

## $\theta$ -CLOSED SETS IN $T_0$ -ALEXANDROFF SPACES

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ABSTRACT. In this paper we introduce notations, elementary facts and characterizations of  $\theta$  - closure,  $\theta$ - interior,  $\theta$ - border,  $\theta$ - frontier,  $\theta$ - exterior, and generalized closed sets along with some of their properties in the class of  $T_0$  -Alexandroff spaces.

### 1. INTRODUCTION

The subject of  $\theta$ -closed sets was first studied in 1966 by Velicko [24] for the purpose of studying the important class of H-closed spaces in terms of arbitrary fiberbases.

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Some authors (see [[5], [6], [7], [14], [23], [11] ,and [17]]) studied the subject. They got several interesting results, and they introduced some new notations as  $\theta$ -derived,  $\theta$ -border,  $\theta$ -frontier and  $\theta$ -exterior of a set. They also have obtained several new and interesting results related to these sets.

M. Caldas, S.Jafari, M. M. Kovár [5] proved that some of the properties of  $\theta$ -derived and  $\theta$ -border are analogous to those for open sets. They continued the study of [24] and they gave some additional properties of  $\theta$ -closure and  $\theta$ -interior of a set.

Dontachev and Maki, [8], had shown that if  $A$  and  $B$  are subsets of a space  $(X, \tau)$ , then  $Cl_\theta(A \cup B) = Cl_\theta(A) \cup Cl_\theta(B)$  and  $Cl_\theta(A \cap B) = Cl_\theta(A) \cap Cl_\theta(B)$ . Dickman and Porter [6] proved that a compact subspace of a Hausdorff space is  $\theta$ -closed. Moreover, they showed that a  $\theta$  - closed subspace of a Hausdorff space is closed. Jankovic [13] proved that a space  $(X, \tau)$  is Hausdorff if and only if every compact set is  $\theta$ -closed.

A subset  $S$  of a topological space  $X$  is called a *generalized closed set* [15] if  $Cl(A) \subseteq U$  whenever  $A \subseteq U$  and  $U$  is open. Recently, this notion has been studied extensively since generalized closed sets do not only include the closed sets, but they also suggest several new concepts of topological spaces.

## 2. DEFINITIONS AND ELEMENTARY RESULTS

**Definition 2.1.** A space  $(X, \tau)$  is called *Alexandroff space* if any arbitrary intersection of open sets is open. Equivalently, each element of  $x$  has a minimal neighborhood base. If  $(X, \tau)$  is an Alexandroff space, we may define an order called (*Alexandroff*) *specialization order* as  $x \leq y$  iff  $x \in cl(y)$ .

The specialization order is reflexive and transitive. It turns out that it is antisymmetric -and hence a partial order - if and only if  $X$  is  $T_0$ . If

$\leq$  is a partial order on  $P$ , we say that  $(P, \leq)$  is a partially ordered set (abbreviated: poset). If  $(P, \leq)$  is a poset, and if  $\mathfrak{B} = \{\uparrow x : x \in P\}$ , then  $\mathfrak{B}$  forms a base for a topology on  $P$ . This topology - denoted by  $\tau(\leq)$  - is a  $T_0$  Alexandroff space. Moreover, if  $(X, \leq)$  is a poset and if  $\tau(\leq)$  is the induced  $T_0$ - Alexandroff topology, then the specialization order of  $\tau(\leq)$  is the order  $\leq$  itself. That is;  $\leq_{\tau(\leq)} = \leq$ . On the other hand, if  $(X, \tau)$  is a  $T_0$  Alexandroff space and if  $\leq_\tau$  is its specialization order, then the induced topology by the specialization order is the original topology. That is;  $\tau(\leq_\tau) = \tau$  [10]. Thus,  $T_0$ - Alexandroff spaces can be completely determined by their specialization orders. A poset  $(X, \leq)$  satisfies *the ascending chain condition (ACC)* if for every nondecreasing sequence  $x_1 \leq x_2 \leq \dots \leq x_i \leq \dots$  in  $X$  there exists  $n$  such that  $x_n = x_{n+1} = \dots$ . A  $T_0$ - Alexandroff space whose corresponding poset satisfies the *ACC* is called *Artinian  $T_0$ - Alexandroff space* [18]. It is known that the property *ACC* is equivalent to the *maximal condition (MXC)*, namely; every subset of a poset has a maximal element. We can define *the descending chain condition (DCC)* dually. It is known that the property *DCC* is equivalent to the *minimal condition (MNC)*, namely; every non-empty subset has a minimal element. A  $T_0$ - Alexandroff space whose corresponding poset satisfies the *DCC* is called *Noetherian  $T_0$ - Alexandroff space* [18].

