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ON DOUBLE NÖRLUND SUMMABILITY OF FOURIER-JACOBI SERIES

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ABSTRACT

In this paper we have proved a theorem on double Nörlund summability of Fourier-Jacobi series, which generalizes various known results. However, our theorem is as follows:

Theorem : Let (N,p_m,q_n) be a double Nörlund method defined by a real non-negative, non-increasing sequence $\{p_m\}$ and a real non-negative,non-decreasing sequence$\{q_n\}$.
Let $\psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t, such that

$$\Psi(n) \log n = O\,[\lambda(P_n)]$$

$$q_n P_n = O\,[(p*q)_n \log n]$$

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{(p*q)_n^{1-c}}{q_n n^{(2\alpha+1)/2}}\right)$$

as n → ∞, where c is a parameter with the restriction that $0 \le c \le 1$. If

$$F_1(t) = \int_0^t |F(\Phi)|\, d\Phi = o\left(\frac{t^{(2\alpha+2)}\psi(t)}{\lambda(P_\tau)}\right)$$

as t → 0, where $\tau = [1/t]$ then the Fourier-Jacobi series is summable (N,p_m,q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2,\ \beta > -1/2$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)|\, dx < \infty$$

are satisfied, where b is fixed.

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1. Definition and Notations:

Let f(x) be a function defined on the interval $-1 \leq x \leq 1$ such that the integral

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)\, dx \tag{1.1}$$

exists in the sense of Lebesgue for $\alpha > -1$, $\beta > -1$. The Fourier-Jacobi series corresponding to the function f(x) is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{1.2}$$

where

$$a_n = g_n \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha,\beta)}(x)\, f(x)\, dx \tag{1.3}$$

with

$$g_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \tag{1.4}$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials defined by the generating function

$$2^{\alpha+\beta}(1-2xt+t^2)^{-\frac{1}{2}}\,[1-t+(1-2xt+t^2)^{\frac{1}{2}}]^{-\alpha}\,[1+t+(1-2xt+t^2)^{\frac{1}{2}}]^{-\beta} = \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, t^n \tag{1.5}$$

Let us write

$$F(\Phi) = \{ f(\cos\Phi) - A\} \left(\sin\frac{\Phi}{2}\right)^{2\alpha+1} \left(\cos\frac{\Phi}{2}\right)^{2\beta+1}$$

A being a fixed constant.

Let $[s_n]$ be the sequence of partial sums of a given infinite series Σa_n. Let $\{p_m\}$ and $\{q_n\}$ be any two sequences of constants with P_m and Q_n as their partial sums respectively and

let

$$(p*q)_n = \sum_{k=0}^{n} p_{n-k} q_k = \sum_{k=0}^{n} p_k q_{n-k} \tag{1.6}$$

tends to infinity as $n \to \infty$.

If the sequence-to-sequence transformation defined by

$$t_n^{p,q} = \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_{n-k} q_k s_k \tag{1.7}$$

tends to a fixed limit s as $n \to \infty$, then the sequence $\{s_n\}$ or the series Σa_n is said to be summable by double Nörlund method (N, p_m, q_n) to s, Borwein [2].

2. Introduction : The study of summability of Fourier-Jacobi series by ordinary Nörlund summability method has been made by several workers ([1],[5],[9],[16]). In the present paper we study the summability of Fourier-Jacobi series by double Nörlund summability method.

Dealing with the Nörlund summability of Fourier-Jacobi series, Prasad and Saxena [10] have established the following :

Theorem A : If

$$F_1(t) = \int_0^t |F(\Phi)| \, d\Phi = O\left(\frac{\Psi(t)\, t^{2\alpha+2}}{\theta(P_n)}\right) \quad \text{as } t \to 0 \tag{2.1}$$

where

$$F(\Phi) = \{ f(\cos\Phi) - A \} (\sin\frac{\Phi}{2})^{2\alpha+1} (\cos\frac{\Phi}{2})^{2\beta+1},$$

$\Psi(t)$ and $\theta(t)$ are non-negative monotonic increasing functions of t such that

$$\Psi(n) \log n = O(\theta(P_n)) \qquad \text{as } n \to \infty \tag{2.2}$$

$$n^{(2\alpha+1)/2} = o(P_n) \qquad \text{as } n \to \infty \tag{2.3}$$

and

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{P_n}{n^{(2\alpha+1)/2}}\right) \qquad \text{as } n \to \infty \tag{2.4}$$

then the Fourier-Jacobi series (1.2) is summable (N,p_n) at the point x= +1, to sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(2\beta-3)/4} |f(x)| \, dx < \infty \tag{2.5}$$

are satisfied, where b is fixed and (N,p_n) is regular Nörlund method defined by the real non-negative and non-increasing sequence $\{p_n\}$ such that $p_n \to \infty$ as $n \to \infty$.

