MacDonald codes over the ring $F_3 + vF_3$

Yasemin Cengellenmis
Mathematics Department
Trakya University, Edirne, TURKEY.
E.mail:ycengellenmis@yahoo.com

Mohammed M. AL-Ashker
Mathematics Department
Islamic University of Gaza P.O.Box 108, Gaza, Palestine
E.mail:mashker@iugaza.edu.ps

Abstract:
In this paper, we construct MacDonald codes of type $\alpha$ over the ring $F_3 + vF_3$, where $v^2 = 1$, $F_3 = \{0, 1, 2\}$ is the field of three elements and investigate some of their properties such as torsion codes and weight distributions.

AMS, Mathematics Classification: Primary 94B05, secondary 51E22.

Key words: MacDonald codes, simplex codes over finite rings.
1. Introduction

The binary MacDonald codes were introduced in [9] and $q$–ary version ($q \geq 2$) MacDonald code over the finite field $F_q$ was studied in [10]. In [5], C.J.Colbourn and M.Gupta obtained two families of MacDonald codes over the ring $Z_4$ from $Z_4$-simplex codes of types $\alpha$ and $\beta$, $S_\alpha^k$ and $S_\beta^k$. They studied some fundamental properties of the codes. In [1], it was shown that the results of [5] concerning the codes over the ring $Z_4$ are valid for the ring $F_2 + uF_2$ where $u^2 = 0$ and $F_2$ is a field of two elements. In [2], the MacDonald codes over the ring $F_2 + uF_2 + u^2F_2$ were constructed, where $u^3 = 0$ and $F_2 = \{0, 1\}$ by using simplex codes over the ring $F_2 + uF_2 + u^2F_2$. Their properties were described. In [6], the MacDonald codes over $F_2 + vF_2$ were constructed where $v^2 = v$.

In [3], the simplex codes of type $\alpha$ over the ring $F_3 + vF_3$ where $v^2 = 1$, $F_3 = \{0, 1, 2\}$ were introduced and the minimum Hamming, Lee and Bouch weights of these codes were obtained.

In this paper, we construct MacDonald codes over the ring $F_3 + vF_3$ by using the simplex codes over the ring $F_3 + vF_3$ of type $\alpha$, where $v^2 = 1$ and we study torsion codes and weight distributions.

2. Preliminaries

The alphabet $R = F_3 + vF_3 = \{0, 1, 2, v, 2v, a = 1 + v, b = 2 + v, c = 1 + 2v, d = 2 + 2v\}$ is a commutative ring with nine elements where $v^2 = 1$ and $F_3 = \{0, 1, 2\}$. The elements $1, 2, v, 2v$ are units. Addition and multiplication operation over $R$ are given in the following tables,

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>v</th>
<th>2v</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>v</td>
<td>2v</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>a</td>
<td>c</td>
<td>b</td>
<td>v</td>
<td>d</td>
<td>2v</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>b</td>
<td>d</td>
<td>v</td>
<td>a</td>
<td>2v</td>
<td>c</td>
</tr>
<tr>
<td>v</td>
<td>v</td>
<td>a</td>
<td>b</td>
<td>2v</td>
<td>0</td>
<td>c</td>
<td>d</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2v</td>
<td>2v</td>
<td>c</td>
<td>d</td>
<td>0</td>
<td>v</td>
<td>1</td>
<td>2</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>v</td>
<td>c</td>
<td>1</td>
<td>d</td>
<td>2v</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>v</td>
<td>a</td>
<td>d</td>
<td>2</td>
<td>2v</td>
<td>c</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>d</td>
<td>2v</td>
<td>1</td>
<td>a</td>
<td>2</td>
<td>0</td>
<td>b</td>
<td>v</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>2v</td>
<td>c</td>
<td>2</td>
<td>b</td>
<td>0</td>
<td>1</td>
<td>v</td>
<td>a</td>
</tr>
</tbody>
</table>
A linear code $C$ of length $n$ over $R$ is an $R$-submodule of $R^n$. An element of $C$ is called a codeword of $C$. There are three well known different weights for codes over $R$, namely Hamming, Lee and Bachoc weights.

The Hamming weight $\text{wt}_H(x)$ of a codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is the number of nonzero components. The minimum weight $\text{wt}_H(C)$ of a code $C$ is the smallest weight among all its nonzero codewords.

