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Explicit Equations to Determine the Variances of Regression Coefficients of OLS and GLS Estimators In An Auto-Correlated Regression Models

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Abstract: We have derived explicit equations to determine the variances of the regression coefficients of ordinary least squares (OLS) and generalized least squares (GLS) estimators in regression models containing an auto-correlated disturbance term for any covariance matrix and design vectors. In addition, we have proved that scaling or shifting the design vector has no effect in the relative efficiency of the variance of GLS to that of OLS.

Keywords: Variance; Ordinary Least Squares; Generalized Least Squares; Efficiency.

ملخص: في هذا البحث تم اشتقاق معادلات رياضية محددة لحساب تباين معاملات الانحدار للمقدرين GLS و OLS في حالة وجود الارتباط الذاتي في نماذج الانحدار وذلك لأي مصفوفة تغاير وتصميم متجهات. بالإضافة تم إثبات أن تغير قياس أو إزاحة تصميم المتجهات لا يؤثر على الكفاءة النسبية لتباين مقدر GLS بالنسبة لتباين مقدر OLS.

1 - Introduction

Let the relationship between the response variable Y and k explanatory variables $X_1, X_2, \ldots, X_k$ in a T-finite system be specified in the following linear regression model

$$\mathbf{Y} = \mathbf{X}\,\beta + U \tag{1.1}$$

where $\mathbf{Y}$ is a $(T \times 1)$ vector of observations on a dependent variable, $\mathbf{X}$ is a $(T \times k)$ design matrix, β is a $(k \times 1)$ vector of unknown regression parameters, and $\mathbf{U}$ is a $(T \times 1)$ random vector of disturbances. For convenience we assume that X is full column rank $k < T$ and its first column is 1's.

In the classical linear regression model it is standard to assume that given any value of the explanatory variables the disturbances are uncorrelated with zero mean and constant variance. Under these general assumptions it can be shown that ordinary least squares (OLS) estimators are optimal.

The OLS estimator of β in the regression model (1.1) is

$$\hat{\beta} = \left(X^T X\right)^{-1} X^T Y \tag{1.2}$$

In problems concerning time series, it is often the case that the disturbances are, in fact, correlated. If the disturbance term has mean zero, i.e. $E\,(\mathbf{U}) = \mathbf{0}$, but is in fact, auto-correlated, i.e. $\mathrm{Cov}(\mathbf{U}) = \sigma_U^2\,\Sigma$, where Σ is a $T \times T$

positive definite matrix and the variance σ_U^2 is either known or unknown positive and finite scalar, then the OLS parameter estimates will continue to be unbiased, i.e. $E(\hat{\beta}) = \beta$. But it has a different covariance matrix;

$$\mathrm{Cov}\left(\hat{\beta}_{OLS}\right) = \sigma_U^2 \left(X^T X\right)^{-1} X^T \Sigma X \left(X^T X\right)^{-1} \tag{1.3}$$

The estimator $\hat{\beta}$ ignores the auto-correlated nature of disturbances and is therefore not efficient. This is accounted for in the generalized least squares (GLS) estimator given by

$$\tilde{\beta} = \left(X^T \Sigma^{-1} X\right)^{-1} X^T \Sigma^{-1} Y \tag{1.4}$$

which is unbiased, i.e. $E(\tilde{\beta}) = \beta$, with covariance matrix

$$\mathrm{Cov}\left(\tilde{\beta}_{GLS}\right) = \sigma_U^2 \left(X^T \Sigma^{-1} X\right)^{-1} \tag{1.5}$$

The relative efficiency of the OLS in a linear regression containing an auto-correlated disturbance term depends on the structure of the matrix of observations on the independent variables, i.e. it depends on a specific design which makes it difficult to characterize in general

There are numerous articles describing the efficiency of the OLS coefficient estimator, which ignores the correlation of the error, relative to the GLS estimator which takes this correlation into account.