The object of this paper is to generalize the above theorem to a more general class of double Nörlund summability of Fourier-Jacobi series.

3. Statement of the theorem :

We establish our result in the form of the following theorem.

Theorem : Let (N, p_m, q_n) be a double Nörlund method defined by a real non-negative, non-increasing sequence $\{p_m\}$ and a real non-negative, non-decreasing sequence $\{q_n\}$.

Let $\Psi(t)$ and $\lambda(t)$ be non-negative monotonic increasing functions of t such that

$$\Psi(n) \log n = O[\lambda(P_n)] \tag{3.1}$$

$$q_n P_n = O[(p^*q) \log n] \tag{3.2}$$

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} = O\left(\frac{(p^*q)_n^{1-c}}{q_n n^{(2\alpha+1)/2}}\right) \tag{3.3}$$

as $n \to \infty$, where c is a parameter with the restriction that $0 \le c \le 1$.

If

$$F_1(t) = \int_0^t |F(\Phi)| \, d\Phi = o\left(\frac{t^{(2\alpha+2)}\Psi(t)}{\lambda(P_\tau)}\right) \tag{3.4}$$

as $t \to 0$, where $\tau = [1/t]$ then the Fourier-Jacobi series (1.2) is summable (N, p_m, q_n) at the point $x = +1$ to the sum A, provided that the condition

$$-1/2 \le \alpha < 1/2, \quad \beta > -1/2$$

and the antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} |f(x)| \, dx < \infty \tag{3.5}$$

are satisfied, where b is fixed.

4. The following lemmas are needed for the proof of our theorem:

LEMMA1 : ([15] p. 167 & 196) : For $\alpha > -1$, $\beta > -1$

$$P_n^{(\alpha,\beta)}(\cos\Phi) = \left| \begin{array}{l} O(n^\alpha), \text{ when } 0 \le \Phi \le 1/n \\ O(n^\beta), \text{ when } \pi - 1/n \le \Phi \le \pi \\ \frac{1}{(n\pi)^{1/2}} \left(\sin\frac{\Phi}{2}\right)^{-(2\alpha+1)/2} \left(\cos\frac{\Phi}{2}\right)^{-(2\beta+1)/2} \\ \cdot \left(\cos\left\{\frac{(2n+\alpha+\beta+1)}{2}\Phi - (2\alpha+1)\frac{\pi}{4}\right\} + \frac{O(1)}{n\sin\Phi}\right) \\ \text{when } 1/n \le \Phi \le \pi - 1/n. \end{array} \right.$$

LEMMA 2 : [10] The antipole condition

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \,|\, f(x) \,|\, dx < \infty$$

is equivalent to

$$\int_{-1}^{b} (1+x)^{(\beta-\alpha-1)/2} \,|\, f(x) - A \,|\, dx < \infty$$

which is further equivalent to

$$\int_{n}^{\pi} |\, F(\Phi) \,|\, (\cos \frac{\Phi}{2})^{-\alpha-\beta-1} \, d\Phi < \infty, \qquad 0 < n < \pi .$$

LEMMA 3 . [7]. If $\{p_n\}$ is a non-negative, non-increasing sequence then for large n, uniformly in $0 < \Phi \leq \pi$, $0 \leq a \leq b \leq n$,

$$\left| \sum_{k=a}^{n} p_k \cos \{(n-k+\rho)\,\Phi - \gamma\} \,(n-k)^{(2\alpha+1)/2} \right| = O\,(n^{(2\alpha+1)/2} \rho_{(1/\Phi)})$$

Where

$$\rho = \frac{\alpha+\beta+2}{2} \quad , \quad \gamma = \frac{(2\alpha+3)\pi}{4} \quad , \quad \alpha \geq -1/2$$

LEMMA 4. [7] : If $\{p_n\}$ is a non-negative, non-increasing and $\{q_n\}$ is a non-negative, non-decreasing sequence then

$$\sum_{k=0}^{n-1} p_k q_{n-k} (n-k)^{(2\alpha-1)/2} = O\,(\,(p^*q)_n \, n^{(2\alpha-1)/2})$$