The Lee weight for the codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is defined by,

$$\text{wt}_L(x) = \sum_{i=1}^{n} \text{wt}_L(x_i)$$

where,

$$\text{wt}_L(x_i) = \begin{cases} 
0 & \text{if } x_i = 0 \\
1 & \text{if } x_i = 1, 2, v \text{ or } 2v \\
2 & \text{if } x_i = 1 + v, 2 + v, 1 + 2v \text{ or } 2 + 2v
\end{cases}$$

In [4], the Bachoc weight for the codeword $x = (x_1, x_2, \ldots, x_n) \in R^n$ is defined by, $\text{wt}_B(x) = \sum_{i=1}^{n} \text{wt}_B(x_i)$ where,

$$\text{wt}_B(x_i) = \begin{cases} 
0 & \text{if } x_i = 0 \\
1 & \text{if } x_i = 1 + v, 2 + v, 1 + 2v \text{ or } 2 + 2v \\
3 & \text{if } x_i = 1, 2, v \text{ or } 2v
\end{cases}$$

The minimum Lee weight $\text{wt}_L(C)$ and the minimum Bachoc weight $\text{wt}_B(C)$ of code $C$ are defined analogously.

For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in R^n$, $d_H(x, y) = |\{i| x_i \neq y_i\}|$ is called distance between $x$ and $y \in R^n$ and it is denoted by,

$$d_H(x, y) = \text{wt}_H(x - y)$$
The minimum Hamming distance between distinct pairs of codewords of a code $C$ is called the minimum distance of $C$ and denoted by $d_H(C)$ or shortly $d_H$.

The Lee distance and Bachoc distance between $x$ and $y \in \mathbb{R}^n$ is defined by,

$$d_L(x, y) = wt_L(x - y) = \sum_{i=1}^{n} wt_L(x_i - y_i)$$

$$d_B(x, y) = wt_B(x - y) = \sum_{i=1}^{n} wt_B(x_i - y_i)$$

respectively.

The minimum Lee and Bachoc distance between distinct pairs of codewords of a code $C$ are called the minimum distance of $C$ and denoted by $d_L(C)$ and $d_B(C)$ or shortly $d_L$ and $d_B$, respectively.

If $C$ is a linear code, then $d_H(C) = wt_H(C)$, $d_L(C) = wt_L(C)$, $d_B(C) = wt_B(C)$.

A generator matrix of $C$ is a matrix whose rows generate $C$.

Two codes are equivalent if one can be obtained from the other by permuting the coordinates.

In [4], it was shown that the ring $R$ has two maximal ideals. These are $m_1 = \langle b \rangle = \langle v - 1 \rangle = \langle v + 2 \rangle = \{0, v + 2, 1 + 2v\}$ and $m_2 = \langle v + 1 \rangle = \{0, 1 + v, 2 + 2v\}$. Moreover $m_1 \cap m_2 = \{0\}$. The following map,

$$\phi : R \rightarrow R/m_1 \times R/m_2$$

$$a \mapsto (\phi_1(a), \phi_2(a))$$

is an isomorphism where these maps $\phi_i : R \rightarrow R/m_i$ are canonical homomorphisms for $i = 1, 2$. It is easy to see that $R/m_i$ is isomorphic to $F_3$, for $i = 1, 2$. The map $\phi^{-1}$ is a ring isomorphism by the generalized Chinese Remainder Theorem and $R$ is isomorphic to $R/m_1 \times R/m_2 \cong F_3 \times F_3$, see [8]. This map can be extended from $R^n$ to $F_3^{2n}$ in the following way:

The Gray map $\phi$ from $R^n$ to $F_3^{2n}$ is defined as

$$\phi : R^n \rightarrow F_3^{2n}$$

$$x + vy \mapsto (x, y)$$

where $x, y \in F_3^n$. The Lee weight of $x + vy$ is the Hamming weight of its Gray image. Note that $\phi$ is linear.
Let \( w_1, w_2, \ldots, w_k \) be vectors in \( \mathbb{R}^n \). Then \( w_1, w_2, \ldots, w_k \) are independent if \( \sum a_j w_j = 0 \) implies that \( a_j w_j = 0 \) for all \( j \). The vectors \( w_1, w_2, \ldots, w_k \) in \( \mathbb{R}^n \) are modular independent if \( \phi(w_1), \phi(w_2), \ldots, \phi(w_k) \) are linearly independent for some \( i \), see [7].

In [7], it was shown that a generating set that is both independent and modular independent is a minimal generating set.

Let \( w = (a_1, \ldots, a_n) \) be a nonzero vector then \( \langle a_1, \ldots, a_n \rangle \) is either \( m_1, m_2 \) or \( R \). Let \( I(w) = |\langle a_1, \ldots, a_n \rangle| \). Hence \( I(w) = 3 \) or 9.