Chipman et al. (1968) give a lower bound for the relative efficiency of the sample mean, that is, they considered the special case in which X consists of a column of ones. Ullah et al. (1983) derived the large sample asymptotic approximation of the variance-covariance matrix of the Prais-Winston (PW) estimator of the regression coefficient. Safi (2006) derived explicit formulas for the relative Efficiencies of the GLS estimator to that of OLS estimator in linear and quadratic design vectors in the presence of AR(1) disturbances with and without an intercept term included in the design.

In this paper we derive explicit equations for the variances of the regression coefficients of OLS and GLS estimators in regression models containing an auto-correlated disturbance term. The features of these explicit equations will, first and foremost, valid for any covariance matrix for the disturbance term, Σ and, second, can be used for any design vector, **X**. In addition, we prove that scaling or shifting the design vector has no effect in the relative efficiency of the variance of GLS to that of OLS. In other words, the relative efficiency is invariant to scaling and shifting of the design vectors. We consider a design vector with an intercept term included in the design. Finally, we offer some conclusion remarks and suggestions for future research on the explicit equations for the variances of the regression coefficients of OLS and GLS in regression models.

This article is organized as follows. In section 2, we derive explicit equations for the variance of the regression coefficients of OLS and GLS estimators in regression models containing an auto-correlated disturbance term for any covariance matrix and design vectors. Section 3 presents the relative efficiency of GLS to OLS. We prove that scaling or shifting the design vector has no effect in the relative efficiency of the variance of GLS to that of OLS. Numerical illustrations are considered in section 4. Finally, section 5 summarizes the results and offer some conclusion remarks and suggestions for future research on the explicit equations for the variances of the regression coefficients of OLS and GLS in regression models.

2- The Variances of OLS and GLS Estimators

In this section we derive explicit equations for the variances of the regression coefficients of OLS and GLS estimators. In general, the variance of an estimator in regression models containing an auto-correlated disturbance term depends on the structure of the matrix of observations on the independent variables, i.e. it depends on a specific design which makes it difficult to characterize in general. We consider a design vectors with an intercept term included in the design.

2.1 The Variance of OLS Estimator

Consider the simple linear trend

$$\mathbf{Y_t} = \mathbf{X}\,\boldsymbol{\beta} + \mathbf{U_t}\,,\ t = 1, 2, \ldots, T$$

Suppose $\mathbf{X}$ is a vector with an intercept term, i.e. $\mathbf{X}^{\mathrm{T}} = \begin{pmatrix} \mathbf{J}^{\mathrm{T}} \\ \mathbf{x}^{\mathrm{T}} \end{pmatrix}$ where $\mathbf{J}^{\mathrm{T}}$ is a vector of all one's, i.e. $\mathbf{J}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}$, $\mathbf{x^T} = \begin{pmatrix} x_1 & x_2 & \ldots & x_T \end{pmatrix}$. Let $\mathbf{X}^{*\mathrm{T}}$ be a scaled vector such that $\mathbf{X}^{*\mathrm{T}} = \begin{pmatrix} \mathbf{J}^{\mathrm{T}} \\ \mathrm{b}\mathbf{x}^{\mathrm{T}} \end{pmatrix}$, $\mathrm{b} \in \mathbb{R}, \mathrm{b} \neq 0$, $\mathrm{b}\mathbf{x}^{\mathrm{T}} = \begin{pmatrix} \mathrm{b}x_1 & \mathrm{b}x_2 & \ldots & \mathrm{b}x_T \end{pmatrix}$.

We derive the variances of OLS for an intercept and the slope of the regression coefficients for both design vectors, $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$.