**Definition 2.2.** (a) A poset  $(X, \leq)$  satisfies the finite chain condition (*FCC*) if it satisfies both *ACC* and *DCC*.

(b) A  $T_0$ -Alexandroff space whose corresponding posets satisfies *FCC* is called *generalized locally finite (g-locally finite)*.

Needless to say that, finite and locally finite topologies are Alexandroff topologies. More precisely, finite topologies are locally finite and  $T_0$ -locally finite topologies are g-locally finite. The converse is not always true as the following examples show.

*Example 2.3.* (1) [18] Let  $X = \mathbb{N} \cup \{\perp\}$  with order  $\leq$  defined as follows  $\perp \leq x \forall x \in \mathbb{N}$  and  $x$  is incomparable with  $y$  for all  $x, y \in \mathbb{N}$ . Then  $X$  is g-locally finite space. Since there is no finite open set contains  $\perp$ ,  $X$  is not locally finite.

(2) Consider the digital line  $\mathbb{Z}$  with minimal neighborhood base for  $n \in \mathbb{Z}$ ,  $V(n) = \{n\}$  if  $n$  is odd and  $V(n) = \{n-1, n, n+1\}$  if  $n$  is even. Then  $\mathbb{Z}$  is  $T_0$ -locally finite space which is not finite.

Let  $(X, \tau(\leq))$  be a  $T_0$ - Alexandroff space where  $(X, \leq)$  is its corresponding poset,  $A \subseteq X$ , and  $x \in X$ . The following notations have the following meanings:

$A^c$  = The complement of  $A$

$\uparrow x$  =  $\{y \in X : y \geq x\}$ .

$\uparrow A$  =  $\{y \in X : y \geq x \forall x \in A\} = \bigcup_{x \in A} \uparrow x$ .

$\downarrow x$  =  $\{y \in X : y \leq x\}$ .

$\downarrow A$  =  $\{y \in X : y \leq x \forall x \in A\} = \bigcap_{x \in A} \downarrow x$ .

If  $(X, \tau(\leq))$  is Artinian [18], then we define the following:

$M$  = the set of maximal elements of  $X$ .

$M(A)$  = the set of maximal elements of  $A$  with respect to induced order.

$\hat{x} = \uparrow x \cap M$ .

$\hat{A} = \uparrow A \cap M$ .

In the case where  $M$  contains only one element, we denote  $\top$  to be the maximum element.

Dually, if  $(X, \tau(\leq))$  is Noetherian [18], we define the following:

$m$  = the set of minimal elements of  $X$ .  $m(A)$  = the set of minimal elements of  $A$  with respect to induced order.

$\check{x} = \downarrow x \cap m$ .

$\check{A} = \downarrow A \cap m$ .

In the case where  $m$  contains only one element, we denote  $\perp$  to be the minimum element.

**Definition 2.4.** [5] A subset  $A$  of a space  $X$  is called *preopen* ( resp. *semi-open*,  *$\alpha$ -open*) if  $A \subseteq \text{Int}(\text{Cl}(A))$  (resp.  $A \subseteq \text{Cl}(\text{Int}(A))$ ,  $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ ). The complement of a preopen (resp. semi-open,  $\alpha$ -open) is called *preclosed* (resp. *semi-closed*,  *$\alpha$ -closed*).

