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On Primary Compactly Packed Modules Over Noncommutative Rings

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Abstract: This research considers three basic concepts concerning primary submodules, which are important, at least to some mathematicians.

The first concept is the concept of primary radical of a submodule, the second concept is the concept of minimal primary submodules, and the third one is the concept of primary compactly packed modules.

In this paper, we generalize these three concepts over noncommutative rings and generalize many results concerning them, in the rings that are not necessary commutative.

المقاسات الابتدائية المحزومة الرص المعرفة على الحلقات غير الابدالية

ملخص: تهتم هذه الدراسة بثلاثة مفاهيم رئيسية تتعلق بالمقاسات الجزئية الابتدائية، و التي تعتبر ذات أهمية كبرى، على الأقل لبعض الرياضيين.

المفهوم الأول هو مفهوم الجذر الابتدائي للمقاسات الجزئية، المفهوم الثاني هو مفهوم المقاسات الجزئية الابتدائية الأصغرية. أما المفهوم الثالث فهو يتعلق بالمقاسات الابتدائية المحزومة الرص.

نقوم في هذا البحث بتعميم المفاهيم الثلاثة السابقة على الحلقات غير الابدالية و نعمم العديد من النتائج المتعلقة بها .

1. Introduction

Primary modules have long been known in commutative rings for their center stage role in some theories in commutative algebra. As long as for their importance in generalizing many concepts associated prime submodules such as the concepts of prime radical of a submodule, and compactly packed modules.

More recently, primary submodules have been introduced in noncommutative rings in the hope of extending and generalizing the concepts associated primary submodules over noncommutative rings.

Let R be an arbitrary ring. A nonzero submodule N of an R-module M is primary

if for every nonzero submodule

, where $rann\ (N) = rann\ (\overline{N})$ $\overline{N} \subseteq N$,

for some positive integer n }. This definition $r^n N = 0$, $rann(N) = \{r \mid r \in R$

of a primary submodule over noncommutative rings was introduced recently in [1]. In fact, this definition is a generalization of the definition of prime submodules in an arbitrary ring and an extension of the definition of primary submodules over commutative rings, see[1]. The definition of primary submodules over noncommutative rings was a motive for us to generalize many concepts and results associated to prime submodules over noncommutative rings, and primary submodules over commutative rings.

In the beginning of this paper , in Section 2, we introduce the definition of primary radical of a submodule as follows: Let N be a submodule of an R-module M. If there exist primary submodules containing N , then the intersection of all primary submodules containing N is called the primary radical of N and is denoted by *prad*(N). If there is no primary submodule containing N , then *prad*(N)=M. In particular *prad*(M)=M.

Also in this section, we study some properties of primary radical of a submodule.

In[2], and [3], Chin Pi Lu proved some results on minimal prime submodules in commutative rings. We generalize the concept of minimal prime submodules to the concept of minimal primary submodules over commutative rings, see[4],[5].

In Section three we define the minimal primary submodules over non commutative rings. Thus we define a primary submodule Q of an R-module M to be a minimal primary over a submodule N if $N \subseteq Q$ and we show that there is no smaller primary submodule with this property. We also prove some results concerning minimal primary submodules over noncommutative rings.

Key Words: Primary submodules over noncommutative rings, primary radical of a submodule over noncommutative rings, minimal primary submodules over noncommutative rings, primary compactly packed modules over noncommutative rings.

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The compactness property of prime ideals was studied by Ries, C.M. and Viswanthan, T.M. in 1970, see[6]. Then in 1995, Yong Hwan Cho., defined the comprimely packed rings, see[7]. The previous studies, in [6] and [7], were generalized to modules in 2002, on the rings that are commutative, and the concept of compactly packed modules was introduced, see[8]. Then, we generalized these results on primary modules, and defined the primary compactly packed modules over commutative rings in [4],[5],[9]. In Section 4, we study more properties of primary radical of a submodule, and introduce the concept of primary compactly packed modules over any arbitrary ring. We define a proper submodule N of an R-module M to be primary compactly packed if for each family $\{P_a\}_{a\in l}$ of primary submodules of M with $N \subseteq \bigcup_{a\in l} P_a$, $N \subseteq P_b$ for some $b \in l$. Moreover, the module M is called primary compactly packed if every submodule of M is primary compactly packed. Also, we give equivalent definitions of primary compactly packed modules and study some various properties of primary compactly packed modules.

Finally, we remark that throughout this paper, we will work exclusively with left unitary modules and all rings are assumed to be rings with identity.