Theorem 1 Suppose $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$ are the design vectors in the simple linear trend as described in the above notations. Let $\mathrm{Var}\left(\hat{\beta}_{0.\mathrm{OLS}}\right)$, $\mathrm{Var}\left(\hat{\beta}^{*}_{0.\mathrm{OLS}}\right)$, $\mathrm{Var}\left(\hat{\beta}_{1.\mathrm{OLS}}\right)$, and $\mathrm{Var}\left(\hat{\beta}^{*}_{1.\mathrm{OLS}}\right)$ represent the variances of OLS for an intercept and the slope of the regression coefficients for the design vectors $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$, respectively. Then

(a) $\mathrm{Var}\left(\hat{\beta}_{0.\mathrm{OLS}}\right)=\mathrm{Var}\left(\hat{\beta}^{*}_{0.\mathrm{OLS}}\right)=\sigma_{\mathrm{U}}^{2}d^{2}a_{1}^{\mathrm{T}}\Sigma\, a_{1}$

(b) $\mathrm{Var}\left(\hat{\beta}_{1.\mathrm{OLS}}\right)=b^{2}\mathrm{Var}\left(\hat{\beta}^{*}_{1.\mathrm{OLS}}\right)=\sigma_{\mathrm{U}}^{2}d^{2}a_{2}^{\mathrm{T}}\Sigma\, a_{2}$

where σ_{U}^{2} is the disturbance variance, $d=\left(\|\mathbf{J}\|^{2}\|\mathbf{x}\|^{2}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)^{2}\right)^{-1}$, $a_{1}^{\mathrm{T}}=\|\mathbf{x}\|^{2}\mathbf{J}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{x}^{\mathrm{T}}$, and $a_{2}^{\mathrm{T}}=\|\mathbf{J}\|^{2}\mathbf{x}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{J}^{\mathrm{T}}$.

Proof

$$\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}=\frac{1}{\mathbf{J}^{\mathrm{T}}\mathbf{J}\,\mathbf{x}^{\mathrm{T}}\mathbf{x}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)^{2}}\begin{pmatrix}\mathbf{x}^{\mathrm{T}}\mathbf{x} & -\mathbf{J}^{\mathrm{T}}\mathbf{x}\\ -\mathbf{J}^{\mathrm{T}}\mathbf{x} & \mathbf{J}^{\mathrm{T}}\mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{J}^{\mathrm{T}}\\ \mathbf{x}^{\mathrm{T}}\end{pmatrix},$$

$$\begin{aligned}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}&=\frac{1}{\mathbf{J}^{\mathrm{T}}\mathbf{J}\,\mathbf{x}^{\mathrm{T}}\mathbf{x}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)^{2}}\begin{pmatrix}\mathbf{x}^{\mathrm{T}}\mathbf{x} & -\mathbf{J}^{\mathrm{T}}\mathbf{x}\\ -\mathbf{J}^{\mathrm{T}}\mathbf{x} & \mathbf{J}^{\mathrm{T}}\mathbf{J}\end{pmatrix}\begin{pmatrix}\mathbf{J}^{\mathrm{T}}\\ \mathbf{x}^{\mathrm{T}}\end{pmatrix}\\ &=\left(\|\mathbf{J}\|^{2}\|\mathbf{x}\|^{2}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)^{2}\right)^{-1}\begin{pmatrix}\|\mathbf{x}\|^{2}\mathbf{J}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{x}^{\mathrm{T}}\\ \|\mathbf{J}\|^{2}\mathbf{x}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{J}^{\mathrm{T}}\end{pmatrix}\\ &=d\times\begin{pmatrix}a_{1}^{\mathrm{T}}\\ a_{2}^{\mathrm{T}}\end{pmatrix}\end{aligned}$$

where

$$d=\left(\|\mathbf{J}\|^{2}\|\mathbf{x}\|^{2}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)^{2}\right)^{-1},\ a_{1}^{\mathrm{T}}=\|\mathbf{x}\|^{2}\mathbf{J}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{x}^{\mathrm{T}},\ a_{2}^{\mathrm{T}}=\|\mathbf{J}\|^{2}\mathbf{x}^{\mathrm{T}}-\left(\mathbf{J}^{\mathrm{T}}\mathbf{x}\right)\mathbf{J}^{\mathrm{T}}.$$