LEMMA 5 . Let

$$N_n(\Phi) = \frac{2^{\alpha+\beta+1}}{(p^*q)_n} \sum_{k=0}^{n-1} p_k \, q_{n-k} \, \delta_{n-k} P_{n-k}^{(\alpha+1,\beta)} (\cos \Phi)$$

where

$$\delta_n = \frac{2^{-\alpha-\beta-1} \,\Gamma(n+\alpha+\beta+2)}{\Gamma(n+1)\,\Gamma(n+\beta+1)} \simeq \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} \quad , \quad n^{\alpha+1}$$

Then for $-1/2 \le \alpha < 1/2$, $\beta > -1/2$ and $\{p_m\},\{q_n\}$ satisfying the conditions of the theorem, we have

$$N_n(\Phi) = \left| \begin{array}{ll} O(n^{2\alpha+2}), \text{ when } 0 \le \Phi \le 1/n & (4.1) \\ O(n^{2\alpha+\beta+1}), \text{ when } \pi-1/n \le \Phi \le \pi & (4.2) \\ O\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} (\sin \frac{\Phi}{2})^{-(2\alpha+3)/2} (\cos \frac{\Phi}{2})^{-(2\beta+1)/2} \right) & (4.3) \\ +\ O[\, n^{(2\alpha-1)/2} (\sin \frac{\Phi}{2})^{-(2\alpha+5)/2} (\cos \frac{\Phi}{2})^{-(2\beta+3)/2} & \\ \text{when } 1/n \le \Phi \le \pi - 1/n. & \end{array} \right.$$

Proof . Using Lemma 1 for $0 \le \Phi \le 1/2$ together with Lemma 4, the required estimate in (4.1) follows. For the estimate in (4.2), we use Lemma 1 for $\pi - 1/n \le \Phi < \pi$ together with Lemma 4.

For $1/n \le \Phi \le \pi - 1/n$, we have from Lemma 1.

$$N_n(\Phi) = \frac{O(1)}{(p^*q)_n} \sum_{k=0}^{(n-1)} p_k\, q_{(n-k)}\, (n-k)^{(2\alpha+1)/2} \cdot (\sin \frac{\Phi}{2})^{-(2\alpha+1)/2} (\cos \frac{\Phi}{2})^{-(2\beta+1)/2} \cdot \left(\cos\{(n-k+\rho)\Phi - \gamma\} + \frac{O(1)}{(n-k)\sin\Phi} \right)$$

since for fixed n, $\{q_{n-k}\}$ is non increasing, we can deal with the first term of the right by first using the second mean value theorem and then applying Lemma 3 to deal with the second term on the right, we apply the result of Lemma 4 and the required estimate follows.

LEMMA 6. The condition

$$q_n\, n^{(2\alpha+1)/2} = O(p^*q)_n^{1-c}$$

For $0 \le c \le 1$ under the hypothesis of the theorem.

Proof . The expression on the left of (3.3) is increasing and hence greater than or equal to a positive constant. Hence (3.4) implies that for some positive constant A,

$$\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k} \ge \sum_{k=a}^{n} \frac{P_k}{k^{(2\alpha+1)} \log k} \quad \text{where } a > 2$$

(note that $k\, p_k \leq p_k \Rightarrow p_k \geq k\, p_k$, by the condition on $\{p_n\}$)

$$= \sum_{k=a}^{n} \frac{P_k q_{n-k}}{q_{n-k} k^{(2\alpha+3)/2} \log k}$$

$$> \frac{(p^*q)_n}{q_n n^{(2\alpha+1)/2}} \qquad \text{by the condition on } \{q_n\}$$

$$> \frac{(p^*q)_n^{1-c}}{q_n n^{(2\alpha+1)/2}} \qquad \{\therefore (p^*q)_n > (p^*q)_n^{1-c} \text{ for } 0 \leq c \leq 1\}$$

From which the result in Lemma 6 follows.

