The Islamic University Journal (Series of Natural Studies and Engineering)
Vol.14, No.1, P.63-72, 2006, ISSN 1726-6807, http//www.iugzaza.edu.ps/ara/research/

NONCONSECUTIVE MAGIC SQUARES 4x4[(1)]

Saleem Al-ashhab
Department of Mathematics,
Al al-bayt University,
Mafraq, Jordan
email: SALEEM_AL_ASHHAB@HOTMAIL.COM

received:1/9/2005, accepted: 5/10/2005

Abstract: We present a list of the number of magic 4x4 squares using sets of nonconsecutive integers and develop theory for this kind of squares.

Introduction

A 4x4 magic square is a square of 16 distinct integers, where the sum of the entries in all rows, columns and both diagonals is constant. This constant is called the magic sum. For example,

0	7	10	13
11	12	1	6
5	2	15	8
14	9	4	3

is a 4x4 magic square using the consecutive integers 0, ..., 15 with magic sum 30. A 4x4 Nasik (diabolic) square is a magic square having the additional property that the sum of the entries in the six (broken) off-diagonals is also the magic sum. The above square is diabolic since

(0) + (6 + 15 + 9) = (7 + 11) + (8 + 4) = (10 + 12 + 5) + (3) = 30

and

(13) + (11 + 2 + 4) = (10 + 6) + (5+9) = (7 + 1 + 8) + (14) = 30

We consider here the problem of counting 4x4 magic squares using a set of nonconsecutive integers. It is well known that there are 880 unique 4x4 magic squares using the consecutive integers 1, ..., 16 (see [6]). These 880 are generated by a smaller set of 220 magic squares by applying row and column transformations (see [3]). Each square belonging to this set will be called a fundamental magic square.

A set of 16 integers can not generate 4x4 magic squares unless the sum of its elements yields by division over 4 an integer, which represents the magic sum. All considered sets in this paper will satisfy this condition. We can assume without loss of generality that the smallest integer in any set is one. When we

---

([1]) This paper was Presented at the 2nd Islamic University Conference of Mathematics(2002)

consider the magic squares generated by a certain set, then some of these squares have the structure:

1	a	*	*
b	c	*	*
*	*	d	*
*	*	*	*

where a < b and c < d, and some of these squares have the structure:

a	1	*	*
c	b	*	*
*	*	*	d
*	*	*	*

where a < b and c < d. The squares, which fall into one of these two classes, are called the fundamental squares of this set. It is proven in [1], that the fundamental squares generate the whole set of magic squares. The number of fundamental magic squares of a set will be denoted by f, which refers to “fruitfulness” by generating magic squares of this set.

The dual set of a set of 16 integers is the set obtained by replacing each integer of the set by the value

largest integer + 1 - the integer.

A set is called symmetric (selfdual), if the set and its dual are identical. This means that the set after arranging its elements is split into two conjugate halves. The magic sum of a selfdual set is 2*(largest integer + 1). The value of f for a set and its dual are the same, since the number of squares generated by the set and its dual is the same. This is due to the fact that replacing each cell in a magic square by

largest integer + 1 - the cell

will transform a magic square using the integers of the set into a magic square using the integers of the dual and vice versa.

We use the notation {1, ... , 19} \ {6, 15, 17} to represent the set {1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19}. We use the graphical notation

* * * * * | * * * * * * * * | * | * *

to describe this set and call it the shape of the set. Hence, the previous note means that the orientation left/right of the shape does not affect f. The selfdual sets are in this sense the geometrically symmetric sets. Inserting between each two adjacent stars a constant number of dashes will not alter the value of f. This is due to the fact that we can transform the magic squares using the integers

$I + 1$, $I \in A$ (A is a set of 16 distinct integers including zero)

into magic squares using the integers

$n * I + 1$ (n is a positive integer) , $I \in A$

by executing the following steps:

1) Subtracting one from each cell,

2) multiplying each cell by n,
3) adding one to all cells.
Since these operations are reversible, we deduce that the number of magic squares using both sets is equal.

