IUG Journal of Natural Studies
Peer-reviewed Journal of Islamic University-Gaza

ISSN 2790-1653

IUGNS
Vol 31, No 01, 2022, pp 14-28

Received on (12-04-2022) Accepted on (13-02-2023)

On the Geometry of the Flag Varieties of Orthogonal group and Symplectic group

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https://doi.org/10.33976/IUGNS.31.2/2023/2

Abstract:

The aim of this paper is to discuss an important discourse, as most literatures highlighted the dimension of partial flag in a certain manner, according to a fixed definition that does not include all possible cases. In the present paper the existence of two different cases of flags in the partial flag varieties of the groups $SO(n,\mathbb{C})$ and $SP(2n,\mathbb{C})$ is explained wholly. In addition to this we will proceed by calculating the dimensions for all the given cases after pertaining the signature of these flags that are studied.

Keywords:

flag varieties, partial flag varieties, orthogonal group, symplectic group, semisimple Lie group, homogeneous space, flag manifold.

1. Introduction

A flag manifold Z of a semisimple complex Lie group G is a (compact) projective algebraic variety Z where G acts transitively on Z. It is easy to prove that the G-action on Z is algebraic action and thus $Z = G/P$, where P is an algebraic subgroup of G as the stabilizer of a point called parabolic subgroup.

In other fashion $Z = G/P$ is called a flag variety, which is a homogeneous space whose points are flags in a finite dimensional vector space V over $\mathbb{C}$.

In fact flag varieties first appeared at the end of the 19th century in the papers of Hermann Günter Grassmann, Julius Plücker, Hermann Schubert and other mathematicians. Between 1987 and 1990, Lakshmibai had studied the geometry of the flag variety G/P of all classical groups in a series of papers, See [17],[18],[19], and [20]. In recent years, several papers had appeared which study flag varieties from various points of view (see [1], [2], [3], [4], [6], [7], [8], [9], [10], [12], [13], [14], [15] and [16]).

The variety of full flags in a vector space V over a field $\mathbb{C}$, which is a flag variety G/B for the special linear group $G = SL(n, \mathbb{C})$ over $\mathbb{C}$ and Borel subgroup B , has a specific form, signature and dimension. One can also consider homogeneous spaces G/B and G/P not only for $GL(V)$, but also for other connected reductive group. These kind of flag varieties arise by restriction from the special linear group $SL(n, \mathbb{C})$ to subgroups such as the orthogonal groups $SO(n, \mathbb{C})$ or symplectic group $SP(2n, \mathbb{C})$.

For the case of orthogonal and symplectic groups, additional conditions must be imposed on the flags in both cases full flag varieties and partial flag varieties. In order to study the flag varieties, we need to know the properties of flags and how can we write their signatures, and we need to know the dimensions of the flag varieties in these cases. Therefore, in this paper, we presented the method of writing the flags for each of the group $SO(n, \mathbb{C})$ and $SP(2n, \mathbb{C})$, and we found the dimensions of full flag varieties and partial flag varieties. In a section 3.2 we proved that there are two different cases of partial flag varieties and each of them has a different dimension.

2. Flag varieties

2.1. Flag variety via Dynkin diagram

Let G be a complex semisimple Lie group and let G/B be the full flag variety where B is a Borel subgroup. If P is a parabolic subgroup that contains B then G/P is called a partial flag variety (where G acts transitively).

Let $\mathfrak{g}$ be a lie algebra of the Lie group G. Define a maximal torus T of G with Lie algebra $\mathfrak{h}$. Let Δ denote the set of roots of $\mathfrak{h}$ and Δ^+, Δ^s denote the set of positive roots and simple roots respectively. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which defined the Borel subgroup B is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{b} = \mathfrak{h} \bigoplus \oplus_{\Delta^+} \mathfrak{g}_\alpha$ for some positive root system $\Delta^+$and a Cartan subalgebra $\mathfrak{h}$. Then a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ which defined the parabolic subgroup P is a subalgebra of $\mathfrak{g}$ that contains a Borel subalgebra $\mathfrak{b}$. This implies that, for a fixed Borel subalgebra $\mathfrak{b} = \mathfrak{h} \bigoplus \oplus_{\Delta^+} \mathfrak{g}_\alpha$ the parabolic subalgebra $\mathfrak{q}$ that contains $\mathfrak{b}$ is of the form $\mathfrak{q} = \mathfrak{h} \bigoplus \oplus_\eta \mathfrak{g}_\alpha$ with $\Delta^+ \subset \eta \subset \Delta$. Notice that, using I as a subset of the set of simple roots Δ^s of Δ^+, the condition of Δ^s implies that

$$\eta = \Delta^+ \cup \{\alpha \in \Delta : \alpha \in \text{span}\,(\Delta^s - I)\}.$$