2. Primary Radical of Submodules

We start this section by the definition of primary radical of a submodule, as follows.

Definition 2.1 Let N be a submodule of an R-module M. If there exist primary submodules containing N, then the intersection of all primary submodules containing N is called the primary radical of N and is denoted by *prad*(N). If there is no primary submodule containing N, then *prad*(N)=M. In particular *prad*(M)=M.

We say that a submodule N is a primary radical submodule if *prad*(N)=N.

Examples 2.2

1) It is clear that every primary submodule is primary radical submodule.

2) Let R=$\boldsymbol{Z}$, the set of integers. Since every ideal of R is a submodule of R, primary ideals of R are primary submodules of R. So for $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, where p_i\`s are prime numbers, and since $(n) = \bigcap_{i=1}^{k} (p_i^{a_i})$, then in $\boldsymbol{Z}$, every ideal is primary radical submodule of R.

The following result can be easily noticed.

Proposition 2.3 Let N and L be submodules of an R-module M. Then

1) $N \subseteq prad(N)$.

2) If $N \subseteq L$, then $prad(N) \subseteq prad(L)$.

3) $prad(prad(N)) = prad(N)$; that is, the primary radical of N is a primary
radical submodule.

4) $prad(N \cap L) \subseteq prad(N) \cap prad(L)$.

5) $prad(N + L) = prad(prad(N) + prad(L))$.

Theorem 2.4 For any R-module M, if M satisfies the ACC for primary radical submodules, then every primary radical submodule of M is the primary radical of a finitely generated submodule.

Proof. Assume that there exists a primary radical submodule N which is not primary radical of a finitely generated submodule. Let $e_1 \in N$ and let $N_1 = prad(e_1 R)$. Then $N_1 \subseteq N$. So there exists $e_2 \in N - N_1$. Let $N_2 = prad(e_1 R + e_2 R)$. Then $N_1 \subset N_2 \subset N$. So that there exists $e_3 \in N - N_2$. Continuing in this process, we will have an ascending chain of primary radical submodules $N_1 \subset N_2 \subset N_3 \subset \ldots$,
which is a contradiction.

The following Theorem follows from Example 2.2 (1).

Theorem 2.5 If every primary radical submodule is the primary radical of a finitely generated submodule, then every primary submodule is the primary radical of a finitely generated submodule.

Proposition 2.6 Let N and L be submodules of an R-module M such that whenever $N \cap L \subseteq Q$, we have $N \subseteq Q$ or $L \subseteq Q$ for any primary submodule Q of M. Then $prad(N \cap L) = prad(N) \cap prad(L)$.

Proof. By part 4 of Proposition 2.3, $prad(N \cap L) \subseteq prad(N) \cap prad(L)$. Now if $prad(N \cap L) = M$, then clearly $prad(N) = prad(L) = M$ and so $prad(N \cap L) = prad(N) \cap prad(L)$. If $prad(N \cap L) \neq M$, then there exists a primary submodule Q such that $(N \cap L) \subseteq Q$. By hypothesis, $N \subseteq Q$ or $L \subseteq Q$, so that $prad(N) \subseteq Q$ or $prad(L) \subseteq Q$. Since this is true for all primary submodule Q containing $N \cap L$, then $(prad(N) \cap prad(L)) \subseteq prad(N \cap L)$ and therefore $prad(N \cap L) = prad(N) \cap prad(L)$.

We generalize Proposition 2.6 as follows.

Proposition 2.7 Let $N_1, N_2, \ldots, N_k$ be submodules of an R-module M such that whenever $N_1 \cap N_2 \cap \ldots \cap N_k \subseteq Q$, we have $N_i \subseteq Q$ for some i

= 1,2,…, k, for any primary submodule Q of M. Then $prad(\bigcap_{i=1}^{k} N_i) = \bigcap_{i=1}^{k} prad(N_i)$.

3. Minimal Primary submodules

We define the minimal primary submodules over noncommutative rings as follows.

Definition 3.1 A primary submodule Q of an R-module M is called a minimal primary submodule over a submodule N if $N \subseteq Q$ and there is no smaller primary submodule with this property. Thus a primary submodule Q is a minimal primary submodule of an R-module M if it does not strictly contain any other primary submodule.

Lemma 3.2 Let $\{Q_i\}_{i\in I}$ be a nonempty family of primary submodules of an R-module M . Then either $\bigcap_{i\in I} Q_i = \{0\}$ or $\bigcap_{i\in I} Q_i$ is a primary submodule of the R-module M .

