$ is $g-$closed. But for arbitrary set A, $cl^*(A)$ need not be always $g-$closed.

Theorem 3.1. If $(X, \tau(\leq))$ is an LB T_0 A-space and $A \subseteq X$, then $cl^*(A) = A \cup \bigcup_{x \in A} \breve{x}$.

Proof. From Corollary 2.4, the set $A \cup \bigcup_{x \in A} \breve{x}$ is a $g-$closed set containing A, so $cl^*(A) \subseteq A \cup \bigcup_{x \in A} \breve{x}$. Let S be any $g-$closed set containing A. Then $\forall x \in S$, $\breve{x} \subseteq S$. If $y \in A \cup \bigcup_{x \in A} \breve{x}$ and $y \notin A$, then $y \in \breve{x}$ for some $x \in A$. As $A \subseteq S$, $y \in S$. Hence $A \cup \bigcup_{x \in A} \breve{x}$ is the smallest $g-$closed set containing A. Therefore $cl^*(A)$ = $A \cup \bigcup_{x \in A} \breve{x}$. □

Corollary 3.2. If X is an LB T_0 A-space, then $\forall A \subseteq X$, $cl^*(A)$ is a $g-$closed.

From this corollary, we deduce that in an LB T_0 $A-$space X, τ^* is precisely the collection of all $g-$open sets in X.

Example 3.3. In Example 2.6, as A $g-$closed, $cl^*(A) = A$. So $A^c = \{b, c, d, h\} \in \tau^*$. While $cl^*(B) = B \cup \{h\} \neq B$, so B is not $g-$closed.

Theorem 3.4. If $(X, \tau(\leq))$ is an LB T_0 A-space, then (X, τ^*) is a $T_{\frac{1}{2}}$ $A-$space.

Proof. Let $\{u_\alpha : \alpha \in \Delta\}$ be a collection of $g-$closed sets in (X, τ) $\left(= \textit{closed in } (X, \tau^*)\right)$, and let $x \in \bigcup_{\alpha \in \Delta} u_\alpha$. Then by Corollary 2.4, $\breve{x} \subseteq u_\alpha$ for some $\alpha \in \Delta$. So, $\breve{x} \subseteq \bigcup_{\alpha \in \Delta} u_\alpha$. Hence $\bigcup_{\alpha \in \Delta} u_\alpha$ is $g-$closed, and (X, τ^*) is $A-$space. □

As (X, τ^*) is a T_0 $A-$space, we denote its specialization order by $\leq^*$. The following theorem describes the partial order $\leq^*$ on X.

Proposition 3.5. Let $(X, \tau(\leq))$ be an LB T_0 A-space and x, y two elements in X. Then $x \leq^* y$ if and only if $x \in \{y\} \bigcup \breve{y}$.

Proof. Direct using the fact $cl^*\{y\} = \{y\} \bigcup \breve{y}$. □

Remark 3.6. With respect to the order $\leq^*$, we will use the following notations:

- M^* to be the set of all maximal elements.
- m^* to be the set of all minimal elements.
- $\uparrow^* x := \{y \in X : x \leq^* y\}$.
- $\downarrow^* x := \{y \in X : y \leq^* x\}$.
- $\hat{x}^* := \uparrow^* x \cap M^*$.
- $\breve{x}^* := \downarrow^* x \cap m^*$.

Theorem 3.7. Let $(X,\tau(\leq))$ be an LB T_0 A-space. Then:

a. each element in X is either maximal or minimal with respect to $\leq^*$.

b. (X,τ^*) is submaximal space.

c. $m^* = m$.

d. $M^* = (X \setminus m) \cup (m \cap M)$.

Proof. The parts (a) and (b) come directly from Theorem 1.1 and the fact that τ^* is $T_{\frac{1}{2}}$.