5. <u>Proof of the theorem</u> : Following Obrechkoff ([8]. p. 99. and Rao [11]) the nth partial sum of the series (1.2) at the point $x = 1$ is given by

$$S_n(1) = 2^{\alpha+\beta+1} \delta_n \int_0^{\pi} \left(\sin \frac{\Phi}{2}\right)^{2\alpha+1} \left(\cos \frac{\Phi}{2}\right)^{2\beta+1} f(\cos \Phi)\; P_n^{(\alpha+1,\beta)}(\cos \Phi)\, d\Phi$$

consequently,

$$S_n(1) - A = 2^{\alpha+\beta+1} \delta_n \int_0^{\pi} F(\Phi)\; P_n^{(\alpha+1,\beta)}(\cos \Phi)\, d\Phi \tag{5.1}$$

using (1.7), the (N, p_m, q_n) mean of the series (1.2) is given by

$$t_n^{p,q} - A = \frac{1}{(p^*q)_n} \sum_{k=0}^{n} p_k q_{(n-k)} \{S_{n-k}(1) - A\}$$

$$= \int_0^{\pi} F(\Phi)\; N_n(\Phi)\, d\Phi$$

$$= I \text{, say}$$

$$= \left(\int_0^{1/n} + \int_{1/n}^{\eta} + \int_{\eta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi} \right) F(\Phi)\; N_n(\Phi)\, d\Phi$$

$$t_n^{p,q} - A = I_1 + I_2 + I_3 + I_4 \tag{5.2}$$

say, where η is a suitable constant such that $0 < \eta < \pi$.
Now in order to prove our theorem, we have to show that

$$I = o(1), \qquad \text{as } n \rightarrow \infty \tag{5.3}$$

for which we need to prove that

$$I_j \;=\; o(1)\,, \qquad \text{as } n \to \infty \tag{5.4}$$

for $j = 1,2,3,4$.

Let us first consider I_1, we have

$$\begin{aligned}
I_1 &= O\left(\int_0^{1/n} \left|F(\Phi)\right| \left|N_n(\Phi)\right| d\Phi\right) \\
&= O\left(n^{2\alpha+2}\right) \int_0^{1/n} \left|F(\Phi)\right| d\Phi && \text{by (4.1) of Lemma 5.} \\
&= O\left(n^{2\alpha+2}\right).\ o\left(\frac{\Psi(n)\, n^{-2\alpha-2}}{\lambda(P_n)}\right) && \text{by (3.4)} \\
&= o\left(\frac{1}{\log n}\right) && \text{by (3.1)}
\end{aligned}$$

$$I_1 = o(1)\,, \qquad \text{as } n \to \infty \tag{5.5}$$

considering I_2, we have

$$\begin{aligned}
I_2 = {} & O\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n}\right) \int_{1/n}^{\eta} \frac{\left|F(\Phi)\right| P_{[1/\Phi]}}{\Phi^{(2\alpha+3)/2}}\, d\Phi \\
& + O\left(n^{(2\alpha-1)/2}\right) \int_{1/n}^{\eta} \frac{\left|F(\Phi)\right|}{\Phi^{(2\alpha+5)/2}}\, d\Phi
\end{aligned}$$

using (4.3) of Lemma 5.

$$I_2 \;=\; I_{2.1} + I_{2.2} \qquad \text{say .} \tag{5.6}$$

Given $\varepsilon > 0$, let η be chosen so that

$$\left|F_1(\Phi)\right| \;=\; \frac{\varepsilon\, \Phi^{(2\alpha+2)}\, \delta_{(1/\Phi)}}{\Phi\left(P_{(1/\Phi)}\right)}\,, \qquad 0 \le \Phi \le \eta.$$

Then

$$\begin{aligned}
\left|I_{2.1}\right| &\le \frac{K\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \int_{1/n}^{\eta} \frac{\left|F(\Phi)\right| P_{[1/\Phi]}}{\Phi^{(2\alpha+3)/2}}\, d\Phi \\
&\le \frac{K\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \left\{ \left(\frac{\left|F_1(\Phi)\right| P_{(1/\Phi)}}{\Phi^{(2\alpha+3)/2}}\right)_{1/n}^{\eta} - \int_{1/n}^{\eta} F^{1}(\Phi)\, d\left(\frac{P_{(1/\Phi)}}{\Phi^{(2\alpha+3)/2}}\right) \right\}
\end{aligned}$$

$$\left|\, I_{2.1} \,\right| \;\le\; I_{2.1.1} + I_{2.1.2} \qquad \text{(say)} \tag{5.7}$$

where K is absolute constant, not necessarily same at each occurrence. If K (η) denotes a constant depending on η, we see that, for fixed η,

$$\left| I_{2.1.1} \right| = K(\eta) \frac{q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} + O\left(\frac{q_n\, P_n\, \Psi(n)}{(p^*q)_n\, \lambda(P_n)} \right)$$

$$= K(\eta)\ o\left(\frac{1}{(p^*q)_n^{\mathbf{c}}} \right) + o(1)$$

for $0 \le c \le 1$.

$$\left| I_{2.1.1} \right| = o(1)\,, \quad \text{as } n \to \infty \tag{5.8}$$

by using Lemma 6, (3.4), (3.1) and (3.2).

