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On Almost 2-Absorbing Primary Sub-modules

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Abstract

Let R be a commutative ring with identity and M be a unitary R-module, In this paper we introduce the concept of almost 2-absorbing primary sub-modules as a new generalization of 2-absorbing sub-modules. We study some basic properties of almost 2-absorbing primary sub-modules and give some characterizations of them, especially for (finitely generated faithful) multiplication modules.

Keywords:
2-absorbing sub-modules,
Weakly 2-absorbing sub-modules,
Almost 2-absorbing sub-modules,
Almost 2-absorbing primary sub-modules,

1. Introduction:

Throughout this article, we consider all rings as commutative rings with identity and all modules as unital. For any two sub-modules N and K of an R-module M, the residual of N by K is defined as the set $(N:K)=\{r\in R : rK\subseteq N\}$ which is clearly an ideal of R. In particular, the ideal $(0:M)$ is called the annihilator of M and is denoted by *Ann zero* ideal of R. Let N be a sub-module of M and I be an ideal of R. The residual sub-module of N by I is defined as $(N:I)=\{m\in M : I(m)\subseteq N\}$. These two residual ideal and sub-module were proved to be useful in studying many concepts of modules, see, for example, (Anderson, 2000; Bast and Smith, 1988; Naom and Hasan, 1986).

Recall that a proper sub-module N of an R-module M is a 2-absorbing (resp. weakly 2-absorbing) sub-module of M (Darani and Soheilnia, 2011) if, whenever $abm\in N$ (resp. $0\neq abm\in N$) for $a,b\in R$ and $m\in M$, then $am\in N$ or $bm\in N$ or $ab\in(N:M)$. A proper sub-module K is maximal in M if there is no proper sub-module of M containing K properly. A local module is a module with unique maximal sub-module. An R-module M is called a multiplication module provided that, for every sub-module N of M, there exists an ideal I of R so that $N=IM$ (or equivalently, $N=(N:M)M$). Multiplication modules were studied extensively in (Bast and Smith, 1988; Low and Smith, 1990; Smith, 1988). An R-module M is called a cancellation module of R if, for all ideals I and J of R, $IM=JM$ implies that $I = J$.

The class of 2-absorbing sub-modules of modules was introduced as a generalization of the class of 2-absorbing ideals of rings. Then, many generalizations of 2-absorbing sub-modules were studied such as weakly 2-absorbing, primary 2-absorbing (Dubey and Aggarwal, 2015), classical 2-absorbing (Mostafanasab and Tekir, 2015) and almost 2-absorbing (Ashour, Al-Ashker and Naji, 2016). In this article, we introduce the concept of almost 2-absorbing primary sub-modules as one of the generalizations of 2-absorbing (and weakly 2-absorbing) sub-modules. We generalize some basic properties of 2-absorbing and weakly 2-absorbing to almost 2-absorbing primary sub-modules. In particular, we give characterizations of almost 2-absorbing primary sub-modules in multiplication modules.

2. Some Properties of Almost 2-absorbing Primary Sub-modules:

Definition: A proper ideal I of R is called an almost 2-absorbing primary ideal if $a,b,c \in R$ with $abc \in I - I^2$ implies that $ab \in I$ or $ac \in \sqrt{I}$, or $bc \in \sqrt{I}$.

Definition: A proper sub-module N of an R-module M is called an almost 2-absorbing primary sub-module of M if, whenever $a,b \in R$ and $m \in M$ such that $abm \in N - (N : M)N$, implies that $ab \in \sqrt{(N : M)}$ or $am \in N$, or $bm \in N$.

Definition: Let M be an R-module and N be a proper sub-module of M. Then N is said to be a weakly 2-absorbing primary sub-module of M if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$.

It is clear that, any 2-absorbing primary sub-module is weakly 2-absorbing primary and any weakly 2-absorbing primary sub-module is almost 2-absorbing primary. However, the converses are not necessarily true.

Example: (1) Consider the Z-module Z_{30}, $N = \{0\}$ is a weakly 2-absorbing primary sub-module but is not 2-absorbing primary, because $0 = 2.3.5 \in N$, but $2.5 \notin N$, $3.5 \notin N$ and $2.3 \notin \sqrt{(N : M)} = \{0\}$.

(2) Let N be any sub-module that is not 2-absorbing primary of R-module M such that $(N : M)N = N$, then N is almost 2-absorbing primary but not 2-absorbing primary.