(c) Let $x \in m^*$, and let $y \leq x$. Then from part (a), either $y = x$ or $y \in \breve{x}$. So $y \leq^* x$, and hence $y = x$. If $x \in m$, then $\breve{x} = \{x\}$ and $\breve{x} \cup \{x\} = \{x\}$. So, if $y \leq^* x$, then $y \in \breve{x} \cup \{x\} = \{x\}$. Hence $y = x$, and $x \in m^*$.

(d) Comes directly from parts (a) and (c). □

Proposition 3.8. If $(X,\tau(\leq))$ is an LB T_0 A-space and $A \subseteq X$, then $int^*(A) = (A \setminus m) \cup \{x \in m : \uparrow x \subseteq A\}$.

Proof. $x \in int^*(A)$ iff $\uparrow^* x \in A$. □

Example 3.9. For the LB T_0 $A-$space described in Example 2.6, the induced order $\leq^*$ on X is given in Figure 2 below:

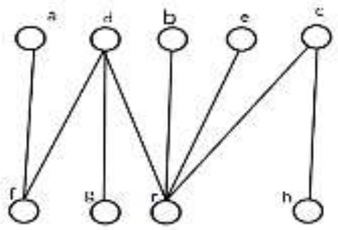

Figure 2: Induced poset $(X,\leq^*)$

Theorem 3.10. If $(X,\tau(\leq))$ is an LB T_0 $A-$space, then:

a. $m \cap M = m^* \cap M^*$.

b. $\downarrow^* x = \{x\} \cup \breve{x}$.

c. if $x \notin m$ then $\uparrow^* x = \{x\}$, and if $x \in m$ then $\uparrow^* x = \uparrow x$.

Proof. The proof comes directly from the description of $\leq^*$ in Proposition 3.5.and Theorem 3.7. □

3. The Relation Between τ^* and τ_α:

If (X,τ) is an $A-$space with a collection F of closed sets, then F is itself an Alexandroff topology on X, called the dual Alexandroff topology of τ on X, and usually denoted by τ_∂. If $(X,\tau(\leq))$ is a T_0 $A-$space, then the Alexandroff dual is also T_0-space. The induced order $\leq_\partial$ is the reverse order of the order $\leq$; that is, $x \leq_\partial y$ iff $y \leq x$. So, for $x \in X$, we have $V(x)$ = $cl_\partial(x)$ and $V_\partial(x) = cl(x)$. Moreover, if $(X,\tau(\leq))$ is a UB (resp. an LB) T_0 $A-$space, then the dual space $(X,\tau_\partial(\leq_\partial))$ is an LB (resp. a UB) T_0 $A-$space. The set of maximal elements M^∂ in the corresponding poset $(X,\leq_\partial)$ equals the set of minimal points m in $(X,\leq)$, and similarly $m^\partial = M$. For $A \subseteq X$, the up (resp. down) set $\uparrow_\partial A = \downarrow A$ (resp. $\downarrow_\partial A = \uparrow A$). For $x \in X$, $\hat{x}_\partial = \uparrow_\partial x \cap M^\partial$ and $\breve{x}_\partial = \downarrow_\partial x \cap m^\partial$. It is clearly that $\hat{x}_\partial = \breve{x}$ and $\breve{x}_\partial = \hat{x}$.

Theorem 4.1. Let $(X,\tau(\leq))$ be an LB T_0 A-space, and A is a subset of X. Then A is g-closed in (X,τ) if and only if A is $\alpha-$open in the dual space $(X,\tau_\partial(\leq_\partial))$.

Proof. A is g-closed set in $(X,\tau(\leq))$ if and only if $\breve{x} \subseteq A$ $\forall x \in A$ (Theorem 2.4) if and only if $\hat{x}_\partial \subseteq A$ $\forall x \in A$ if and only if A is α-open set in (X,τ_∂) (Theorem 1.2 part (c)). □

Proposition 4.2. Let $(X,\tau(\leq))$ be a UB T_0 $A-$space, then $(\tau_\alpha)^* = \tau_\alpha$.