**Theorem 2.1** Let \( C \) be a code with minimal generating set \( \{w_1, w_2, \ldots, w_s\} \), then \( |C| = \prod_{i=1}^{s} I(w_i) \), where \( |C| \) mean the number of codewords in \( C \).

**Proof** The summations \( \sum a_i w_i \) are distinct when each \( a_i w_i \) is not zero and there are 9 choices for \( a_i \) if \( I(w_i) = 9 \) and there are 3 choices for \( a_i \) if \( I(w_i) = 3 \).

**Corollary 2.2** Let \( \{w_1, w_2, \ldots, w_k\} \) be a minimal generating set for a linear code \( C \) over \( \mathbb{R} \) where there are \( k_1 \) vectors having 0, 1, 2, \( 2v, a, b, c \) and \( d \) and \( k_2 \) vectors having only 0, \( b \) and \( c \) or only 0, \( a \) and \( d \) among them. Then \( |C| = 9^{k_1}3^{k_2} \).

In [4], it was shown that any code over \( \mathbb{R} \) is permutation equivalent to a code generated by the following matrix

\[
\begin{pmatrix}
I_{k_1} & (1-v)B_1 & (v+1)A_1 & (1-v)B_2 & (1-v)A_2 & (1+v)A_3 & (1-v)B_3 \\
0 & (1+v)I_{k_2} & 0 & (1+v)A_4 & 0 & & \\
0 & 0 & (1-v)I_{k_3} & 0 & & (1-v)B_4 & \\
\end{pmatrix}
\]

where \( A_i \) and \( B_i \) are ternary matrices over \( F_3 \), by the properties of Chinese Remainder Theorem. Such a code is said to have rank \( \{9^{k_1}, 3^{k_2}\} \). If \( H \) is a code over \( \mathbb{R} \), let \( H^+ \) (resp. \( H^- \)) be the ternary code such that \( (1+v)H^+ \) (resp. \( (1-v)H^- \)) is read \( H \) mod \( (1-v) \) (resp. \( H \) mod \( (1+v) \)).

In [4], it was obtained that,

\[ H = (1+v)H^+ \oplus (1-v)H^- \]

with

\[ H^+ = \{ s \mid \exists t \in F_3^n : (1+v)s + (1-v)t \in H \} \]

\[ H^- = \{ t \mid \exists s \in F_3^n : (1+v)s + (1-v)t \in H \} \]

The code \( H^+ \) is permutation equivalent to a code with generator matrix of the form...
where $A_i$ are ternary matrices for $i = 1, 2, 3, 4$ and ternary code $H^-$ is permutation equivalent to a code with generator matrix of the form

\[
\begin{pmatrix}
I_{k_1} & 2B_1 & 0 & 2B_2 & 2B_3 \\
0 & I_{k_2} & 0 & A_4 & 0
\end{pmatrix}
\]

where $B_i$ are ternary matrices for $i = 1, 2, 3, 4$ in [4].

In [3], the simplex codes over the ring $R$ of type $\alpha$ were constructed as the following:

Let $G_\alpha^k$ be a $k \times 3^{2k}$ matrix over $R$ defined inductively by,

\[
G_\alpha^k = \begin{pmatrix}
G_{\alpha}^{k-1} & 0 & \ldots & 0 \\
G_{\alpha}^{k-1} & 1 & \ldots & 1 \\
G_{\alpha}^{k-1} & 2 & \ldots & 2 \\
\vdots & \vdots & \ddots & \vdots \\
G_{\alpha}^{k-1} & G_{\alpha}^{k-1} & \ldots & G_{\alpha}^{k-1}
\end{pmatrix}
\]

where

\[
G_1^\alpha = \begin{pmatrix}
0 & 1 & 2 & v & 2v & a & b & c & d
\end{pmatrix}
\]

The columns of $G_\alpha^k$ consist of all distinct $k$-tuples over $R$. The code $S_\alpha^k$ generated by $G_\alpha^k$ has length $3^{2k}$, see [3].

In [3], it was shown that the minimum weights of $S_\alpha^k$ are $d_H = 6 \cdot 3^{2(k-1)}, d_L = 4 \cdot 3^{2k-1}$ and $d_B = 2 \cdot 3^{2k-1}$.

Now, some facts about ternary simplex codes, will be given.