**Definition 2.5.** [5] A point  $x \in X$  is called a  $\theta$ -closure ( $\theta$ -adherent) of  $A$  if  $A \cap \text{cl}(V) \neq \phi$  for every open set  $V$  containing  $x$ . The set of all  $\theta$ -closure points of  $A$  is called the  $\theta$ -closure of  $A$  and is denoted by  $\text{cl}_\theta(A)$ . A subset  $A$  of  $X$  is called  $\theta$ -closed if  $A = \text{cl}_\theta(A)$ . The complement of a  $\theta$ -closed set is  $\theta$ -open. The family of all  $\theta$ -open sets forms a topology on  $X$  denoted by  $\tau_\theta$ .

It should be noted that if  $x \in \text{cl}(A)$ , then  $A \cap V \neq \phi$  for each open set  $V$  containing  $x$ , so  $A \cap \text{cl}(V) \neq \phi$ , and this implies that  $x \in \text{cl}_\theta(A)$ . Hence, if  $A$  is  $\theta$ -closed, then  $A$  is closed, and certainly, this implies that  $\tau_\theta \subseteq \tau$ . It is well-known that  $\tau = \tau_\theta$  if and only if the space  $(X, \tau)$  is regular [5]. Moreover Dontachev and Maki [8] showed that, if  $A$  and  $B$  are two subsets of a topological space, then  $\text{cl}_\theta(A \cup B) = \text{cl}_\theta(A) \cup \text{cl}_\theta(B)$ ,  $\text{cl}_\theta(A \cap B) = \text{cl}_\theta(A) \cap \text{cl}_\theta(B)$  and  $\text{cl}_\theta(A \setminus B) = \text{cl}_\theta(A) \setminus \text{cl}_\theta(B)$ . Finally, it is clear that  $A$  is  $\theta$ -closed in  $(X, \tau)$  iff it is closed in  $(X, \tau_\theta)$ .

Note that the  $\theta$ -closure of a given set need not be  $\theta$ -closed but it is always closed. In fact  $\text{cl}_\theta(A) \subseteq \text{cl}_\theta(\text{cl}_\theta(A))$  [5].

Also, as noted above, for any topological space  $X$  where  $A \subseteq X$ , we have that  $\text{cl}(A) \subseteq \text{cl}_\theta(A)$ .

**Lemma 2.6.** *If  $X$  is a  $T_0$ -Alexandroff space and  $A$  is an open subset of  $X$ , then  $\text{cl}_\theta(A) = \text{cl}(A)$ .*

*Proof.* For any topological space  $X$  where  $A \subseteq X$ , we have  $cl(A) \subseteq cl_\theta(A)$ . Now to prove that  $cl_\theta(A) \subseteq cl(A)$  when  $A$  is open, let  $x \in cl_\theta(A)$  and let  $V$  be any open set containing  $x$ . Then  $A \cap cl(V) \neq \phi$ . So there exists  $a \in A$  such that  $a \in cl(V)$ . Therefore  $a \in \downarrow u$  for some  $u \in V$ , and hence  $u \in \uparrow a \cap V$ . Since  $\uparrow a \subseteq A$ , we have that  $A \cap V \neq \phi$ . Therefore  $x \in cl(A)$ .  $\square$

**Theorem 2.7.** *Let  $(X, \tau(\leq))$  be a  $T_0$ -Alexandroff space, and let  $A$  be a non-empty subset of  $X$ . Then all the following are equivalent:*

- 1)  $x \in cl_\theta(A)$ .
- 2)  $cl(\uparrow x) \cap A \neq \phi$ .
- 3)  $\uparrow x \cap \uparrow a \neq \phi$  for some  $a \in A$ .
- 4)  $\uparrow x \cap \uparrow A \neq \phi$ .
- 5)  $U \cap \uparrow A \neq \phi$  for each open set  $U$  containing  $x$ .
- 6)  $x \in cl(\uparrow A)$ .
- 7)  $x \in cl_\theta(\uparrow A)$ .

*Proof.* (1  $\Rightarrow$  2)  $\uparrow x$  is open set contains  $x$  so  $cl(\uparrow x) \cap A \neq \phi$ .

(2  $\Rightarrow$  3) Let  $a \in cl(\uparrow x) \cap A$ . So there exists  $u \in \uparrow x$  such that  $a \in \downarrow u$ . Equivalently,  $u \in \uparrow a$  and hence  $u \in \uparrow x \cap \uparrow a$ .