Set $\Delta' = \{\alpha \in \Delta : \alpha \in \text{span}\,(\Delta^s - I)\}$ then it follows that

$$\eta = \Delta' \cup (\Delta^+ - \Delta')$$

This shows that parabolic subalgebra $\mathfrak{q}$ is of the form

$$\mathfrak{q} = h \oplus \bigoplus_{\alpha \in \Delta'} g_\alpha \oplus \bigoplus_{\alpha \in \Delta^+ - \Delta'} g_\alpha$$

There is a $1-1$ correspondence between the subset $I \subseteq \Delta^s$ and a parabolic subgroup containing a fixed Borel subgroup, i.e. there is a $1-1$ correspondence between the simple roots which define a parabolic subgroup and the nodes of the Dynkin diagram. More precisely, if $\mathfrak{p}$ defined by the set of simple roots $\Delta^s - I$, then $\mathfrak{p}$ is defined by removing the simple roots I from the Dynkin diagram. For more information see [5].

2.2. Full Flag varieties G/B

Let (G, b) be a semisimple complex Lie group which leaves invariant a non degenerate complex bilinear form $b(v, w)$ for $v, w \in V$ where V is a 2 n-dimensional complex vector space.

Definition 2.1. If $W \subset V$, we write $W^\perp = \{v \in V : b(v, w) = 0, \forall w \in W\}, W^\perp$ is called the orthogonal subspace of W with respect to b. And we call W isotropic if $W \subset W^\perp$.

Definition 2.2. [8] Let G be the orthogonal group $SO(m, \mathbb{C})$ or the symplectic group $SP(m, \mathbb{C})$. A maximally b-isotropic full flag F with respect to bilinear form b of G is a sequence of $m+1$ vector spaces

$$F = (\{0\} = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{C}^m)$$

such that $\dim V_i = i$, for all $0 \le i \le m$ and $V_{m-i} = V_i{}^{\perp}$, for all $1 \le i \le m$. This means that V_i for all $1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$ are isotropic subspaces.

In consonance with orthogonal and symplectic groups, let $V = U \bigoplus U^*$ where U is b isotropic and U^* is its dual with respect to b. Let $r_1, \dots, r_n$ be a basis of U and $s_1, \dots, s_n$ its dual basis, then $r_1, \dots, r_n, s_1, \dots, s_n$ is a basis of V. A flag F is called maximally isotropic flag in this case if and only if $V_i \subset V_i^{\perp}$ and $V_{2n-i}^{\perp} \subset V_{2n-i}$ $\forall 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor$. Consider the basis $r_i, s_i, 1 \le i \le n$ of eigenvectors of a maximal torus with distance eigenvalues. A space E which generated by n vectors of the basis r_i, s_i is isotropic if contains at most one and not both of the elements r_i, s_i for each i.

Assuming that we have a maximally isotropic flag associated to some permutation of the basis r_i, s_i, then in the first n positions of the flag the above condition holds: if r_i appears, then s_i does not, and vice versa. Now the full flag is determined by the first n positions; regard it as a permutation of the r_i and, if s_i occurs instead of an r_i, where that spot is marked with a minus sign . We denote the set of all maximally b-isotropic flag by Z_I, see ([9]). In fact Z_I is a complex manifold and it is called a flag variety of the Lie group G. The manifold structure of Z_I arises from the fact that the Lie group G acts transitively on it.

Example 2.3. Let $G = SO(2n, \mathbb{C})$ the special orthogonal group with non degenerate complex bilinear form $b: \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n}$ defined by

$$b(v, w) = \sum_{i=1}^{2n} v_{2n-i+1} w_i$$

and the Hermitian form

$$h(v, w) = \sum_{i=1}^{n} v_i \bar{w}_i - \sum_{i=n+1}^{2n} v_i \bar{w}_i$$

Consider the standard basis of $\mathbb{C}^{2n}$,

$e_1, e_2, \dots, e_{2n-1}, e_{2n}$

and denote $E_+ =< e_1, \dots, e_n >$, and $E_- =< e_{n+1}, \dots, e_{2n} >$, so $\mathbb{C}^{2n} = E_+ \bigoplus E_-$. Take $n = 3$, then the flag

$\{0\} \subset < e_1 > \subset < e_1, e_4 > \subset < e_1, e_4, e_2 > \subset < e_1, e_4, e_2, e_5 > \subset < e_1, e_4, e_2, e_5, e_3 > \subset \mathbb{C}^6$

is a maximally b-isotropic full flag in $SO(6, \mathbb{C})/B$.