Proof. It is easy to show that $\bigcap_{i\in I} Q_i$ is a submodule of M . Now, suppose that $\bigcap_{i\in I} Q_i \neq \{0\}$. Let N be a nonzero submodule of $\bigcap_{i\in I} Q_i$. Then N is a nonzero submodule of $Q_i, \forall i \in I$. Since Q_i is primary $\forall i \in I$, then $rann(N)=rann(Q_i)$, $\forall i \in I$. Now, $r \in rann(N)$ if and only if $r \in rann(Q_i)$ if and only if $\forall i \in I$ there exists a positive integer n_i such that $r^{n_i} Q_i = 0$ if and only if $r^s(\bigcap_{i\in I} Q_i) = 0$, where $s = \sum_{i\in I} n_i$ if and only if $r \in rann(\bigcap_{i\in I} Q_i)$. Thus $rann(N)=rann(\bigcap_{i\in I} Q_i)$, and $\bigcap_{i\in I} Q_i$ is primary.

Theorem 3.3 If an R-module M satisfies the ACC on submodules, and $0 \neq A$ is a submodule of M that is contained in a primary submodule Q of M , then Q contains a minimal primary submodule over A.

Proof. Denote by Ω, the set of all primary submodules which contain A, and are contained in Q. Then $Q \in \Omega$, and therefore Ω is nonempty. If $\overline{Q}$ and $\overline{\overline{Q}}$ belong to Ω, then we write $\overline{Q} \leq \overline{\overline{Q}}$ if $\overline{\overline{Q}} \subseteq \overline{Q}$. This gives a partial order on Ω. We shall prove that Ω is an inductive system. Let Σ be a nonempty totally ordered subset of Ω. Let $\overline{Q}$ be the intersection of all members of Σ. By the previous Lemma, $\overline{Q}$ is a primary submodule of M , or $\overline{Q}$=0. Since $0 \neq A \subseteq \overline{Q} \subseteq Q$, then $\overline{Q}$ is primary, and $\overline{Q} \in \Omega$. Also since $\overline{Q} \subseteq B$ for every $B \in \Sigma$, we have $B \leq \overline{Q}$ for every $B \in \Sigma$. Thus $\overline{Q}$ is an upper bound for Σ. Therefore, Ω is an inductive system. By Zorn's Lemma, Ω contains a maximal element Q^*. Since $Q^* \in \Omega$, it is primary submodule with $A \subseteq Q^* \subseteq Q$. Suppose now that Q_1 is a primary submodule satisfying $A \subseteq Q_1 \subseteq Q$. Then $Q_1 \in \Omega$ and $Q^* \leq Q_1$. Consequently, since Q^* is maximal in Ω, $Q^* = Q_1$. This shows that Q^* is a minimal primary submodule of A and completes the proof.

4. Primary Compactly Packed Modules

Now, we can generalize the concept of primary compactly packed modules that was introduced in [4] over the rings that are not necessary commutative as follows.

Definition 4.1 A proper submodule N of a unitary R-module M is primary compactly packed if for each family $\{P_a\}_{a\in l}$ of primary submodules of M with $N \subseteq \bigcup_{a\in l} P_a$, $N \subseteq P_b$ for some $b \in l$. Moreover, the module M is called primary compactly packed if every submodule of M is primary compactly packed.

Theorem 4.2 Let M be an R-module. Then the following statements are equivalent.

a) M is primary compactly packed.
b) For each proper submodule N of M, there exists $a \in N$ such that $prad(N) = prad(Ra)$.
c) For each proper submodule N of M, if $\{N_a\}_{a\in l}$ is a family of submodules of M and $N \subseteq \bigcup_{a\in l} N_a$, then $N \subseteq prad(N_b)$ for some $b \in l$.
d) For each proper submodule N of M, , if $\{N_a\}_{a\in l}$ is a family of primary radical submodules of M and $N \subseteq \bigcup_{a\in l} N_a$, then $N \subseteq N_b$ for some $b \in l$.

Proof. (a$\rightarrow$b) : Let N be a proper submodule of M, it is clear that $prad(Ra) \subseteq prad(N)$ for each $a \in N$. Suppose that $prad(N) \not\subset prad(Ra)$ for each $a \in N$. Then for each $a \in N$, there exists a primary submodule P_a for which $Ra \subseteq P_a$ and $N \not\subset Pa$. However, $N = \bigcup_{a \in N} Ra \subseteq \bigcup_{a \in N} P_a$; that is, M is not primary compactly packed, which is a contradiction.