(3  $\Leftrightarrow$  4) The condition " $\uparrow x \cap \uparrow a \neq \phi$  for some  $a \in A$ " is equivalent to saying that " $\uparrow x \cap \uparrow A \neq \phi$ ".

(3  $\Rightarrow$  1) Let  $V$  be any open set containing  $x$ . Since  $\uparrow a \cap \uparrow x \neq \phi$  for some  $a \in A$ , there exist  $u \in \uparrow a$  and  $u \in \uparrow x \subseteq V$ . So,  $a \in \downarrow u \subseteq cl(V)$ . Therefore  $cl(V) \cap A \neq \phi$ .

(4  $\Rightarrow$  5) Let  $U$  be open set containing  $x$ , so  $\uparrow x \subseteq U$  and hence  $\uparrow x \cap \uparrow A \subseteq U \cap \uparrow A$ . This implies that  $U \cap \uparrow A \neq \phi$ .

(5  $\Rightarrow$  6) The definition of  $cl(\uparrow A)$ .

(6  $\Leftrightarrow$  7)  $\uparrow A \in \tau$ , so by Lemma 2.6,  $cl(\uparrow A) = cl_\theta(\uparrow A)$ .

(6  $\Rightarrow$  4)  $\uparrow x$  is open set containing  $x$ , so if  $x \in cl(\uparrow A)$ , then by defenition of closure,  $\uparrow x \cap \uparrow A \neq \phi$ .  $\square$

**Corollary 2.8.** *Let  $X$  be a  $T_0$ -Alexandroff space, and let  $A$  be a non-empty subset of  $X$ . Then  $\forall x \in cl_\theta(A)$ ,  $\downarrow x \subseteq cl_\theta(A)$ ; that is; if  $y \leq x$ , then  $y \in cl_\theta(A)$ .*

*Proof.*  $cl_\theta(A) = cl(\uparrow A)$  is a closed subset of  $(X, \tau)$ .  $\square$

**Corollary 2.9.** *Let  $X$  be a  $T_0$ -Alexandroff, and let  $A$  be a non-empty subset of  $X$ . Then  $\uparrow z \cap cl_\theta(A) = \phi \forall z \notin cl_\theta(A)$*

*Proof.* Suppose to contrary that  $\exists z_0 \notin cl_\theta(A)$  such that  $\uparrow z_0 \cap cl_\theta(A) \neq \phi$ . So, there exists  $u \in \uparrow z_0$  and  $u \in cl_\theta(A)$ . Since from part (6) of Theorem 2.7,  $cl_\theta(A)$  is closed set in  $(X, \tau)$ , we get  $z_0 \in \downarrow u \subseteq cl_\theta(A)$  which is a contradiction.  $\square$

**Theorem 2.10.** *Let  $X$  be an Artinian  $T_0$ -Alexandroff space, and let  $A$  be a non-empty subset of  $X$ . Then all the following are equivalent:*

- 1)  $x \in cl_\theta(A)$ .
- 2)  $\hat{x} \cap \hat{A} \neq \phi$ .
- 3)  $x \in cl(\hat{A})$ .

*Proof.* (1)  $\Leftrightarrow$  (3) Since  $\hat{A} \subseteq \uparrow A$ , then  $cl(\hat{A}) \subseteq cl(\uparrow A) = cl_\theta(A)$ . Now, let  $x \in \uparrow A$ , so there exists  $z \in \hat{A}$  such that  $x \leq z$ , then  $x \in \downarrow z \subseteq \downarrow \hat{A} = cl(\hat{A})$ , then  $\uparrow A \subseteq cl(\hat{A})$ , so  $cl(\uparrow A) \subseteq cl(cl(\hat{A})) = cl(\hat{A})$ . Therefore, by Theorem 2.7  $cl_\theta(A) = cl(\uparrow A) \subseteq cl(\hat{A})$ .