In the case of flag variety G/B, where the Borel subgroup B is the stabilizer of a maximally b-isotropic flag in Z_I. The following statement is well-known. As a sake of convenience for the reader, the proof shall be reviewed.

Lemma 2.4. There exists a G-equivariant natural isomorphism between Z_I and G/B.

Proof. Let $gB \in G/B$, write g as a matrix with column vectors $g = (g_1; g_2; \dots; g_{2n})$ where $g_i \in \mathbb{C}^m$. Then define $\tau: G/B \longrightarrow Z_I$ to be

$$\tau(gB) = F =< g_1B >\subset< g_1B, g_2B >\subset \cdots \subset< g_1B, \dots, g_{2n}B >$$

where F is a flag associated to the ordered basis $g_1B, g_2B, \dots, g_{2n}B$.

2.3. The signature of flags in the Full Flag varieties

In this section we introduce the signature of the flags in the homogenous space $Z_I \cong G/B$ where G acts transitively. Let us consider the case of flags in the flag variety $Z_I \cong G/B$. Let $b(v, w)$ be the bilinear form of G with hermiation form $h(v, w)$. Fix a flag $F = (0 \subset V_1 \subset \cdots \subset V_m) \in Z_I$, define the following sequences:

$$a: 0 \le a_1 \le a_2 \le \cdots \le a_m$$
$$b: 0 \le b_1 \le b_2 \le \cdots \le b_m$$
$$d: 0 \le d_1 \le d_2 \le \cdots \le d_m$$

with $\text{sign}\,(V_i) = (a_i, b_i, d_i)$ with respect to h where a_i (resp. b_i) denotes the dimension of a maximal negative subspace and b_i denotes the dimension of a maximal positive subspace and d_i define the degeneracy of the restriction of h to V_i with $a_i + b_i + d_i = i \; \forall 1 \le i \le m$. The 3 -tuple (a, b, d) is called the signature of the flag F. If $d \ne 0, i.e. \exists d_i \ne 0$ we refer to F as being non-degenerate and write $\text{sign}\,(F) = (a, b)$.

Let us define a symbol parametrization of the flag $F \in G/B$ as follows: Consider a sequence of m-empty boxes, if $a_{i+1} = a_i + 1$ then place - in the box i, and if $b_{i+1} = b_i + 1$ then place $+$ in the box i.

Definition 2.5. Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ be a sequence of m signs. Define the transpose of α to be $\alpha^t = (\alpha_m, \dots, \alpha_2, \alpha_1)$. If $\alpha = -\alpha^t$ then α is called skew-symmetric and if $\alpha = \alpha^t$ then α is called symmetric.

Example 2.6. Let $\alpha_1 = + - + + - - + -$ and $\alpha_2 = + - + - - + - +$. Note that $\alpha_1 = -\alpha_1^t$ so α_1 is skew-symmetric and $\alpha_2 = \alpha_2^t$ so α_2 is symmetric.

Note that, once we know $\left(\alpha_1, \alpha_2, \dots, \alpha_{\lfloor\frac{m}{2}\rfloor}\right)$ in a skew-symmetric sequence or in a symmetric sequence α, then we know $\left(\alpha_{\lfloor\frac{m}{2}\rfloor+1}, \alpha_{\lfloor\frac{m}{2}\rfloor+2}, \dots, \alpha_m\right)$, therefore $\alpha = \left(\alpha_1, \alpha_2, \dots, \alpha_{\lfloor\frac{m}{2}\rfloor}\right)$ can be considred to be as symbol parametrization of the flags' signature.