(2.1)

Recall the definition for the variance of OLS estimator in equation (1.3) and collecting terms. We find that

$$\begin{aligned}\mathrm{Cov}\left(\hat{\beta}_{\mathrm{OLS}}\right)&=\sigma_{\mathrm{U}}^{2}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\Sigma\,\mathbf{X}\left(\mathbf{X}^{\mathrm{T}}\mathbf{X}\right)^{-1}\\ &=\sigma_{\mathrm{U}}^{2}d^{2}\begin{pmatrix}a_{1}^{\mathrm{T}}\\ a_{2}^{\mathrm{T}}\end{pmatrix}\Sigma\left(a_{1}\quad a_{2}\right)\\ &=\sigma_{\mathrm{U}}^{2}d^{2}\begin{pmatrix}a_{1}^{\mathrm{T}}\Sigma\, a_{1} & a_{1}^{\mathrm{T}}\Sigma\, a_{2}\\ a_{2}^{\mathrm{T}}\Sigma\, a_{1} & a_{2}^{\mathrm{T}}\Sigma\, a_{2}\end{pmatrix}\end{aligned}$$

Then the variances of an intercept and the slope for the OLS estimator, respectively, are

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$$\mathrm{Var}\left(\hat{\beta}_{0.\mathrm{OLS}}\right)=\sigma_U^2 d^2 a_1^T \Sigma\, a_1, \tag{2.2}$$

$$\mathrm{Var}\left(\hat{\beta}_{1.\mathrm{OLS}}\right)=\sigma_U^2 d^2 a_2^T \Sigma\, a_2 \tag{2.3}$$

where d, a_1^T, a_2^T are defined in (2.1).

Similarly, for the scaled design vector $\mathbf{X}^{*^T}$,

$$\left(\mathbf{X}^{*T}\mathbf{X}^{*}\right)^{-1}\mathbf{X}^{*T}=\left[b^2\left(\|\mathbf{J}\|^2\|\mathbf{x}\|^2-\left(\mathbf{J}^T\mathbf{x}\right)^2\right)\right]^{-1}\begin{pmatrix} b^2\|\mathbf{x}\|^2\mathbf{J}^T-b^2\left(\mathbf{J}^T\mathbf{x}\right)\mathbf{x}^T \\ b\|\mathbf{J}\|^2\mathbf{x}^T-b\left(\mathbf{J}^T\mathbf{x}\right)\mathbf{J}^T \end{pmatrix}$$

$$=b^{-2}d\times\begin{pmatrix} b^2 a_1^T \\ b\, a_2^T \end{pmatrix}$$

$$=d\times\begin{pmatrix} a_1^T \\ b^{-1} a_2^T \end{pmatrix}$$

Recall the definition for the variance of OLS estimator in equation (1.3) and collecting terms. We find that

$$\mathrm{Cov}\left(\hat{\beta}^*_{\mathrm{OLS}}\right)=\sigma_U^2\left(\mathbf{X}^{*T}\mathbf{X}^{*}\right)^{-1}\mathbf{X}^{*T}\,\Sigma\;\mathbf{X}^{*}\left(\mathbf{X}^{*T}\mathbf{X}^{*}\right)^{-1}$$

$$=\sigma_U^2 d^2\begin{pmatrix} a_1^T \\ b^{-1}a_2^T \end{pmatrix}\Sigma\left(a_1 \quad b^{-1}a_2\right)$$

$$=\sigma_U^2 d^2\begin{pmatrix} a_1^T\Sigma\, a_1 & b^{-1}a_1^T\Sigma\, a_2 \\ b^{-1}a_2^T\Sigma\, a_1 & b^{-2}a_2^T\Sigma\, a_2 \end{pmatrix}$$