1. The list

We computed using the computer the value of f for different sets of integers and classified the list according to the obtained values of f:

f	**the set**
0	{1, ... , 19} \ {6, 15, 17}
1	{1, ... , 19} \ {8, 14, 16}
2	{1, ... , 19} \ {8, 11, 15}
3	{1, ... , 19} \ {5, 12, 13}
4	{1, ... , 19} \ {12, 13, 17}
5	{1, ... , 19} \ {13, 16, 17}
6	{1, ... , 18} \ {12, 15}
7	{1, ... , 18} \ {11, 16}
8	{1, ... , 19} \ {11, 14, 17}
9	{1, ... , 18} \ {10, 13}
10	{1, ... , 19} \ {8, 13, 17}
11	{1, ... , 19} \ {14, 15, 17}
12	{1, ... , 19} \ {12, 16, 18}
13	{1, ... , 18} \ {6, 17}
14	{1, ... , 18} \ {8, 15}
15	{1, ... , 19} \ {12, 14, 16}
16	{1, ... , 18} \ {14, 17}
17	{1, ... , 20} \ {9, 11, 12, 18}
18	{1, ... , 19} \ {7, 14, 17}
19	{1, ... , 18} \ {15, 16}
20	{1, ... , 20} \ {11, 13, 16, 18}
21	{1, ... , 20} \ {7, 12, 13, 18}
22	{1, ... , 22} \ {13, 14, 18, 19, 20, 21}
23	{1, ... , 20} \ {16, 17, 18, 19}
24	{1, ... , 18} \ {3, 16}
25	{1, ... , 19} \ {7, 15, 16}
26	{1, ... , 20} \ {8, 9, 12, 13}
27	{1, ... , 21} \ {2,15,16,18,20}
28	{1, ... , 18} \ {4, 15}
29	{1, ... , 20} \ {7, 15, 17, 19}
30	{1, ... , 20} \ {9, 15, 16, 18}
31	{1, ... , 21} \ {2, 3, 16, 18, 20}
32	{1, ... , 18} \ {8, 11}
34	{1, ... , 19} \ {4, 16, 18}
36	{1, ... , 18} \ {6, 13}
37	{1, ... , 23} \ {3, 11, 17, 18, 20, 21, 22}
38	{1, ... , 21} \ {5, 8, 11, 16, 19}