Remark 2.7. If G is an orthogonal group then the flags in G/B have symmetric parametrization, and if G is symplectic then the flags in G/B have skew symmetric parametrization

2.4 A partial flag variety G/P

Given a sequence $(d_1, \dots, d_s)$ of positive integers with sum $\left\lfloor\frac{m}{2}\right\rfloor$, a partial flag in $\mathbb{C}^m$ of type $(d_1, \dots, d_s)$ is an increasing sequence of linear subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = \mathbb{C}^m$$

where $\dim\left(V_j/V_{j-1}\right) = d_j$.

Since G acts transitively on the space of maximally b-isotropic partial flags (lemma 2.4) and for the reasons explained in section 2.1 two kinds of maximally b-isotropic partial flags are defined according to which nodes are removed from the Dynkin diagram, i.e. If we remove the last root of the Dynkin diagram (see [5], page 114) or not. The first kind of the partial flags is defined if the removed roots dose not contain the last root as follows: A partial flag of type $(d_1, \dots, d_s)$ in $\mathbb{C}^m$ is an increasing sequence of linear subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s \subset V_{s-1}^{\perp} \subset \cdots \subset V_1^{\perp} \subset \mathbb{C}^m$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s-1$ and $\dim V_s = \left\lceil\frac{m}{2}\right\rceil + d_s$. This type of flags satisfy that $V_s = V_s^{\perp}$, so we write only V_s in the flag. In this case, the partial flag variety is $Z_I \cong G/P$ where P is the parabolic subgroup of G which stabilizes the above flags (1).

The second kind of partial flags is defined if the removed roots contains the last root as follows: A partial flag of type $(d_1, \dots, d_s)$ in $\mathbb{C}^m$ which is an increasing sequence of linear subspaces

$$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_s^{\perp} \subset \cdots \subset V_1^{\perp} \subset \mathbb{C}^m$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s$. In this case, our partial flag variety is $Z_I \cong G/P$ where P is the parabolic subgroup of G which stabilizes the above flags (2).

2.5. The signature of flags in the partial flag variety

Finding a signature for the flags is an essential point to give the flag variety a clear picture in order to deal with it and study its geometric properties. Previously, in section 2.3 the signature of full flag varieties were studied. This signature was distinguished by the ease of use, because the difference between the dimension of V_{i+1} and the dimension of V_i is exactly 1 for each i. But the matter is different in the case of partial flag variety, and the reason for this is that the difference between the dimension of V_{i+1} and the dimension of V_i is greater than 1 for some $\mathbf{i}$, and this is what makes the matter more complicated and studying it must be done accurately so that we take into account this change.

Based on the definition of a partial flag in $\mathbb{C}^m$ of type $(d_1, \dots, d_s)$ which is an increasing sequence of linear subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_s = \mathbb{C}^m$$

where $\dim\left(V_j/V_{j-1}\right) = d_j$, which means that the number of vectors in the base of V_j exceeds the number of vectors in base of V_{j-1} by d_j and which means an increase in negative and positive vectors. Therefore, when expressing the signature for this flag, we will use which called block form for signature,

$$B_1 B_2 \dots \dots B_{s-1} B_s$$

where B_i is a set of d_i of $+$ and $-$ signs according to the vectors in B_i, and called a block. To facilitate dealing with the block form, we will adopt a specific order of the signs inside any block, where the negative signs are placed in the far right of the block and positive signs in the far left of the block.

Let us discuss the parametrization of the flags in the partial flags in our consideration. Firstly, consider the partial flag $Z_I = G/P$ which consists of the maximally b isotropic flags

$$0 = V_0 \subset V_{d_1} \subset \cdots \subset V_{d_{s-1}} \subset V_{d_s} \subset V_{d_{s-1}}^{\perp} \subset \cdots \subset V_{d_1}^{\perp} \subset \mathbb{C}^m$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s-1$ and $\dim\, V_s = n + d_s$. Then we have two cases, skew-symmetric flags and symmetric flags [1] :