Then the variances of an intercept and the slope for the OLS estimator for the scaled design vector, respectively, are

$$\mathrm{Var}\left(\hat{\beta}^*_{0.\mathrm{OLS}}\right)=\sigma_U^2 d^2 a_1^T \Sigma\, a_1, \tag{2.4}$$

$$\mathrm{Var}\left(\hat{\beta}^*_{1.\mathrm{OLS}}\right)=b^{-2}\sigma_U^2 d^2 a_2^T \Sigma\, a_2 \tag{2.5}$$

From equations (2.2) through (2.5), we get

$$\mathrm{Var}\left(\hat{\beta}_{0.\mathrm{OLS}}\right)=\mathrm{Var}\left(\hat{\beta}^*_{0.\mathrm{OLS}}\right)=\sigma_U^2 d^2 a_1^T \Sigma\, a_1$$

$$\mathrm{Var}\left(\hat{\beta}_{1.\mathrm{OLS}}\right)=b^2\,\mathrm{Var}\left(\hat{\beta}^*_{1.\mathrm{OLS}}\right)=\sigma_U^2 d^2 a_2^T \Sigma\, a_2 \qquad \blacksquare$$

2.2 The Variance of GLS Estimator

We derive the variances of GLS for an intercept and the slope of the regression coefficients for both design vectors, $\mathbf{X}^T$ and $\mathbf{X}^{*^T}$.

Theorem 2 Suppose $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$ are the design vectors in the simple linear trend as described in the above notations. Let $\operatorname{Var}\left(\tilde{\beta}_{0.\mathrm{GLS}}\right)$, $\operatorname{Var}\left(\tilde{\beta}^{*}_{0.\mathrm{GLS}}\right)$, $\operatorname{Var}\left(\tilde{\beta}_{1.\mathrm{GLS}}\right)$, and $\operatorname{Var}\left(\tilde{\beta}^{*}_{1.\mathrm{GLS}}\right)$ represent the variances of GLS for an intercept and the slope of the regression coefficients for the design vectors $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$, respectively. Then

(a) $\operatorname{Var}\left(\tilde{\beta}_{0.\mathrm{GLS}}\right)=\operatorname{Var}\left(\tilde{\beta}^{*}_{0.\mathrm{GLS}}\right)=\sigma_{\mathrm{U}}^{2}\mathrm{g}\|\mathbf{x}\|^{2}\,\Sigma^{-1}$

(b) $\operatorname{Var}\left(\tilde{\beta}_{1.\mathrm{GLS}}\right)=\mathrm{b}^{2}\operatorname{Var}\left(\tilde{\beta}^{*}_{1.\mathrm{GLS}}\right)=\sigma_{\mathrm{U}}^{2}\mathrm{g}\|\mathbf{J}\|^{2}\,\Sigma^{-1}$

where σ_{U}^{2} is the disturbance variance,

$$\mathrm{g}=\left(\|\mathbf{J}\|^{2}\,\Sigma^{-1}\|\mathbf{x}\|^{2}\,\Sigma^{-1}-(\mathbf{J},\mathbf{x})^{2}_{\Sigma^{-1}}\right)^{-1},(\mathbf{J},\mathbf{x})_{\Sigma^{-1}}=\mathbf{J}^{\mathrm{T}}\Sigma^{-1}\mathbf{x}$$