(b$\rightarrow$c): Let N be a proper submodule of M, and let $\{N_a\}_{a \in l}$ be a family of submodules of M such that $N \subseteq \bigcup_{a \in l} N_a$. By (b) , there exists $a \in N$ such that $prad(N) = prad(Ra)$. Then $a \in \bigcup_{a \in l} N_a$ and hence $a \in N_b$ for some $b \in l$, so that $Ra \subseteq N_b$ for some $b \in l$, and hence , $N \subseteq prad(N) = prad(Ra) \subseteq prad(N_b)$ for some $b \in l$.

(c$\rightarrow$d) : Let N be a proper submodule of M and let $\{N_a\}_{a \in l}$ be a family of primary radical submodules of M such that $N \subseteq \bigcup_{a \in l} N_a$. Then by (c), there exists $b \in l$ such that $N \subseteq prad(N_b)$. Since N_b is primary radical submodule of M, then $N \subseteq N_b$.

(d$\rightarrow$a) Let N be a proper submodule of M and suppose that $\{N_a\}_{a \in l}$ is a family of submodules of M that satisfies $N \subseteq \bigcup_{a \in l} N_a$. Since N_a is primary submodule of M for each $a \in l$, $N_a = prad(N_a)$ for each $a \in l$. Thus $N \subseteq \bigcup_{a \in l} N_a = \bigcup_{a \in l} prad(N_a)$. By (d), there exists $b \in l$ such that $N \subseteq prad(N_b) = N_b$. Thus M is primary compactly packed.

Theorem 4.3 If M is primary compactly packed which has at least one maximal submodule, then M satisfies the ACC on primary radical submodules.

Proof. Let $N_1 \subseteq N_2 \subseteq \ldots$ be an ascending chain of primary radical submodules of M and let $L = \bigcup_i N_i$. If $L = M$ and H is a maximal submodule of M, then $H \subset \bigcup_i N_i$. Since M is primary compactly packed, by Theorem 4.2, $H \subseteq N_j$ for some j. Therefore, by maximality of H, $H = N_j$ for some j. Since $N_j \subseteq N_{j+n} \subseteq \bigcup_i N_i$, $n = 1,2,\ldots$, and N_j is maximal, either $N_j = N_{j+n}$ for every $n = 1,2,\ldots$, thus $N_j = \bigcup_i N_i = M$, which is impossible, or $N_{j+n} = \bigcup_i N_i = M$, which is also impossible. Thus L is a proper submodule of M. Since M is primary compactly packed, by Theorem 4.2, $L \subseteq N_j$ for some j and hence $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_j = N_{j+1} = N_{j+2} = \ldots$, therefore the ACC is satisfied on primary radical submodules.

Theorem 4.4 Let $\Phi : M \to \overline{M}$ be an R-module isomorphism. If M is primary compactly packed, then $\overline{M}$ is primary compactly packed.

Proof. Let M be primary compactly packed, and suppose that $\overline{N} \subseteq \bigcup_{a \in l} K_a$ where $\overline{N}$ is a proper submodule of $\overline{M}$ and K_a is a primary submodule of $\overline{M}$ for each $a \in l$. Since Φ is an R-module isomorphism, then $\Phi^{-1}(\overline{N}) \subseteq \Phi^{-1}(\bigcup_{a \in g} K_a) = \bigcup_{a \in g}(\Phi^{-1}(K_a))$

Since K_a is a primary submodule of $\overline{M}$ for each $a \in l$, by [1], $\Phi^{-1}(K_a)$ is a primary submodule of M for each $a \in l$. But M is primary compactly packed. Thus

there exists $b \in l$ such that $\Phi^{-1}(\overline{N}) \subseteq \Phi^{-1}(K_b)$. Therefore $\overline{N} \subseteq K_b$ for some $b \in l$, and hence $\overline{N}$ is primary compactly packed.

Thus $\overline{M}$ is primary compactly packed.

The following definition was introduced in the rings that are commutative, see[9].

We give a generalization to it in noncommutative rings.

Definition 4.5 A module M of an arbitrary ring R is called a Bezout module if every finitely generated submodule of M is cyclic.

Theorem 4.6 Let M be a Bezout module. If M satisfies the ACC on primary radical submodules, then M is primary compactly packed.

Proof. Let N be a proper submodule of M. By Proposition 2.3, *prad* (N) is a primary radical submodule; hence, by Theorem 2.4, there exists afinitely generated submodule L of M such that *prad* (N)=*prad* (L) and hence L is a cyclic submodule of M, because M is Bezout. It follows from Theorem 4.2 that M is primary compactly packed.

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