Case 1 : If the flag is skew-symmetric, then the parametrization of the flag can be written in the block form

$$B_1 B_2 \dots B_{s-1} [B_s] (-B_{s-1}) \dots (-B_2)(-B_1)$$

where each block $B_i, 1 \le i \le s-1$ consists of d_i entries except the block B_s, it has $2d_s$ entries, d_s of positive's entries and d_s of negative's entries. It is clear that if the first sings of the d_s entries is known then the other signs can be found, without loss of generality the positives of the block B_s can be put in the beginning. Therefore the parametrization of the flag becomes

$$B_1B_2 \dots B_{s-1}[+ + \cdots + + - - \cdots - -](-B_{s-1}) \dots (-B_2)(-B_1)$$

and to make it easy for understanding it is recommended to write the first half of this parametrization

$$B_1B_2 \dots B_{s-1}[+ + \cdots + +]$$

Case 2 : If the flag is symmetric, then the parametrization of the flag can be written in the block form

$$B_1B_2 \dots B_{s-1}[B_s]B_{s-1} \dots B_2B_1$$

, where each block $B_i, 1 \leq i \leq s-1$ consists of d_i entries except the block B_s has $2d_s$ or $2d_s + 1$ entries. In this case the block B_s consists of even number of negative signs and even number of positive signs if B_i has $2d_s$ entries but if B_i has $2d_s + 1$ then it has even number of negative signs and even number of positive signs and unique zero entry which will be placed in the middle between the positive and negative signs. Because the first half of the parametrization is used then the block B_s is divided into two similar blocks

$$B_s = B_{s_1}B_{s_1}$$

[1] The skew symmetric flags appears in the case of $G = SP(2n, \mathbb{C})$ and the symmetric flags appear in the case of $G = SO(m, \mathbb{C})$ each block B_{s_1} has half of the negative signs and half of positive signs and the zero is omitted if it exists, and then the signature becomes;

$$B_1B_2 \dots B_{s-1}B_{s_1}B_{s_1}(B_{s-1}) \dots (B_2)(B_1)$$

Simply just the first half of these parametrizations is written

$$B_1B_2 \dots B_{s-1}B_{s_1}$$

On the other hand, if the partial flag $Z_I = G/P$ which consists of the maximally b - isotropic flags

$$0 = V_0 \subset V_{d_1} \subset \cdots \subset V_{d_s} \subset V_{d_s}^{\perp} \subset \cdots \subset V_{d_1}^{\perp} \subset \mathbb{C}^{2n}$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s$. Then two cases are achieved: Case 1 : If the flag is skew-symmetric, then the parametrization of the flag can be written in the block form

$$B_1B_2 \dots B_s(-B_s) \dots (-B_2)(-B_1)$$

where each block $B_i, 1 \leq i \leq s$ consists of d_i entries and simply the first half of this parametrization can be written

$B_1 B_2 \dots B_{s-1} B_s$

Case 2 : If the flag is symmetric, then the parametrization of the flag can be written in the block form

$B_1 B_2 \dots B_s B_s \dots B_2 B_1$

, where each block $B_i, 1 \leq i \leq s-1$ consists of d_i entries. For ease we write just the first half of these parameterizations

$B_1 B_2 \dots B_{s-1} B_s$

3. Example of the group $SO(7, \mathbb{C})$

Consider the semisimple Lie group $G = SO(7, \mathbb{C})$ with complex bilinear form defined by

$$b(v, w) = -\sum_{i=1}^{3} v_i w_i + \sum_{i=4}^{7} v_i w_i$$

and Hermitian form

$$h(v, w) = -\sum_{i=1}^{3} v_i \bar{w}_i + \sum_{i=4}^{7} v_i \bar{w}_i$$

Let Z_I denote the flag variety for the group $SO(7, \mathbb{C})$, which parameterizes full flags of maximally b-isotropic subspaces in $\mathbb{C}^7$.

Define the maximally isotropic flag with respect to the ordered basis

$e_1 + ie_2, e_4 + ie_5, e_6 + ie_7, e_3, e_6 - ie_7, e_4 - ie_5, e_1 - ie_2$

or any of its permutations, for example the flag induced by the basis

$e_4 - ie_5, e_1 + ie_2, e_6 + ie_7, e_3, e_6 - ie_7, e_1 - ie_2, e_4 + ie_5$

this flag has signature $\alpha = -++---+$, simply can be written as $\alpha = -++$. Note to have maximally b-isotropic subspaces in $\mathbb{C}^7$, the vector e_3 must stay always in the middle of the basis defined the flag, and therefore the sign of this vector can be ignored from the signature when the half of it is written for easiness purpose.