(1)  $\Leftrightarrow$  (2) Since  $x \in cl_\theta(A)$  is equivalent to  $\uparrow x \cap \uparrow A \neq \phi$ ,  $\hat{A} \subseteq \uparrow A$ , and  $\hat{x} \subseteq \uparrow x$ , we have  $\hat{x} \cap \hat{A} \neq \phi$ .  $\square$

**Proposition 2.11.** *Let  $X$  be an Artinian  $T_0$ -Alexandroff space, and let  $A$  be a non-empty subset of  $X$ . If  $\hat{A} \subseteq A$ , then  $cl_\theta(A) = cl(A)$ .*

*Proof.* It follows from the inclusions  $cl_\theta(A) = cl(\hat{A}) \subseteq cl(A) \subseteq cl_\theta(A)$ . □

**Proposition 2.12.** *If  $X$  is Noethirian  $T_0$ -Alexandroff space, and if  $A$  is a non-empty subset of  $X$ , then  $cl_\theta(A) = cl(\uparrow m(A)) = cl_\theta(m(A))$ .*

*Proof.*  $X$  is Noethirian, so  $\uparrow A = \uparrow (m(A))$  [18]. Thus,  $cl_\theta(A) = cl(\uparrow A) = cl(\uparrow m(A)) = cl_\theta(m(A))$ . □

**Proposition 2.13.** *If  $X$  is a finite  $T_0$ -Alexandroff space, then  $cl(\hat{A}) = cl(\uparrow m(A))$ .*

*Proof.* Finite  $T_0$ -Alexandroff spaces satisfy FCC, so from Theorem 2.10 and Proposition 2.12, we get that  $cl(\hat{A}) = cl(\uparrow m(A))$ . □

Let  $PO(X)$  denote the set of all preopen sets in  $X$ .

**Theorem 2.14.** [9] *In any topological space  $(X, \tau)$ , if  $U \in PO(X)$ , then  $cl_\alpha(U) = cl_\theta(U) = cl(U)$ .*

**Corollary 2.15.** [9] *If  $A$  is preopen, then the following are equivalent:*

- 1)  $A$  is  $\alpha$  - closed.
- 2)  $A$  is  $\theta$  - closed.
- 3)  $A$  is closed.

**Lemma 2.16.** *Let  $X$  be a  $T_0$ -Alexandroff space and let  $A$  be a non-empty subset of  $X$ . If  $A$  is clopen (= closed and open), then  $A$  is  $\theta$ -closed.*

*Proof.* Suppose that  $A$  is clopen, and consider  $cl_\theta(A) = cl(\uparrow A) = cl(A) = A$ . Therefore  $A$  is  $\theta$ -closed. □

*Remark 2.17.* Using Corollary 2.15, we can prove Lemma 2.16 for any topological space as follows:

Since  $A$  is clopen,  $A$  is open and hence preopen. Since  $A$  is closed, it is  $\theta$ -closed.

Note that if  $X$  is a  $T_0$ -Alexandroff space and  $A$  is a non-empty clopen subset of  $X$ , then  $A$  is  $\alpha$ -closed.

**Definition 2.18.** A subset  $A$  of a space  $(X, \tau)$  is called:

- 1) a *generalized closed set* (= g-closed) [9] if  $cl(A) \subseteq U$  whenever  $A \subseteq U$ ,  $U$  is open.
- 2) a  *$\theta$ -generalized closed set* (=  $\theta$ -g-closed) [9] if  $cl_\theta(A) \subseteq U$  whenever  $A \subseteq U$ ,  $U$  is open.
- 3) an  *$\alpha$ -generalized closed set* (=  $\alpha$ -g-closed) [9] if  $cl_\alpha(A) \subseteq U$  whenever  $A \subseteq U$ ,  $U$  is open.

**Proposition 2.19.** [9] *Let  $X$  be any topological space,  $A \subseteq X$ . If  $A$  is a preopen set, then the following are equivalent:*

- 1)  $A$  is  $g$ -closed.
- 2)  $A$  is  $\theta$ - $g$ -closed.
- 3)  $A$  is  $\alpha$ - $g$ -closed.

### 3. $\theta$ - INTERIOR POINTS AND $\theta$ - OPEN SETS

**Definition 3.1.** [14] A point  $x \in X$  is said to be a  $\theta$  - interior point of  $A$  if there exists an open set  $U$  containing  $x$  such that  $U \subset cl(U) \subset A$ . The set of all  $\theta$  - interior points of  $A$  is said to be the  $\theta$  - interior of  $A$  and it is denoted by  $Int_\theta(A)$ .