Remark: Since $(N : M) \subseteq \sqrt{(N : M)}$ for any sub-module N of an Rmodule M, then any almost 2-absorbing sub-module is an almost 2-absorbing primary sub-module of M.

Proposition 2.1: *Let N be a sub-module of an R-module M, and $(N : M)$ be a radical ideal in R, then N is almost 2-absorbing primary if and only if N is almost 2-absorbing sub-module.*

Proof: (⇒) Suppose N is almost 2-absorbing primary sub-module, let $a,b \in R$, $m \in M$ with $abm \in N - (N : M)N$, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)} = (N : M)$ (since $(N : M)$ is radical), hence N is almost 2-absorbing sub-module.
(⇐) It is trivial.

Corollary 2.2: *Let N be a sub-module of an R-module M, and $(N : M)$ be a prime ideal in R, then N is almost 2-absorbing primary if and only if N is almost 2-absorbing sub-module.*

Proof: By Proposition 2.1, since every prime ideal is radical.

Theorem 2.3: *Let N, K be R-sub-modules of M with $K \subseteq N$. If N is an almost 2-absorbing primary sub-module of M then N/K is an almost 2-absorbing primary R-sub-module of M/K.*

Proof: Let $a, b \in R$ and $m + K \in M/K$ such that $ab(m + K) \in (N/K) - (N/K : M/K)N/K$. Since $(N : M) = (N/K : M/K)$ then, $abm + K \in N/K - (N : M)N/K$ and so $abm \in N - (N : M)N$. As N is almost 2-absorbing primary in M, then $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N : M)}$. Therefore, $a(m + K) \in N/K$ or $b(m + K) \in N/K$ or $ab \in \sqrt{(N/K : M/K)}$, and hence N/K is almost 2-absorbing primary in M/K.

Proposition 2.4: *Let N be an almost 2-absorbing primary sub-module of R-module M. If S is a multiplicatively closed subset of R, then $S^{-1}N$ is almost 2-absorbing primary sub-module in R-module $S^{-1}M$.*

Proof: Let $a, b \in R$, $s \in S$ and $m \in M$ such that $ab(\frac{m}{s}) \in S^{-1}N - (S^{-1}N : S^{-1}M)S^{-1}N$. Then, $\frac{abm}{s} \in S^{-1}N - S^{-1}((N : M)N)$. Indeed, if $\frac{abm}{s} \in S^{-1}((N : M)N)$, then there is $t \in S$ such that

$\frac{abm}{s} = \frac{r_1n_1+r_2n_2+\ldots+r_kn_k}{t} = r_1\frac{n_1}{t}+r_2\frac{n_2}{t}+\ldots+r_k\frac{n_k}{t}$ where $r_i \in (N : M)$ and $n_i \in N$, $i = 1, 2,\ldots, k$. Thus, $\frac{abm}{s} \in (N : M)(S^{-1}N) \subseteq (S^{-1}N : S^{-1}M)S^{-1}N$, which is a contradiction. As $\frac{abm}{s} \in S^{-1}N$, there is $t \in S$, such that $ab(tm) \in N - (N : M)N$, since N is almost 2-absorbing primary then $a(tm) \in N$ or $b(tm) \in N$ or $ab \in \sqrt{(N : M)} \subseteq \sqrt{(S^{-1}N : S^{-1}M)}$ and hence $\frac{atm}{ts} = a\frac{m}{s} \in S^{-1}N$ or $\frac{btm}{ts} = b\frac{m}{s} \in S^{-1}N$ or $ab \in \sqrt{(S^{-1}N : S^{-1}M)}$.

Proposition 2.5: *Let Q be a sub-module of R-module M, N be an any R-module. If $Q \oplus N$ is an almost 2-absorbing primary sub-module of $M \oplus N$ then Q is an almost 2-absorbing primary sub-module of M.*

Proof: Suppose $Q \oplus N$ is almost 2-absorbing primary sub-module in $M \oplus N$. Let $a, b \in R$, $m \in M$ such that $abm \in Q - (Q : M)Q$. Then we get $ab(m, 0) \in (Q \oplus N) - (Q \oplus N : M \oplus N)(Q \oplus N)$. Since $Q \oplus N$ is almost 2-absorbing primary, then $ab \in \sqrt{(Q \oplus N : M \oplus N)}$ or $a(m, 0) \in Q \oplus N$ or $b(m, 0) \in Q \oplus N$, that is, $am \in Q$ or $bm \in Q$ or $ab \in \sqrt{(Q : M)}$. Hence, Q is almost 2-absorbing primary sub-module in M.