On the other hand, two types of partial flag varieties can be written,

$\{0\} \subset < e_4 - ie_5, e_1 + ie_2 > \subset < e_1 + ie_2, e_4 + ie_5, e_6 + ie_7, e_3, e_6 - ie_7 > \subset \mathbb{C}^7$

with signature $\alpha = (-+)(+--)(-+)$ and for ease we write $\alpha = (-+)(-)$, and ,

$\{0\} \subset < e_4 - ie_5, e_1 + ie_2 > \subset < e_1 + ie_2, e_4 + ie_5, e_6 + ie_7, e_3 > \subset < e_1 + ie_2, e_4 + ie_5, e_6 + ie_7, e_3, e_6 - ie_7 > \subset \mathbb{C}^7$
with signature $\alpha = (-+)(-) + (-)(-+)$ and for ease we write $\alpha = (-+)(-)$,

4. The dimension of the flag varieties

As we go, through the partial flag varieties of orthogonal and symplectic groups, it is indispensable to calculate the dimension of these flag varieties. For these types the flag varieties are related to two different kinds of maximally isotropic flags. According to the simple roots that are removed from the Dynkin diagram we have 2 kinds of partial flag varieties: The first one is the partial flag variety G/P where P is the parabolic subgroup obtained by removing $(s-1)$-nodes plus the last node from the Dynkin diagram and stabilizes the flag

$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_s^{\perp} \subset \cdots \subset V_d^{\perp} \subset \mathbb{C}^m$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s$.

While the other one is the partial flag variety G/P where P is the stabilizer of the flags

$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{s-1} \subset V_s \subset V_{s-1}^{\perp} \subset \cdots \subset V_1^{\perp} \subset \mathbb{C}^m$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s-1$ and $\dim V_s = \left\lfloor \frac{m}{2} \right\rfloor + d_s$, this parabolic subgroup obtained by removing (s)-nodes from the first $\left\lfloor \frac{m}{2} \right\rfloor - 1$-nodes from the Dynkin diagram.

4.1. The dimension of the full flag varieties

To keep in mind that the Borel subgroup is the smallest parabolic subgroup, or in other words, the Borel subgroup is the parabolic subgroup resulting from deleting all nods in the Dynkin diagram (see [5] and [11]). Therefore, the dimensions of the full flag variety spaces became specified in all cases of classical groups.

It is known that the dimensions of the full flag varieties G/B where G is the orthogonal group $G = SO_{2n}$ is $n(n-1)$, but if the group is $G = SO_{2n+1}$ then the dimensions of the full flag varieties G/B is $\frac{2n^2-1}{2}$. On the other hand, if G is the symplectic group $SP(2n)$ then the dimensions of the full flag varieties G/B is n^2.

4.2. The dimension of the partial flag varieties

In contrast to the case G/B, the dimension of the partial flag variety G/P is dependent on the removed nodes, or other words, the dimension of the partial flag variety G/P depends on the type of the flags defined on this homogenous space.

Let us now calculate the dimension for each case of partial flag variety. Consider $Z_I = G/P$, the set of all maximal b-isotropic flags of signature $(d_1, d_2, \dots, d_s)$ of type

$$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_s^{\perp} \subset \cdots \subset V_d^{\perp} \subset \mathbb{C}^m$$

such that $\dim (V_j/V_{j-1}) = d_j$ for all $j = 1, \dots, s$. If $G = SP(2n, \mathbb{C})$ or $G = SO(2n+1, \mathbb{C})$, then by using what we know about the dimension of Grassmannin variety in [18] and [19], the dimension of the partial flag variety G/P is ;

$$\begin{aligned}
\dim Z_I &= \dim \left(\mathfrak{F}(d_1, d_2, \dots, d_s; n)\right) \\
&= \dim \left(Gr(d_1, n)\right) + \dim \left(\mathfrak{F}(d_2, d_3, \dots, d_s; n - d_1)\right) \\
&= 2d_1(n - d_1) + \frac{d_1(d_1+1)}{2} + \dim \left(\mathfrak{F}(d_2, d_3, \dots, d_s; n - d_1)\right) \\
&= 2d_1(n - d_1) + \frac{d_1(d_1+1)}{2} + 2d_2(n - d_2) + \frac{d_2(d_2+1)}{2} + \dim \left(\mathfrak{F}(d_3, d_4, \dots, d_s; n - d_2)\right) \\
&= 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s} \left(\frac{d_i(d_i+1)}{2}\right)
\end{aligned}$$

By using similar calculation, if $G = SO(2n, \mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is

$$\dim Z_I = 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s} \frac{d_i(d_i-1)}{2}$$