$A$  is  $\theta$ -open iff  $A = Int_\theta(A)$ . If  $(X, \tau)$  is a topological space,  $A \subseteq X$ , then  $A$  is  $\theta$ -open in  $\tau$  iff  $A$  is open in  $\tau_\theta$ . In general,  $Int_\theta(A)$  need not be  $\theta$ -open, since  $Int_\theta(Int_\theta(A)) \subseteq Int_\theta(A)$  [14].

**Lemma 3.2.** *In a  $T_0$ - Alexandroff space  $X$  where  $A \subseteq X$ ,  $A$  is  $\theta$ -open iff  $cl(\uparrow A) = A$  iff  $A$  is  $\theta$ -closed.*

*Proof.*  $A$  is  $\theta$ -open iff  $\forall x \in A, cl(\uparrow x) \subseteq A$  iff  $cl(\uparrow A) \subseteq A$ . But  $A \subseteq cl(\uparrow A)$ , so  $A$  is  $\theta$ -open iff  $cl(\uparrow A) = A$ . From Theorem 2.7,  $cl(\uparrow A) = A$  iff  $cl_\theta(A) = A$ .  $\square$

**Lemma 3.3.** *Let  $X$  be a  $T_0$ -Alexandroff space,  $A$  be a non-empty subset of  $X$ . Then  $x \in Int_\theta(A)$  iff  $cl(\uparrow x) \subseteq A$ . That is;  $Int(A) = \{x : cl(\uparrow x) \subseteq A\}$ .*

*Proof.*  $(\Rightarrow)$  Assume  $x \in Int_\theta(A)$ , then  $x \in U \subseteq cl(U) \subseteq A$  for some open set  $U$ . So,  $\uparrow x \subseteq U$ , and hence  $cl(\uparrow x) \subseteq cl(U) \subseteq A$ .

$(\Leftarrow)$  This follows from the fact  $\uparrow x$  is an open set containing  $x$ .  $\square$

Recall that  $\tau_\theta$  - the set of all  $\theta$  - open subsets of  $X$  - forms a topology on  $X$ , and  $\tau_\theta \subseteq \tau$ . Also, recall that, if  $\tau$  is a  $T_0$ -Alexandroff space, then  $\tau$  is discrete iff the corresponding poset  $(X, \leq_\tau)$  is antichain.

**Theorem 3.4.** *If  $X$  is a  $T_0$ -Alexandroff space with a top element  $\top$ , then for each non-empty subset  $A$  of  $X$ ,  $cl_\theta(A) = X$ . That is; each non-empty subset of a  $T_0$ -Alexandroff with a top element is dense with respect to  $\tau_\theta$ .*

*Proof.* Let  $\top$  be a top element of  $X$ . Since  $\top \in \uparrow A$ , then  $X = cl(\uparrow A) = cl_\theta(A)$ .  $\square$

**Definition 3.5.** [16] A topological space  $(X, \tau)$  is called a *regular space* iff for each closed subset  $F \subseteq X$  and each point  $x \notin F$ , there exist open set  $U$  and  $V$  such that  $x \in U$ ,  $F \subseteq V$ , and  $U \cap V = \phi$ .

**Theorem 3.6.** [19] *A  $T_0$ -Alexandroff space  $X$  is regular iff  $X$  is discrete.*

**Theorem 3.7.** *Let  $X$  be a  $T_0$ -Alexandroff space. Then  $\tau_\theta = \tau$  iff  $\tau$  is discrete.*

*Proof.*  $\tau = \tau_\theta$  iff  $X$  is regular iff  $X$  is discrete.  $\square$

**Proposition 3.8.** *Let  $X$  be a  $T_0$ -Alexandroff space,  $A \subseteq X$ . Then the following are equivalent:*

- a)  $A$  is  $\theta$ -closed.
- b)  $A$  is  $\theta$ -open.
- c)  $A$  is clopen.
- d)  $A$  is preclopen.