Further

$$I_{2.1.2} \le \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \int_{1/n}^{\eta} \frac{\Phi^{2\alpha+2}\ \Psi_{(1/\Phi)}}{\lambda(P_{[1/\Phi]})} \left| d\left(\frac{P_{[1/\Phi]}}{\Phi^{(2\alpha+3)/2}} \right) \right|$$

$$\le \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \int_{1/n}^{\eta} \frac{x^{-2\alpha-2}}{\log x}\ d\left| P_{[x]}\, x^{(2\alpha+3)/2} \right|$$

using (3.1)

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \left(\int_{1/n}^{\eta} \frac{x^{-(2\alpha+1)/2}}{\log x}\, d\,P_{[x]} + \frac{(2\alpha+3)}{2} \int_{1/n}^{\eta} \frac{x^{-(2\alpha+3)/2}}{\log x}\, P_{[x]}\, dx \right)$$

$$= \frac{K\,\varepsilon\, q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n} \left(L + \frac{(2\alpha+3)}{2} M \right), \ \text{say}$$

Now

$$L = \sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+1)/2} \log k}$$

$$L = O\left(\sum_{k=2}^{n} \frac{P_k}{k^{(2\alpha+3)/2} \log k} \right)$$

and

$$M \leq \sum_{k=1}^{n-1} P_k \int_k^{k+1} \frac{x^{-(2\alpha+3)/2}}{\log x}\, dx$$

$$= O\left(\sum_{k=1}^{n-1} \frac{P_k}{k^{(2\alpha+3)/2} \log k} \right) \qquad \text{by using (3.3).}$$

Hence

$$|I_{2.1.2}| \leq \frac{K\varepsilon}{(p*q)_n^c} = o(1), \text{ as } n \to \infty. \tag{5.9}$$

From (5.7),(5.8) and (5.9) it follows that

$$I_{2.1} = o(1), \text{ as } n \to \infty. \tag{5.10}$$

Next, considering $I_{2.2}$, we have

$$|I_{2.2}| \leq K\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} |F(\Phi)|\, \Phi^{-(2\alpha+5)/2}\, d\Phi$$

$$= n^{(2\alpha-1)/2} \left\{ K\, [F_1(\Phi)\, \Phi^{-(2\alpha+5)/2}]_{1/n}^{\eta} + K \int_{1/n}^{\eta} F_1(\Phi)\, \Phi^{-(2\alpha+7)/2}\, d\Phi \right\}$$

$$|I_{2.2}| \leq I_{2.2.1} + I_{2.2.2}, \qquad \text{(say)} \tag{5.11}$$

Hence

$$|I_{2.2.1}| = K(\eta)\, n^{(2\alpha-1)/2} + o(1)$$

$$|I_{2.1.1}| = o(1),, \text{ as } n \to \infty. \tag{5.12}$$

$$|I_{2.2.2}| \leq K\varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{\Phi^{(2\alpha+2)/2}\, \Psi_{(1/\Phi)}\, \Phi^{-(2\alpha+7)/2}}{\lambda\, (P_{|1/\Phi|})}\, d\Phi$$

$$= K\varepsilon\, n^{(2\alpha-1)/2} \int_{1/n}^{\eta} \frac{x^{-(2\alpha+1)/2}}{\log x}\, dx$$

$$|I_{2.2.2}| \leq K\varepsilon\,; \text{ since } \alpha < 1/2. \tag{5.13}$$

Hence from (5.11), (5.12) and (5.13), it follows that

$$I_{2.2} = o(1), \text{ as } n \to \infty. \tag{5.14}$$

Thus from (5.6),(5.10) and (5.14), we have

$$I_2 \;=\; o(1)\,, \quad \text{as } n \to \infty\,. \tag{5.15}$$

considering I_3, we have

$$I_3 \;= O\left(\frac{(p^*q)_n\, n^{(2\alpha+1)/2}}{(p^*q)_n}\right) \int_{\eta}^{\pi-1/n} |\,F(\Phi)\,|\,(\sin\frac{\Phi}{2})^{-(2\alpha+3)/2}(\cos\frac{\Phi}{2})^{-(2\alpha+1)/2}$$
$$P_{[1/\Phi]}\;\; d\Phi + \int_{\eta}^{\pi-1/n} |\,F(\Phi)\,|\,(\sin\frac{\Phi}{2})^{-(2\alpha+5)/2}(\cos\frac{\Phi}{2})^{-(2\alpha+3)/2}$$

$$I_3 \;=\; I_{3.1} + I_{3.2}\,, \qquad \text{say} \tag{5.16}$$

since $(\sin\frac{\Phi}{2})^{-(2\alpha+3)/2}$ is bounded for $\eta \le \Phi \le \pi$ and since $P_{[1/\Phi]}$ is bounded and