On the other hand, if $Z_I = G/P$, the set of all maximal b-isotropic flags of signature $(d_1, d_2, \dots, d_s)$ of type

$$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_{s-1}^{\perp} \subset \cdots \subset V_d^{\perp} \subset \mathbb{C}^{2n}$$

such that $\dim (V_j/V_{j-1}) = d_j$ for all $j = 1, \dots, s$ and $\dim V_s = n + d_s$. If $G = SP(2n, \mathbb{C})$ or $G = SO(2n+1, \mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is ;

$$
\begin{aligned}
\dim Z_I &= \dim\left(\mathfrak{F}(d_1,d_2,\dots,d_s;n)\right)\\
&= \dim\left(Gr(d_1,n)\right) + \dim\left(\mathfrak{F}(d_2,d_3,\dots,d_s;n-d_1)\right)\\
&= 2d_1(n-d_1) + \frac{d_1(d_1-1)}{2} + \dim\left(\mathfrak{F}(d_2,d_3,\dots,d_s;n-d_1)\right)\\
&= 2d_1(n-d_1) + \frac{d_1(d_1-1)}{2} + 2d_2(n-d_2) + \frac{d_2(d_2-1)}{2} + \dim\left(\mathfrak{F}(d_3,d_4,\dots,d_s;n-d_2)\right)\\
&= 2\sum_{1\le i<j\le s} d_i d_j + \sum_{i=1}^{s-1}\left(\frac{d_i(d_i-1)}{2}\right)
\end{aligned}
$$

and if $G = SO(2n,\mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is

$$
\dim Z_I = 2\sum_{1\le i<j\le s} d_i d_j + \sum_{i=1}^{s-1}\frac{d_i(d_i-1)}{2}
$$

This difference between the cases mentioned is apparent because the dimension of the Grassmanian space depends on the type of the flags in the flag variety,i.e. if the flags are of the type

$$
\{0\} = V_0 \subset V_{d_1} \subset \cdots \subset V_{d_s} \subset V_{d_s}^{\perp} \subset \cdots \subset V_{d_1}^{\perp} \subset \mathbb{C}^{2n}
$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1,\dots,s$, then the Grassmanian space $Gr(n;2n)$ has flags of the form

$$
\{0\} = V_0 \subset V_d \subset V_d^{\perp} \subset \mathbb{C}^{2n}
$$

such that $\dim\,(V_d) = n$, but if the flags are of the type

$$
\{0\} = V_0 \subset V_{d_1} \subset \cdots \subset V_{d_{s-1}} \subset V_{d_s} \subset V_{d_{s-1}}^{\perp} \subset \cdots \subset V_{d_1}^{\perp} \subset \mathbb{C}^{2n}
$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1,\dots,$ sand $\dim\, V_s = n + d_s$, then the Grassmanian space $Gr(d;2n)$ has flags of the form

$$
\{0\} = V_0 \subset V_d \subset V_S \subset V_d^{\perp} \subset \mathbb{C}^{2n}
$$

such that $\dim\,(V_d) = d$ and $\dim\,(V_S) = 2n - d$.

5. Results and discussion

It is clear from the previous sections that we have two types of partial flag varieties G/P for $G = SP(2n,\mathbb{C})$ and $G = SO(m,\mathbb{C})$ depending on if the removed roots from the Dynkin digram that defined the parabolic subgroup P contained the last root or not.

These two types can be defined as follows:

If $Z_I = G/P$, the set of all maximal b-isotropic flags of signature $(d_1, d_2, \dots, d_s)$ of type

$$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_s^{\perp} \subset \cdots \subset V_d^{\perp} \subset \mathbb{C}^m$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s$. If $G = SP(2n, \mathbb{C})$ or $G = SO(2n+1, \mathbb{C})$, then the dimension of the partial flag variety G/P is ;

$$\dim Z_I = 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s} \left(\frac{d_i(d_i+1)}{2}\right).$$

and if $G = SO(2n, \mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is

$$\dim Z_I = 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s} \frac{d_i(d_i-1)}{2}$$