Proof

$$\begin{aligned}
\left(\mathbf{X}^{\mathrm{T}}\Sigma^{-1}\mathbf{X}\right)^{-1}&=\left(\begin{pmatrix}\mathbf{J}^{\mathrm{T}}\\ \mathbf{x}^{\mathrm{T}}\end{pmatrix}\Sigma^{-1}\begin{pmatrix}\mathbf{J} & \mathbf{x}\end{pmatrix}\right)^{-1}\\
&=\begin{pmatrix}\mathbf{J}^{\mathrm{T}}\Sigma^{-1}\mathbf{J} & \mathbf{J}^{\mathrm{T}}\Sigma^{-1}\mathbf{x}\\ \mathbf{x}^{\mathrm{T}}\Sigma^{-1}\mathbf{J} & \mathbf{x}^{\mathrm{T}}\Sigma^{-1}\mathbf{x}\end{pmatrix}^{-1}\\
&=\begin{pmatrix}\|\mathbf{J}\|^{2}\,\Sigma^{-1} & (\mathbf{J},\mathbf{x})_{\Sigma^{-1}}\\ (\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & \|\mathbf{x}\|^{2}\,\Sigma^{-1}\end{pmatrix}^{-1}\\
&=\left(\|\mathbf{J}\|^{2}\,\Sigma^{-1}\|\mathbf{x}\|^{2}\,\Sigma^{-1}-(\mathbf{J},\mathbf{x})^{2}_{\Sigma^{-1}}\right)^{-1}\begin{pmatrix}\|\mathbf{x}\|^{2}\,\Sigma^{-1} & -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}}\\ -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & \|\mathbf{J}\|^{2}\,\Sigma^{-1}\end{pmatrix}\\
&=\mathrm{g}\times\begin{pmatrix}\|\mathbf{x}\|^{2}\,\Sigma^{-1} & -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}}\\ -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & \|\mathbf{J}\|^{2}\,\Sigma^{-1}\end{pmatrix}
\end{aligned}$$

where

$$\mathrm{g}=\left(\|\mathbf{J}\|^{2}\,\Sigma^{-1}\|\mathbf{x}\|^{2}\,\Sigma^{-1}-(\mathbf{J},\mathbf{x})^{2}_{\Sigma^{-1}}\right)^{-1},\ (\mathbf{J},\mathbf{x})_{\Sigma^{-1}}=\mathbf{J}^{\mathrm{T}}\Sigma^{-1}\mathbf{x} \qquad (2.6)$$

Recall the definition for the variance of GLS estimator in equation (1.5) and collecting terms. We find that

$$\operatorname{Cov}\left(\tilde{\beta}_{\mathrm{GLS}}\right)=\sigma_{\mathbf{U}}^{2}\left(\mathbf{X}^{\mathrm{T}}\,\Sigma^{-1}\mathbf{X}\right)^{-1}$$

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$$= \sigma_{\mathbf{U}}^2 \, g \times \begin{pmatrix} \|\mathbf{x}\|^2 \Sigma^{-1} & -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} \\ -(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & \|\mathbf{J}\|^2 \Sigma^{-1} \end{pmatrix}$$

Then the variances of an intercept and the slope for the GLS estimator, respectively, are

$$\operatorname{Var}\left(\tilde{\beta}_{0.\mathrm{GLS}}\right) = \sigma_{\mathrm{U}}^2 g \|\mathbf{x}\|^2 \Sigma^{-1}, \tag{2.7}$$

$$\operatorname{Var}\left(\tilde{\beta}_{1.\mathrm{GLS}}\right) = \sigma_{\mathrm{U}}^2 g \|\mathbf{J}\|^2 \Sigma^{-1} \tag{2.8}$$

where g is defined in (2.6).

Similarly, for the scaled design vector $\mathbf{X}^{*^{\mathrm{T}}}$,

$$\left(\mathbf{X}^{*\mathbf{T}}\Sigma^{-1}\mathbf{X}^{*}\right)^{-1} = g \times \begin{pmatrix} \|\mathbf{x}\|^2 \Sigma^{-1} & -b^{-1}(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} \\ -b^{-1}(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & b^{-2}\|\mathbf{J}\|^2 \Sigma^{-1} \end{pmatrix}$$

Recall the definition for the variance of GLS estimator in equation (1.5) and collecting terms. We find that

$$\mathbf{Cov}\left(\tilde{\beta}^{*}_{\mathbf{GLS}}\right) = \sigma_{\mathrm{U}}^2 \left(\mathbf{X}^{*\mathbf{T}}\Sigma^{-1}\mathbf{X}^{*}\right)^{-1}$$

$$= \sigma_{\mathrm{U}}^2 g \times \begin{pmatrix} \|\mathbf{x}\|^2 \Sigma^{-1} & -b^{-1}(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} \\ -b^{-1}(\mathbf{J},\mathbf{x})_{\Sigma^{-1}} & b^{-2}\|\mathbf{J}\|^2 \Sigma^{-1} \end{pmatrix}$$