39 {1, ..., 24} \ {3, 6, 9, 10, 15, 18, 21, 22}
40 {1, ... , 20} \ {2, 6, 15, 19}
41 {1, ... , 20} \ {5, 16, 18, 19}
42 {1, ... , 20} \ {7, 10, 11, 18}
43 {1, ... , 19} \ {13, 15, 18}
44 {1, ... , 18} \ {7, 16}
45 {1, ... , 19} \ {5, 14, 15}
46 {1, ... , 20} \ {9, 14, 15, 16}
47 {1, ... , 19} \ {13, 14, 15}
48 {1, ..., 21} \ {11, 12, 15, 18, 19}
49 {1, ... , 19} \ {5, 12, 17}
50 {1, ... , 20} \ {15, 16, 17, 18}
51 {1, ... , 20} \ {3, 14, 18, 19}
52 {1, ... , 22} \ {7, 8, 11, 12, 19, 20}
53 {1, ... , 17} \ {13}
54 {1, ... , 18} \ {9, 14}
55 {1, ... , 22} \ {2, 9, 12, 14, 19, 21}
56 {1, ... , 20} \ {6, 9, 16, 19}
57 {1, ... , 20} \ {13, 14, 15, 16}
58 {1, ... , 19} \ {3, 17,18}
59 {1, ... , 19} \ {9, 11, 18}
60 {1, ... , 21} \ {9, 10, 11, 16, 17}
61 {1, ... , 22}\{2, 13, 15, 16, 18, 21}
62 {1, ... , 20} \ {10, 12, 17, 19}
63 {1, ... , 18} \ {13, 14}
64 {1, ... , 19} \ {7, 10, 13}
65 {1, ... , 24} \ {5, 10, 11, 12, 13, 18, 19, 20}
67 {1, ... , 22} \ {3, 14, 17, 18, 20, 21}
68 {1, ... , 19} \ {3, 10, 17}
69 {1, ... , 20} \ {5, 14, 15, 16}
71 {1, ... , 18} \ {10, 17}
72 {1, ... , 21} \ {2, 7, 11, 15, 20}
76 {1, ... , 21} \ {4, 9, 11, 13, 18}
132 {1, 3, 6, 7, 8, 9, 12, 14, 21, 23, 26, 27, 28, 29, 32, 34}
144 {1, 2, 4, 5, 9, 10, 12, 13, 19, 20, 22, 23, 27, 28, 30, 31}
146 {1, ... , 30} \ {2, 4, 5, 10, 11, 13, 15, 16, 18, 20, 21, 26, 27, 29}
148 {1, ... , 26} \ {5, 6, 11, 12, 13, 14, 15, 16, 21, 22}
156 {1, ... , 29} \ {2, 4, 5, 10, 11, 13, 15, 17, 19, 20, 25, 26, 28}
158 {1, ... , 27} \ {3, 6, 7, 8, 11, 14, 17, 20, 21, 22, 25}
160 {1, ... , 25} \ {5, 6, 11, 12, 13, 14, 15, 20, 21}
162 {1, ... , 24} \ {2, 9, 11, 12, 13, 14, 16, 23}
164 {1, ... , 22} \ {5, 6, 11, 12, 17, 18}
168 {1, ... , 23} \ {5, 10, 11, 12, 13, 14, 19}
170 {1, ... , 26} \ {3, 6, 7, 8, 11, 16, 19, 20, 21, 24}
172 {1, ... , 24} \ {3, 8, 11, 12 ,13, 14, 17, 22}
174 {1, ... , 22} \ {2, 9, 11, 12, 14, 21}

176 {1, ... , 26} \ {2, 3, 4, 6, 10, 17, 21, 23, 24, 25}
178 {1, ... , 20} \ {9, 10, 11, 12}
180 {1, ... , 22} \ {5, 10, 11, 12, 13, 18}
182 {1, ... , 24} \ {3, 6, 7, 10, 15, 18, 19, 22}
184 {1, ... , 22} \ {2, 4, 9, 14, 19, 21}
186 {1, ... , 22} \ {5, 6, 7, 16, 17, 18}
188 {1, ... , 20} \ {5, 6, 15, 16}
190 {1, ... , 22} \ {2, 3, 6, 17, 20, 21}
192 {1, ... , 25} \ {2, 4, 5, 7, 13, 19, 21, 22, 24}
194 {1, ... , 20} \ {5, 10, 11, 16}
196 {1, ... , 20} \ {2, 9, 12, 19}
198 {1, ... , 18} \ {9, 10}
200 {1, ... , 20} \ {3, 8, 13, 18}
202 {1, ... , 20} \ {3, 4, 17, 18}
204 {1, ... , 20} \ {2, 4, 17, 19}
206 {1, ... , 19} \ {9, 10, 11}
208 {1, ... , 19} \ {5, 10, 15}
212 {1, ... , 22} \ {2, 3, 5, 18, 20, 21}
218 {1, ... , 18} \ {2, 17}
220 {1, ... , 16}
260 {1, ... , 17} \ {9}

It is easy to note that f > 71 corresponds to symmetric sets, only. Further, we note that all these values of f are even. This was the case for all symmetric sets during our experiments. Thus, we conjecture that f is even for any symmetric set. Note that the maximal value of f is 260. It corresponds to the symmetric set with the shape

* * * * * * * * | * * * * * * * *.