Let $G(S_k)$ (columns consisting of all non zero ternary $k$-tuples) be a generator matrix for an $[n, k]$ ternary simplex code $S_k$. Then the extended ternary simplex code $\hat{S}_k$ generated by the matrix

\[
G(\hat{S}_k) = (0|G(S_k))
\]

Inductively,

\[
G(\hat{S}_k) = \begin{pmatrix}
0 \ldots 0 & 1 \ldots 1 & 2 \ldots 2 \\
G(\hat{S}_{k-1}) & G(\hat{S}_{k-1}) & G(\hat{S}_{k-1})
\end{pmatrix}
\]

with

\[
G(\hat{S}_1) = (012)
\]
Lemma 2.2 The $H^+$ (or $H^-$) ternary codes of $S^\alpha_k$ are equivalent to the $3^k$ copies of $\hat{S}_k$.

Proof. It will be proved by induction, firstly for $H^+$. Observe that the ternary $H^+$ code of $S^\alpha_k$ is the set of codewords obtained by replacing $a$ by 1 and $d$ by 2 in all $a$-linear combination of the rows of the matrix $aG_k$. For $k = 2$, the result holds.

$$G_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\end{pmatrix}$$

$$H^+ = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
\end{pmatrix}$$

If $aG_{k-1}$ is permutation equivalent $3^{k-1}$ copies of a $aG(\hat{S}_{k-1})$, then the matrix $aG_k$ takes the form

$$\begin{pmatrix}
0 & 0 & \cdots & 0 & a & a & a & \cdots & a & a & a & a & a & a \\
a & a & a & \cdots & a & a & a & a & \cdots & a & a & a & a & a \\
a & a & a & a & \cdots & a & a & a & a & \cdots & a & a & a & a \\
a & a & a & a & a & \cdots & a & a & a & a & \cdots & a & a & a \\
\end{pmatrix}$$

Regrouping the columns gives the desired result. The proof for the $H^-$ case is similar to the above case.

3. MacDonald codes of type $\alpha$

In [3], the simplex codes had been obtained. A simplex code $S^\alpha_k$ of type $\alpha$ is a linear $[3^{2k}, 2k, 6.3^{2(k-1)}, 4.3^{2k-1}, 2.3^{2k-1}]$ and inductive generator matrix given by

$$G^\alpha_k = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & 1 & 1 & 2 & \cdots & 2 & \cdots & 2 & \cdots & 2 & \cdots \\
G^\alpha_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}$$

with

$$G^\alpha_1 = \begin{pmatrix}
0 & 1 & 2 & v & 2v & a & b & c & d \\
\end{pmatrix}$$

We define the MacDonald codes via the generator matrices of simplex codes. Let $G^\alpha_{k,u}$ be the matrix obtained from $G^\alpha_k$ by deleting columns corresponding to the columns of $G^\alpha_u$, for $1 \leq u \leq k - 1$ i.e.

$$G^\alpha_{k,u} = \begin{pmatrix}
G^\alpha_k \\
0 \\
\end{pmatrix}$$
where 0 is a \((k-u) \times 3^{2u}\) zero matrix and \((A \setminus B)\) denotes the matrix obtained from the matrix \(A\) by deleting the columns of the matrix \(B\).

The code \(M_{k,u}^\alpha\) generated by the matrix \(G_{k,u}^\alpha\) is the punctured code of \(S_k^\alpha\) and is a MacDonald code.

\[ M_{k,u}^\alpha \text{ is a code of length } n = 3^{2k} - 3^{2u} \text{ and dimension } 2k_1 + k_2. \]

**Remark** We define \(H^+\) or \(H^-\) of \(M_{k,u}^\alpha\) as torsion code for the code \(M_{k,u}^\alpha\).

**Lemma 3.1** The Torsion code of \(M_{k,u}^\alpha\) is ternary linear \([3^{2k} - 3^{2u}, 2k_1 + k_2, \sum_{n=1}^{k-u} 6.8.3^{2u-2+(2n-2)}]\) code with weight distribution \(A_H(0) = 1, A_H(6,3^{2k-2}) = 3^{k-u} - 1\) and \(A_H(\sum_{n=1}^{k-u} 6.8.3^{2u-2+(2n-2)}) = 3^{k-u}(3^u - 1)\)

**Proof** First we will prove the \(H^+\) case by induction on \(k\). Since the \(H^+\) code of \(M_{k,u}^\alpha\) is the set of codewords obtained by replacing \(a\) by 1 and \(d\) by 2 in all \(\alpha\)-linear combination of the rows of the matrix \(aG_{k,u}^\alpha\). For \(k = 2\) and \(u = 1\) the result holds. Suppose that the result holds for \(k - 1\) and \(1 \leq u \leq k - 2\). Then for \(k\) and \(1 \leq k \leq k - 1\) the matrix \(aG_{k,u}^\alpha\) takes the form

\[
aG_{k,u}^\alpha = \left( \begin{array}{c} aG_{k}^\alpha \\ \frac{0}{aG_{k}^\alpha} \end{array} \right).
\]