Then the variances of an intercept and the slope for the OLS estimator for the scaled design vector, respectively, are

$$\operatorname{Var}\left(\tilde{\beta}^{*}_{0.\mathrm{GLS}}\right) = \sigma_{\mathrm{U}}^2 g \|\mathbf{x}\|^2 \Sigma^{-1}, \tag{2.9}$$

$$\operatorname{Var}\left(\tilde{\beta}^{*}_{1.\mathrm{GLS}}\right) = b^{-2}\sigma_{\mathrm{U}}^2 g \|\mathbf{J}\|^2 \Sigma^{-1} \tag{2.10}$$

From equations (2.7) through (2.10), we get

$$\operatorname{Var}\left(\tilde{\beta}_{0.\mathrm{GLS}}\right) = \operatorname{Var}\left(\tilde{\beta}^{*}_{0.\mathrm{GLS}}\right) = \sigma_{\mathrm{U}}^2 g \|\mathbf{x}\|^2 \Sigma^{-1}$$

$$\operatorname{Var}\left(\tilde{\beta}_{1.\mathrm{GLS}}\right) = b^2 \operatorname{Var}\left(\tilde{\beta}^{*}_{1.\mathrm{GLS}}\right) = \sigma_{\mathrm{U}}^2 g \|\mathbf{J}\|^2 \Sigma^{-1} \qquad \blacksquare$$

3- Relative Efficiency of GLS and OLS

Statisticians are often interested in the relative efficiency of different estimators when the underlying assumptions of least squares breakdown. In particular, we are interested in the relative efficiency of GLS to OLS when the disturbance vector, $\mathbf{U}$, has mean zero but the covariance of the disturbance vector $\mathbf{U}$ is not a scalar multiple of the identity matrix, $\mathbf{I}$, i.e. $\operatorname{Cov}(\mathbf{U}) = \sigma_{\mathrm{U}}^2 \Sigma \neq \sigma_{\mathrm{U}}^2 \mathbf{I}$.

Definition 1 (Relative Efficiency): Let $\mathrm{Var}\left(\hat{\beta}\right)$ and $\mathrm{Var}\left(\tilde{\beta}\right)$ represent the variances of the OLS and the GLS estimators, respectively. When the disturbance term is auto-correlated, then the relative efficiency of GLS to that of OLS, denoted by $\mathrm{EFF}\left(\beta\right)$, is defined as

$$\mathbf{EFF}\left(\beta\right)=\frac{\left(\mathbf{X}^{\mathrm{T}}\ \Sigma^{-1}\ \mathrm{X}\right)^{-1}}{\left(\mathbf{X}^{\mathrm{T}}\ \mathrm{X}\right)^{-1}\mathbf{X}^{\mathrm{T}}\ \Sigma\ \mathrm{X}\left(\mathbf{X}^{\mathrm{T}}\ \mathrm{X}\right)^{-1}} \tag{3.1}$$

(See for example Greene, 2000).

The relative efficiency in regression models containing an auto-correlated disturbance term depends on the structure of the matrix of observations on the independent variables, i.e. it depends on a specific design which makes it difficult to characterize in general. In addition, it depends also on the structure of the covariance matrix, Σ.

The previous derived equations are designed to compute the relative efficiencies of the variances of regression coefficients of GLS to that of OLS for design vectors $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$ when the disturbances are auto-correlated.