Proof. Let $(X,\tau(\leq))$ be a UB T_0 $A-$space. Then by Theorem 1.1 and Theorem 1.2 (e), (X,τ_α) is $T_{\frac{1}{2}}$. So any g-closed set in τ_α is closed set. Equivalently, any g-open is open in τ_α. Therefore $(X,(\tau_\alpha)^*) = (X,\tau_\alpha)$. □

Proposition 4.3. Let $(X,\tau(\leq))$ be an LB T_0 $A-$space then $(\tau_\partial)_\alpha = (\tau^*)_\partial$.

Proof. U is $\alpha-$open in (X,τ_∂) iff U is g-closed in (X,τ) (Theorem 4.1) if and only if U^c is open in τ^* iff $U\in(\tau^*)_\partial$. □

Proposition 4.4. Let $(X,\tau(\leq))$ be an LB T_0 $A-$space and $A\subseteq X$. Then A is clopen if and only if A is g-clopen.

Proof. A is clopen in τ iff A is clopen in τ_∂ iff A is clopen $(\tau_\partial)_\alpha=(\tau^*)_\partial$ (Theorem 1.2 (d) and Theorem 4.3) iff A is clopen in τ^*. □

References:

Mashhour, A. S., Abd El-Monsef, M.E., & El-Deeb, S.N. (1982). On precontinuous and weak precontinuous mappings. *Proc. Math. Phys. Soc. Egypt 53*, 47–53.

Rose, D., Scible, G., & Walsh, D. (2012). Alexandroff spaces. *Journal of Advanced studies in Topology, 24*, 41–43.

Mahdi, H., & Elostath L. (2014). On Upper Bounded T_0 Alexandroff Spaces. *Int. J. Contemp. Math. Sciences, 9*(8), 361–374.

Mahdi, H., & El atrash M. (2005). On T_0-Alexandroff spaces. *The Islamic University Journal, 13*(2), 19–46.

Mahdi, H., & Alatrash M. (2006). Caractrization of Lower Separation Axioms in T_0 Alexandroff spaces. Proceeding of the first conference on mathematical science, Zarga Private Uni., Jordan, 77–89.

Reilly, I., & Vamanamurthy M. (1990). On some questions concerning preopen sets, *Kyungpook Math. J. 3*(1).

Levine N. (1970). Generalized closed sets in topology. *Rend. Circ. Math. Palermo, 19*(2), 89–96.

Levine N. (1963). Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly 70*, 36–41.

Njastad O. (1965). On some classes of nearly open sets. *Paci c J. Math. 15*, 961–970.

Alexandroff, P. (1937). Diskrete R¨aume. *Mat. Sb. (N.S.) 2*, 501-518.

Noiri T. (1984). On $\alpha-$continuous functions. *Cas. pest. mat., 109*, 118–126.

Dunham W. (1977). $T_{\frac{1}{2}}-$spaces. *Kyungpook Math. J., 17*(2), 161-169.

Dunham, W. (1982). A new closure operator for non-T_1 topologies. *Kyungpook Math. J., 22*, 55–60.

التوبولوجي τ^* على الفضاءات الكساندرووف

مؤثر الإغلاق المعمم cl^* يولد توبولوجي τ^*. في هذا البحث قمنا بدراسة التوبولوجي τ^* على الفضاءات الكساندرووف المحدودة من أسفل. تم إثبات أن الفضاء التوبولوجي (X,τ^*) يكون فضاء submaximal T_0- الكساندرووف. إضافة لذلك حصلنا على نتائج جديدة عن العلاقة بين التوبولوجيين τ^* و τ_α. بعد ذلك قمنا بإثبات أن مجموعة A في الفضاء المحدود من اسفل تكون مغلقة معممة إذا وفقط اذا كانت $\alpha-$مفتوحة في الفضاء التوبولوجي المناظر.

الكلمات المفتاحية:

فضاءات الكساندرووف،
الفضاءات المحدودة من أسفل ومن أعلى،
المجموعات $\alpha-$ مفتوحة،
المجموعات المغلقة المعممة،
التوبولوجي τ^*.

تصنيف مواضيع الرياضيات:

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