Each non zero codeword of \(aM_{k,u}^\alpha\) has Hamming weight either \(6,3^{2k-2}\) or \(\sum_{n=1}^{k-u} 6.8.3^{2u-2+(2n-2)}\), then there will be \(3^{k-u} - 1\) codewords of hamming weight \(6,3^{2k-2}\) and the number of codewords with Hamming weight \(\sum_{n=1}^{k-u} 6.8.3^{2u-2+(2n-2)}\) is \(3^{k-u}(3^u - 1)\). The prove for the \(H^-\) case is similar to the above case

**Remark** Each of the first \(k-u\) rows has total number of units \(4.3^{2k-2}\) and total number of non-unit elements \(4.3^{2k-2}\). Each of the last \(u\) rows has total number of units \(\sum_{n=1}^{k-u} 4.8.3^{(2u-2)+(2n-2)}\) and total number of non-unit elements \(\sum_{n=1}^{k-u} 4.8.3^{(2u-2)+(2n-2)}\).

**Lemma 3.2** Let \(t \in M_{k,u}^\alpha, t \neq 0\). If at least one component of \(t\) elements is a unit then there are four type of codewords:

I. \(w_1(t) = w_2(t) = w_3(t) = w_4(t) = w_5(t) = w_6(t) = 3^{2k-2}, w_0(t) = 3^{2k-2} - 3^{2u}\)
II. \(w_1(t) = w_2(t) = w_3(t) = w_4(t) = w_5(t) = w_6(t) = 3^{2k-2}, w_u(t) = w_d(t) = w_0(t) = 3^{2k-2} - 3^{2u-1}\)
III. \(w_1(t) = w_2(t) = w_3(t) = w_4(t) = w_5(t) = w_6(t) = 3^{2k-2}, w_c(t) = w_0(t) = 3^{2k-2} - 3^{2u-1}\)
IV. \(w_0(t) = w_1(t) = w_2(t) = w_3(t) = w_4(t) = w_5(t) = w_6(t) = w_c(t) = 3^{2k-2} - 3^{2u-1}\)
\[ w_d(t) = 3^{2k-2} - 3^{2u-2} \]

otherwise

I. \( w_a(t) = w_d(t) = 3^{2k-1}, w_0(t) = 3^{2k-1} - 3^{2u} \)

II. \( w_c(t) = w_b(t) = 3^{2k-1}, w_0(t) = 3^{2k-1} - 3^{2u-1} \)

III. \( w_a(t) = w_d(t) = w_0(t) = 3^{2k-1} - 3^{2u-1} \)

IV. \( w_c(t) = w_b(t) = w_0(t) = 3^{2k-1} - 3^{2u-1} \)

\[ \text{Proof} \]

By induction on \( k \).

**Theorem 3.3** The Hamming and Lee weight distributions of \( M_{k,u}^\alpha \) are

\[
\begin{align*}
A_H(0) &= 1 \\
A_H(8.3^{2k-2}) &= 4 \\
A_H(6.3^{2k-2} + 2(3^{2k-2} - 3^{2k-1})) &= 4(3^{2k-2} - 3) \\
A_H(8(3^{2k-2} - 3^{2u-2})) &= 3(3^{2k-2} + 3) \\
A_H(2.3^{2k-1}) &= 4 \\
A_H(2(3^{2k-1} - 3^{2u-1})) &= 2(3^{2k-2} - 3)
\end{align*}
\]

\[
\begin{align*}
A_L(0) &= 1 \\
A_L(4.3^{2k-2} + 4.2.3^{2k-2}) &= 3^{2(k-u)} - 1 \\
A_L(4(3^{2k-2} - 3^{2u-2}) + 4.2(3^{2k-2} - 3^{2u-2})) &= 3^{2k-2} - 3^{2u} - 1
\end{align*}
\]

**Proof** By Lemma 3.2, each non-zero codeword of \( M_{k,u}^\alpha \) has Hamming weight either \( 8.3^{2k-2}, 6.3^{2k-2} + 2(3^{2k-2} - 3^{2k-1}), 8(3^{2k-2} - 3^{2u-2}), (2.3^{2k-1}) \) or \( 2(3^{2k-1} - 3^{2u-1}) \) and Lee weight either \( (4.3^{2k-2} + 4.2.3^{2k-2}) \) or \( 4(3^{2k-2} - 3^{2u-2}) + 4.2(3^{2k-2} - 3^{2u-2}) \). The method for counting the weight are similar to one used for \( S_k^\alpha \) in [3]

**References**