Corollary 1

Suppose $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$ are the design vectors in the simple linear trend as described in the above notations. Then the relative efficiencies of the variance of GLS to that of OLS of an intercept β_0 and the slope β_1 for both design vectors $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$ are given by

$$\left[\mathrm{EFF}\left(\beta_0\right)\right]_{\mathbf{X}^{\mathrm{T}}}=\left[\mathrm{EFF}\left(\beta_0\right)\right]_{\mathbf{X}^{*\mathrm{T}}}=\frac{\mathrm{g}\left\|\mathbf{x}\right\|^2\Sigma^{-1}}{\mathrm{d}^2\mathrm{a}_1^{\mathrm{T}}\ \Sigma\ \mathrm{a}_1}$$

$$\left[\mathrm{EFF}\left(\beta_1\right)\right]_{\mathbf{X}^{\mathrm{T}}}=\left[\mathrm{EFF}\left(\beta_1\right)\right]_{\mathbf{X}^{*\mathrm{T}}}=\frac{\mathrm{g}\left\|\mathbf{J}\right\|^2\Sigma^{-1}}{\mathrm{d}^2\mathrm{a}_2^{\mathrm{T}}\ \Sigma\ \mathrm{a}_2}$$

respectively, where $\mathrm{d}, \mathrm{a}_1^{\mathrm{T}}$, $\mathrm{a}_2^{\mathrm{T}}$ and g are given in (2.1) and (2.6).

Proof. The proof can be reached directly by applying equation (3.1) and using equations (2.2) through (2.5) and (2.7) through (2.10). ■

Thus scaling or shifting the design vector has no effect in the relative efficiency of the variance of GLS to that of OLS for both design vectors $\mathbf{X}^{\mathrm{T}}$ and $\mathbf{X}^{*\mathrm{T}}$. In other words, the relative efficiency is invariant to scaling and shifting of the design vectors.

4- Numerical Illustrations

In this section we present numerical illustrations using the derived equations in sections 2 and 3. We will focus on autoregressive error processes of order one and two. We compute the variances of OLS and GLS estimators and the relative efficiencies when the structure of the design vector, X, is linear. with an intercept term included in the design vector when the sample size n = 500.

Example 1:

For autoregressive error process of order one, AR(1), of the form

$$u_t = \rho u_{t-1} + \varepsilon_t,\ \varepsilon_t \sim \text{i.i.d. } N\left(0, \sigma_\varepsilon^2\right) \tag{4.1}$$

with first order autoregressive disturbance parameter, ρ, equals 0.9.

For intercept term, the variances of GLS and OLS estimators are 0.19305 and 0.19621, respectively. Thus, the relative efficiency of GLS estimator to that of OLS estimator for the intercept term is 0.98389. For slope term, the variances of GLS and OLS estimators are 0.18024 and 0.18903, respectively. Thus, the relative efficiency of GLS estimator to that of OLS estimator for the slope is 0.95350.

Example 2:

For autoregressive error process of order two, AR(2), of the form

$$u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t,\ \varepsilon_t \sim \text{i.i.d. } N\left(0, \sigma_\varepsilon^2\right) \tag{4.2}$$

with second order autoregressive disturbance parameter, $(\phi_1, \phi_2) = (1.7, -0.9)$.

For intercept term, the variances of GLS and OLS estimators are 0.05010 and 0.05095, respectively. Thus, the relative efficiency of GLS estimator to that of OLS estimator for the intercept term is 0.98327. For slope term, the variances of GLS and OLS estimators are 0.05040 and 0.05296, respectively. Thus, the relative efficiency of GLS estimator to that of OLS estimator for the slope is 0.95162.

5- Conclusion Remarks and Future Research

In this article, we have derived explicit equations for the variance of the regression coefficients of OLS and GLS estimators in regression models containing an auto-correlated disturbance term for any covariance matrix and design vectors. In addition, we have proved that scaling or shifting the design vector has no effect in the relative efficiency of the variance of GLS to that of OLS.

Safi (2006), derived explicit formulas to determine the efficiency of OLS in the presence of first order autoregressive disturbances; AR(1). Extend Safi (2006) results for different autoregressive error processes.

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