The maximal value of odd f 's corresponds to the asymmetric set with the shape

* * * * * * * * * * | * * * * * | *.

It seems to be according to our experiments that if we insert more dashes randomly the value of f will get smaller. In the list we summarized the result of searching for f in sets having at most 12 dashes and symmetric sets having at most 24 dashes. There are still “missing” values of f, but we do not think that we can find any new values. Specifically, we conjecture that 260 is the maximal possible value of f.

2. Balanced squares

We consider the set {1, ..., 16} and denote each of first eight integers 1,..., 8 with S (res. the last eight integers 9, ..., 16 with L), which stands for small (res. large). We find that we have amongst the 220 fundamental squares 178 squares having structures similar to

L	S	S	L
S	L	L	S

L	S	S	L
S	L	L	S

This square has in each row/column and both diagonals exactly two S's and two L's. Squares having this property will be called balanced. The set {1, ..., 8, 10, ... , 17} has also amongst its fundamental squares 178 balanced squares. It is easy to see that they are obtainable from the balanced squares of the set {1, ... , 16} by adding one to all large integers. If we add two instead of one, then the squares remain balanced and represent the balanced squares of the set {1, ... , 8, 11, ... , 18}. We found that the set of fundamental squares for the integers {1, ... , 8, 13, ... , 20} consists entirely of balanced squares. The gap between the small and large integers is now big enough to prevent the existence of magic squares having three small integers in a row and three large integers in another row with the same sum- namely, the magic sum. Hence, the sets

$$\{1 , ... , 8 , m , ... , m + 7\} \text{ with } m > 12$$

have f = 178. These 178 fundamental squares are obtainable from the balanced squares of the set {1, ... , 16} by adding a suitable integer to all eight large integers.

We have verified that the number of balanced squares for the other sets in the list is not greater than 178. The number 178, which is the number of balanced squares corresponding to {1, ... , 8, 10, ... , 17} (the set with maximal f), seems to be the maximal number of balanced squares generated by any set.

The value of f for the set {1, ... , 12, 18, 19, 20, 21} is 44. All 44 fundamental squares are balanced in the sense that the four large integers 18, 19, 20 and 21 are distributed in the square in such a manner that each row/column and diagonal contains exactly one of these large integers. Adding one to all four large integers yields to the fundamental squares of the set {1, ... , 12, 19, 20, 21, 22}. Thus, f is 44 for all sets of the structure

$$\{1 , ... , 12 , m , ... , m + 3\} \text{ with } m > 17$$

We verified that the set {1, ... , 15, 28} can not generate any magic squares. This means that it is impossible to insert the large integer 28 into a row such that it has the same sum as the three other rows containing only small integers. With the same reasoning we see that the sets of the structure

$$\{1, ... , 15, 4m\} \text{ with } m > 6$$

have f = 0. Similarly, we verified f = 0 for the set {1, ... , 14, 23, 24} because we can not balance a 4x4 square using only two large integers. Hence, the sets

$$\{1, ... , 14, 2m - 1, 2m\} \text{ with } m > 11$$

have f = 0.

3. Complete balanced squares

If we consider the 178 balanced squares of the set {1, … , 16}, we find a subset of 40 squares having a structure like

S	L	H	S
H	S	S	L
S	H	L	S
L	S	S	H

where L now denotes the integers 9, 10, 11, 12 and H denotes the integers 13, 14, 15, 16 (H stands for huge). We call these squares complete balanced. Hence, complete balanced squares have the property that each row/column and diagonal contains exactly one L and one H. A magic square using any set of 16 increasingly ordered integers a1, …, a16 is complete balanced if it has this property, where L stands for the integers a9, …, a12 and H stands for the integers a13, …, a16. The set {1, … ,8, 13, … ,16, 22, … , 25} has f = 40. We found that these 40 fundamental squares are all complete balanced squares. Also, the set {1, … , 8, 13, … , 16, 23, … , 26} generates 40 fundamental squares, which are all complete balanced squares. The relation between these squares and the complete balanced squares of the set {1, … , 16} is that they are obtainable from the complete balanced squares of the set {1, … , 16} by adding 4 to the large integers and 7 (res. 8) to the huge integers. Adding more to the large or huge integers independently, keeping the difference over 5 between them, will not affect the property of being complete balanced. In this way we obtain actually the fundamental squares of the sets

$$\{1, \dots , 8, \ m + 4 , \dots , m + 7 , m + n + 7, \dots , m + n + 10\}$$

with $m > 8$ and $n > 5$.