$-\beta-1/2 > -\beta-\alpha-1$, we have

$$I_{3.1} \;=\; O\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n}\right) \int_{\eta}^{\pi-1/n} |\,F(\Phi)\,| \;\; (\cos\frac{\Phi}{2})^{-\alpha-\beta-1}\; d\Phi$$
$$=\; O\left(\frac{q_n\, n^{(2\alpha+1)/2}}{(p^*q)_n}\right) \qquad \text{by Lemma 2.}$$

Hence

$$I_{3.1} \;=\; o(1)\,, \quad \text{as } n \to \infty\,. \tag{5.17}$$

Again

$$I_{3.1} \;=\; O(n^{(2\alpha-1)/2}) \left(\int_{\eta}^{\acute{\eta}} + \int_{\acute{\eta}}^{\pi-1/n}\right), \;\text{say} \tag{5.18}$$

Given $\acute{\varepsilon} > 0$, we can choose $\acute{\eta}$ so that

$$\int_{\eta}^{\pi} (\cos\frac{\Phi}{2})^{-\alpha-\beta-1} \;|\,F(\Phi)\,|\;\; d\Phi \;\;\le\;\; \acute{\varepsilon}\,.$$

The contribution of $I_{3.2}$ of the range $(\acute{\eta}, \pi-1/n)$ is

$$\le\; K\, n^{(2\alpha-1)/2} \int_{\acute{\eta}}^{\pi-1/n} |F(\Phi)|\;\; (\cos\frac{\Phi}{2})^{-(2\beta+3)/2}\; d\Phi$$
$$\le\; K\, n^{(2\alpha-1)/2} \int_{\acute{\eta}}^{\pi-1/n} |F(\Phi)|\;\; (\cos\frac{\Phi}{2})^{-\alpha-\beta-1} (\cos\frac{\Phi}{2})^{-(2\alpha-1)/2}\; d\Phi$$
$$\le\; K\;\acute{\varepsilon}\;\;; \tag{5.19}$$

since, in the range considered,

$$(\cos \frac{\Phi}{2})^{-(2\alpha-1)/2} = O\left(\frac{1}{n^{(2\alpha-1)/2}}\right) .$$

Thus the lim sup of the contribution of this range can be made arbitrarily small by suitable choice of $\acute{\varepsilon}$. Thus it is enough to prove that for fixed $\acute{\eta}$, the contribution in the range $(\eta,\acute{\eta})$ is zero.

For fixed $\acute{\eta}$

$$\int_{\eta}^{\acute{\eta}} |F(\Phi)| (\sin \frac{\Phi}{2})^{-(2\alpha+5)/2} (\cos \frac{\Phi}{2})^{-(2\beta+3)/2} \, d\Phi$$

is a constant, so that contribution

$$= O[n^{(2\alpha-1)/2}]$$

$$= o(1) \text{ as } n \to \infty . \qquad \text{for } \alpha < 1/2 \tag{5.20}$$

From (5.18), (5.19) and (5.20), it follows that

$$I_{3.2} = o(1) \text{ as } n \to \infty . \tag{5.21}$$

Hence from (5.16), (5.17) and (5.21), it is obtained that

$$I_3 = o(1) \text{ as } n \to \infty . \tag{5.22}$$

Finally, considering I_4 , we see that

$$I_4 = o(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |F(\Phi)| \, d\Phi ,$$

by (4.2) of Lemma 5.

But

$$n^{\alpha+\beta+1} = o[(\cos \frac{\Phi}{2})^{-\alpha-\beta-1}]$$

uniformly in $\pi-1/n \le \Phi \le \pi$; whence by the use of Lemma 2, it follows immediately that

$$I_4 = o(1) \text{ as } n \to \infty . \tag{5.23}$$

Collecting (5.5), (5.15), (5.22), and (5.23) the required result in (5.4) is established, which, in turn, proves the result in (5.3).

This completes the proof of the theprem.

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