*Proof.* ( $a \Leftrightarrow b$ ) Lemma 3.2.

( $b \Rightarrow c$ ) Let  $A$  be a  $\theta$ -open set. Since  $\tau_\theta \subseteq \tau$ ,  $A$  is open. Moreover from Lemma 3.2,  $cl(A) \subseteq cl(\uparrow A) = A \subseteq cl(A)$ . Therefore,  $A$  is closed, and hence  $A$  is clopen.

( $c \Rightarrow b$ ) If  $A$  is clopen, then  $\downarrow(\uparrow A) = A$ , so by Lemma 3.2  $A$  is  $\theta$ -open.

( $b \Leftrightarrow d$ ) If  $X$  is  $T_0$ -Alexandroff, then  $A$  is clopen iff  $A$  is preclopen [18].  $\square$

**Theorem 3.9.** (*Well-Ordering Principle*) [16] *Every set can be well-ordered.*

**Theorem 3.10.** [16] *Let  $\mathcal{B}$  be a collection of subsets of a nonempty set  $X$ . Then there exists a topology on  $X$  that has  $\mathcal{B}$  as a base iff:*

- (1) *for each  $x \in X$  there is at least one  $U \in \mathcal{B}$  such that  $x \in U$ , and*
- (2) *if  $U \in \mathcal{B}$ ,  $V \in \mathcal{B}$ , and  $x \in U \cap V$ , then there exists a  $W \in \mathcal{B}$  such that  $x \in W \subset U \cap V$ .*

*Observation 3.11.* Any partition  $\mathcal{P}$  of a set  $X$  that satisfies the two conditions of Theorem 3.10 is a base for some topology on  $X$ .

**Theorem 3.12.** *If  $X$  is any set and  $\mathcal{P} = \{A_\alpha : \alpha \in \Delta\}$  is any partition of  $X$ , then  $\mathcal{P}$  is a base for  $\tau_\theta$  for some  $T_0$ -Alexandroff space  $\tau$ .*

*Proof.* Using the Well-Ordering Principle, let  $\leq_\alpha$  be a well order defined on  $A_\alpha$  for each  $\alpha \in \Delta$ . Define a partial order  $\leq$  on  $X$  as follows:

for  $x, y \in X$ , if there exists  $\alpha \in \Delta$  such that  $x, y \in A_\alpha$ , then  $x \leq y$  in  $X$  iff  $x \leq_\alpha y$  in  $A_\alpha$ . Otherwise,  $x, y$  are incomparable. Let  $\tau(\leq)$  be the  $T_0$ -Alexandroff topology on  $X$  induced by  $\leq$ . Then for each  $\alpha$ ,  $A_\alpha$  is connected clopen subset [20]. Hence by proposition 3.8,  $A$  is  $\theta$ -open with respect to  $\tau(\leq)$ . Moreover, if  $U$  is a  $\theta$ -open, then it is clopen and hence  $U$  is a union of elements of  $\mathcal{P}$ . Therefore  $\mathcal{P}$  is a base for  $\tau_\theta$ .  $\square$

**Theorem 3.13.** *If  $\tau$  is a  $T_0$ -Alexandroff space, then*

- a)  $\tau_\theta$  is an Alexandroff space,
- b)  $\tau_\theta$  is  $T_0$  iff  $\tau$  is discrete (In this case,  $\tau_\theta$  is also discrete).

*Proof.* a) let  $\{U_\alpha : \alpha \in \Delta\}$  be a collection of  $\theta$ -open sets. Then by Proposition 3.8,  $U_\alpha$  is clopen for each  $\alpha \in \Delta$ . Since  $\tau$  is Alexandroff,  $\bigcap_{\alpha \in \Delta} U_\alpha$  is clopen and hence  $\bigcap U_\alpha$  is  $\theta$ -open.