Hence, f is 40 for these sets. If we choose for example m = 20 and n = 50, then we can interrupt this result in the following manner: The integers of the set {1, …, 8, 24, …, 27, 79, …, 82} can not generate a magic square unless they balance themselves in a complete balanced square. Since, there are only 40 fundamental complete squares, we get f = 40 for this set.

Now, the set {1, … , 8, 13, … , 16, 22, … , 25} generates only complete balanced squares. By subtracting one from the huge integers we obtain the complete balanced squares of the set {1, … , 8, 13, … , 16, 21, … , 24}. Hence, for this set we have f >= 40. Actually, f is 53 for this set. Thus, there are fundamental squares using these integers, which are not complete balanced. For example, one of these squares is

14	1	7	24
4	21	15	6
23	2	8	13
5	22	16	3

It is just a balanced square. By adding one to the largest eight integers in all 53 fundamental squares, we obtain the 53 fundamental squares of the set {1, … , 8, 14, … , 17, 22, … , 25}. Similarly, we obtain f = 53 for the sets

{1, … , 8, m + 6, … , m + 9, m + 14, … , m + 17} with m > 8.
Similarly, we found that
f = 42 for {1, … , 8, m + 4, … , m + 7, m + 11, … , m + 14} with m > 8,
f = 57 for {1, … , 8, m + 4, … , m + 7, m + 10, … , m + 13} with m > 8,
f = 47 for {1, … , 8, m + 4, … , m + 7, m + 9, … , m + 12} with m > 8.

If we consider the set {1, … , 4, 8, … , 15, 24, … , 27}, then we have 4 small integers, 8 large integers and 4 huge integers. This set has f = 40 and the fundamental squares are complete balanced, where S plays the role of L and vice versa in this case. Hence, the sets

* * * * | | ... | * * * * * * * * | | ... | * * * * ,

where the number of dashes in the second group is greater than the same number in the first group by at least 5 and the first group has at least 3 dashes, have f = 40.

If the number of dashes is equal we get f = 174. For example, the set {1, …, 4, 9, … , 16, 21, … , 24} has f = 174. All these 174 fundamental squares are balanced in the sense that: Replacing S by 0 , L by 1 and H by 2 in each square yields to a new magic square with magic sum 4 like

0	1	1	2
2	1	1	0
2	1	1	0
0	1	1	2

Now, by adding one to all entries of a fundamental square it remains a magic square. Then, by subtracting one from the 4 small integers and adding one to all 4 huge integers it becomes due to its structure a fundamental square of the set {1, …, 4, 10, …, 17, 23, …, 26}. Hence, the sets

{1, … , 4, m , …, m + 7, 2m + 3 , …, 2m + 6} with m > 8

have f = 174.

We can use similar techniques for the sets

{1, 2, m, … , m + 11, 2m + 9, 2m + 10} with m > 11.

They have f = 8 and the fundamental squares in this case transform now into squares like

1	0	2	1
1	1	1	1
1	1	1	1
1	2	0	1

But, if the difference between the small and the large integers is different than the difference between the large and huge integers, then no balance can be reached. Hence, we get f = 0 for the sets

{1 , 2 , m , … , m + 11, 2m + 2n + 9 , 2m + 2n + 10} with m > 6 , n > 0.