On the other hand, if $Z_I = G/P$, the set of all maximal b-isotropic flags of signature $(d_1, d_2, \dots, d_s)$ of type

$$\{0\} = V_0 \subset V_d \subset \cdots \subset V_s \subset V_{s-1}^{\perp} \subset \cdots \subset V_d^{\perp} \subset \mathbb{C}^{2n}$$

such that $\dim\left(V_j/V_{j-1}\right) = d_j$ for all $j = 1, \dots, s$ and $\dim V_s = n + d_s$. If $G = SP(2n, \mathbb{C})$ or $G = SO(2n+1, \mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is ;

$$\dim Z_I = 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s-1} \left(\frac{d_i(d_i-1)}{2}\right).$$

and if $G = SO(2n, \mathbb{C})$, then the dimension of the partial flag variety $Z_I = G/P$ is

$$\dim Z_I = 2 \sum_{1 \le i < j \le s} d_i d_j + \sum_{i=1}^{s-1} \frac{d_i(d_i-1)}{2}$$

7. Acknowledgment

The author would like to thank Prof. Dr. Dr. h.c. mult. Alan T. Huckleberry and Prof. Dr. Peter Heinzner for introducing her this research area and this topic in particular, and would like to thank PD Dr. Stéphanie Cupit-Foutou for her support and advice in finishing these results.

References

[1] A.T. Huckleberry, and A. Simon,(2001). with appendix by D. Barlet. "On cycle spaces of flag domains of $SL_n(\mathbb{R})$ ", J. reine u. angew. Math. 541 , 171-208.

[2] A.T. Huckleberry, (2002). "On certain domains in cycle spaces of flag manifolds", Math. Ann. $323,797-810$.

[3] A.T. Huckleberry, and B. Ntatin, (2003). "Cycle spaces of G-orbits in $G^{\mathbb{C}}$-flag manifolds", Manuscripta Math. 112 , 443-440 (math. RT/0212327).

[4] A.T. Huckleberry, and J.A. Wolf, (2004). "Injectivity of the Double Fibration Transform for Cycle Spaces of Flag Domains", Journal of Lie Theory, 12 , 509-522 (math. RT/0308285)

[5] A. W. Knapp, (2002). "Lie Groups Beyond an Introduction". Progress in Mathematics. Birkhäuser Boston, Boston, 2 edition.

[6] E. V. Ponomareva,(2013). "Classification of double flag varieties of complexity 0 and 1," Izv. Ross. Akad. Nauk Ser. Mat., 77, No. 5, 155-178 .

[7] E. Yu. Smirnov, (2020). "Multiple Flag Varieties", Journal of Mathematical Sciences, Vol. 248, No. 3.

[8] F. S. Abu-Shoga, (2017). "Combinatorial Geometry of Flag Domains in G/B". Ph.D. Thesis, Ruhr University Bochum, Bochum, Germany.

[9] F. S. Abu Shoga, (2020). "Cycle Intersection for Flag Domains ", International Journal of Mathematics and Mathematical Sciences.

[10] J. Hong, and A.T. Huckleberry, (2006). "On boundaries of cycle spaces associated to flag varieties" (Preprint, 26 pages) [11] J. Humphreys, (1975). "Linear Algebraic Groups", Volume 21 of Graduate Texts in Mathematics, Springer.

[12] M. Brion, (2005). "Lectures on the geometry of flag varieties," in: Topics in Cohomological Studies of Algebraic Varieties, Birkh“auser, Basel , pp. 33-85.

[13] P. Magyar,(2005). "Bruhat order for two flags and a line," J. Algebraic Combin., 21, No. 1, $71-101$.

[14] P. Magyar, J. Weyman, and A. Zelevinsky, (1999). "Multiple flag varieties of finite type," Adv. Math.,141, No. 1, 97-118.

[15] P. Magyar, J. Weyman, and A. Zelevinsky,(2000). "Symplectic multiple flag varieties of finite type," J. Algebra, 230, No. 1, 245-265.

[16] R. Devyatov,(2014). "Generically transitive actions on multiple flag varieties," Int. Math. Res. Not., 11, 2972-2989.

[17] V. Lakshmibai,(1987). "Geometry of $G/P - VI$ Bases for Fundamental Representations of Classical Groups", JOURNAL OF ALGEBRA 108, 355-402.

[18] V. Lakshmibai,(1987). "Geometry of G/P - VII The Symplectic Group and the Involution σ", Journal of Algebra, 108, 403-434.

[19] V. Lakshmibai,(1987). "Geometry of $G/P - VIII$ The Group $SO(2n+1)$ and the Involution σ ", Journal of Algebra, 108, 435-471.

[20] V. Lakshmibai,(1990). "Geometry of $G/P - IX$ The Group $SO(2n)$ and the Involution σ ", Journal of Algebra, 130, 122-165.