- b) ( $\Rightarrow$ ) Assume that  $\tau$  is not discrete, so there exist  $x, y \in X$ , such that  $x \leq y$ . Now let  $U$  be any  $\theta$ -open set. If  $x \in U$ , then  $y \in \uparrow x \subseteq \uparrow U$  and hence  $y \in cl(\uparrow U) = U$ . If  $y \in U$ , then  $y \in \uparrow U$  and hence  $x \in \downarrow y \subseteq cl(\uparrow U) = U$ . This proves that, there is no  $\theta$ -open set containing one of the two elements  $x, y$  and not the other. Therefore  $\tau_\theta$  is not  $T_0$ .

( $\Leftarrow$ ) Assume that  $\tau$  is discrete, then  $\leq$  is antichain order. So, for each distinct  $x, y \in X$ ,  $\{x\}$  is  $\theta$ -open contains  $x$  and not  $y$ , since  $\downarrow(\uparrow x) = \{x\}$ . Therefore,  $\tau_\theta$  is  $T_0$ -space.  $\square$

**Definition 3.14.** [16] Let  $X$  be any topological space and  $A \subseteq X$ . Then  $b(A) = A \setminus Int(A)$  is said to be the *border* of  $A$ .

**Definition 3.15.** [5] Let  $X$  be any topological space and  $A \subseteq X$ . Then  $b_\theta(A) = A \setminus Int_\theta(A)$  is said to be the  $\theta$ -*border* of  $A$ .

**Proposition 3.16.** *Let  $X$  be a  $T_0$ - Alexandroff space. Then  $x \in b_\theta(A)$  iff  $x \in A$  and  $\uparrow x \cap \uparrow z \neq \phi$  for some  $z \in A^c$ . That is;  $x \in b_\theta(A)$  iff  $x \in A$  and  $\uparrow x \cap \uparrow A^c \neq \phi$*

*Proof.*  $x \in b_\theta(A)$  iff  $x \in A \setminus Int_\theta(A)$  iff  $x \in A$  and  $x \notin Int_\theta(A)$  iff  $x \in A$  and  $\downarrow (\uparrow x) \not\subseteq A$ , iff  $x \in A$  and  $\exists z \in \downarrow (\uparrow x)$ ,  $z \notin A$ , iff  $x \in A$  and for some  $z \in A^c$ ,  $\uparrow x \cap \uparrow z \neq \phi$ .  $\square$

**Corollary 3.17.** *In a  $T_0$ - Alexandroff space  $X$ ,  $x \in b_\theta(A)$  iff  $x \in A$  and  $U \cap \uparrow A^c \neq \phi$ , for each open set  $U$  containing  $x$ .*

**Definition 3.18.** (a) Let  $X$  be a topological space and  $A \subseteq X$ . The  $\theta$ -exterior of  $A$  [11] is  $Ext_\theta(A) = Int_\theta(A^c)$ .  
 (b) Let  $X$  be a topological space and  $A \subseteq X$ . The  $\theta$ -frontier of  $A$  [5] is  $Fr_\theta(A) = Cl_\theta(A) \setminus Int_\theta(A)$ .

In general  $b_\theta(A) \subseteq Fr_\theta(A)$  for any topological space  $X$ ,  $A \subseteq X$ . If  $A$  is  $\theta$ -closed then  $b_\theta(A) = Fr_\theta(A)$ .

**Proposition 3.19.** *Let  $X$  be a  $T_0$ - Alexandroff space  $X$ . Then  $x \in Ext_\theta(A)$  iff  $cl(\uparrow x) \cap A = \phi$ .*

*Proof.*  $x \in Ext_\theta(A)$  iff  $x \in Int_\theta(A^c)$  iff  $cl(\uparrow x) \subseteq A^c$  iff  $cl(\uparrow x) \cap A = \phi$ .  $\square$

It follows from this proposition that the exterior of a set  $A$  is the largest clopen set  $U$  of  $X$  such that  $U \cap A = \phi$ .

**Corollary 3.20.** *Let  $X$  be a  $T_0$ - Alexandroff space. Then:*

- 1) *if  $x \in Ext_\theta(A)$  then  $\uparrow x \cap A = \phi$ ,*
- 2) *if  $x \in Ext_\theta(A)$  then  $\downarrow x \cap A = \phi$ .*

From this corollary, if either  $\uparrow x$  or  $\downarrow x$  intersects the set  $A$ , then  $x \notin Ext_\theta(A)$ .

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