We consider the set {1, … , 23} \ {3, 6, 9, 12, 15, 18, 21}. It generates 178 fundamental squares balanced in the sense that replacing each integer by its truncation over 3 yields to a magic square using the integers 0, 1, ..., 7 with magic sum 14. If we add to a fundamental square the truncation square, then we obtain a fundamental square of the set

{1, 2, 5, 6, 9, 10, 13, 14, 17, 18, 21, 22, 25, 26, 29, 30}.

Thus, this set has f = 178. By adding a suitable multiple of the truncation square we can generate the fundamental squares of the sets

* * | …| * * | …| * * | …| * * | …| * * | …| * * | …| * * | …| * * ,

where the number of dashes separating the stars is equal. Hence, these shapes have f = 178. We can in a similar way conclude that f for the shapes

* * * * | …| * * * * | …| * * * * | …| * * * *

is the same f of this shape

* * * * | | * * * * | | * * * * | | * * * * ,

which is 164.

4. Symmetric sets

As we mentioned previously we obtained even values of f for all symmetric sets. But, this is not due to a possible transformation amongst the fundamental squares generating half of the fundamental squares from the other halve. For the set {1, ... , 20} \ {9, 10, 11, 12} we had f = 178. These 178 fundamental squares consist of 81 squares of the first class and 97 squares of the second class. Since these numbers are odd, this excludes the possibility of the existence of such a transformation in general. Yet, such a transformation exists in certain cases. For example, the symmetric set {1, ..., 24}\{2, 3, 4, 11, 14, 21, 22, 23} has f = 4. The four fundamental squares are balanced and have the following structure:

e	1	s-1	s-e
d	s-a	a	s-d
s-f	c	s-c	f
s-g	s-b	b	g

where d < f and 2s = the magic sum. Here the small letters represent the small integers. By interchanging the cells in position (2,1) and (3,1) (res. (2,4) and (3,4)) we transform a fundamental square into another fundamental square. Thus, two fundamental squares generate the four fundamental squares.

L. Sallows wrote the Dudeney's analysis of the 1040 magic squares of the set {1, ... , 17}/{9} into the same twelve graphic types as the 880 squares of the set {1, ... , 16}. We present both classifications in the following table:

Type		
I	48	48
II	48	48
III	48	48

IV	96	96
V	96	96
VI	304	480
VII	56	52
VIII	56	52
IX	56	52
X	56	52
XI	8	8
XII	8	8
total	880	1040

Note that type VI is increased while types VII - X decreased. Type I remains unchanged since it represents the Nasik squares. From [5] it is known that there are exactly 48 Nasik 4x4 squares using any set of integers. It is further known that the Nasik 4x4 squares have the structure

a	b	c	d
e	f	g	h
s-c	s-d	s-a	s-b
s-g	s-h	s-e	s-f

where the magic sum is 2s. Thus, the integers of a Nasik square must form a symmetric set. Of course, not every symmetric set produces Nasik squares. For example, the set {1, … , 18}\ {3, 16} does not. The Nasik squares of the set {1, … , 16} can be generated from three squares using row/column transformations (see [4]), which are all balanced. Hence, by adding ones to the eight large integers, we obtain Nasik squares using the integers of each of the sets

{1, … , 8, m, … , m + 7} m > 8.

References

[1] S. Al-ashhab, The Theory of Magic Squares / programming and mathematically (Arabic), 2000.

[2] W. S. Andrews, Magic Squares and Cubes, 1908.

[3] M. Gardner, Mathematical Games, Scientific American, Jan. 1976; Vol. 234, No. 1

[4] A. Grogono, Web Site: www.grogono.com/~magic.

[5] B. Rosser, J. Walker, On The Transformation Group of Diabolic Magic Squares Of Order Four, Bulletin of the American Mathematical Society, 1938; Vol. XLIV.

[6] W. W. Rouse Ball, Mathematical Recreations and Essays, 1911.

[7] L. Sallows, The Lost Theorem, The Mathematical Intelligencer, Number 4 / 1997; Vol. 19, pp 51-54.