If N is a sub-module of R-module M and r ∈ R then a sub-module N_r of M is defined by $N_r = (N : r) = \{m \in M : rm \in N\}$.

Theorem 2.6: Let M be an R-module and N be a proper sub-module of M. The following are equivalent :

(1) N is an almost 2-absorbing primary sub-module.

(2) For $a, b \in R$ such that $ab \notin \sqrt{(N : M)}$, $N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$

Proof: (1 → 2) Let N be an almost 2-absorbing primary sub-module, and assume that $ab \notin \sqrt{(N : M)}$, let $m \in N_{ab}$ then $abm \in N$. If $abm \notin (N : M)N$ then $am \in N$ or $bm \in N$ and hence $m \in N_a$ or $m \in N_b$. If $abm \in (N : M)N$ then $m \in [(N : M)N]_{ab}$. The other containment holds for any sub-module.

(2 → 1) Let $a, b \in R$ and m ∈ M with $abm \in N - (N : M)N$. Assume that $ab \notin \sqrt{(N : M)}$ then $m \in N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$, but $abm \notin (N : M)N$ then $m \in N_a$ or $m \in N_b$, thus $am \in N$ or $bm \in N$.

Proposition 2.7: Let M be an R-module and N be a proper sub-module of M, then N is an almost 2-absorbing primary sub-module in M if and only if for any $a, b \in R$ and sub-module K of M such that, $abK - \{0\} \subseteq N - (N : M)N$ we have $ab \in \sqrt{(N : M)}$ or $aK \subseteq N$ or $bK \subseteq N$.

Proof: (→) Assume that $ab \notin \sqrt{(N : M)}$, then by Theorem 2.6, $K \subseteq N_{ab} = N_a \cup N_b \cup [(N : M)N]_{ab}$, but $abK \not\subset (N : M)N$ then $K \subseteq N_a$ or $K \subseteq N_b$ and hence $aK \subseteq N$ or $bK \subseteq N$.

(←) Suppose that $abm \in N - (N : M)N$ for $a, b \in R$ and m ∈ M. Then, $ab(m) - \{0\} \subseteq N - (N : M)N$ and so $ab \in \sqrt{(N : M)}$ or $a(m) \subseteq N$ or $b(m) \subseteq N$. Therefore, $ab \in \sqrt{(N : M)}$ or $am \in N$ or $bm \in N$, thus N is almost 2-absorbing primary.

Lemma 2.8: (Bataineh, 2011) Let M be an R-module and let N be a proper sub-module of M. Then $(N/((N:M)N) : M/((N : M)N)) = (N:M)$.

Theorem 2.9: Let M be an R-module and N be a proper sub-module of M, then N is almost 2-absorbing primary sub-module in M if and only if $N/(N : M)N$ is weakly 2-absorbing primary sub-module in $M/(N : M)N$.

Proof: (→) Suppose N is almost 2-absorbing primary sub-module in M. Let $a, b \in R$, m ∈ M such that $0 \neq ab(m + (N : M)N) \in N/(N : M)N$.

Then $abm \in N - (N:M)N$, and so $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N:M)}$ since N is almost 2-absorbing primary. So $ab \in \sqrt{(N/(N:M)N : M/(N:M)N)} = \sqrt{(N:M)}$ (by Lemma 2.8) or $a(m + (N:M)N) \in N/(N:M)N$ or $b(m + (N:M)N) \in N/(N:M)N$. Hence $N/(N:M)N$ is weakly 2-absorbing primary in $M/(N:M)N$.

($\leftarrow$) Assume $N/(N:M)N$ is weakly 2-absorbing primary in $M/(N:M)N$. Let $a, b \in R$, $m \in M$ such that $abm \in N - (N:M)N$. So $0 \neq ab(m + (N:M)N) \in N/(N:M)N$. Then we have $ab \in \sqrt{(N/(N:M)N : M/(N:M)N)} = \sqrt{(N:M)}$ or $a(m + (N:M)N) \in N/(N:M)N$ or $b(m + (N:M)N) \in N/(N:M)N$. That is, $am \in N$ or $bm \in N$ or $ab \in \sqrt{(N:M)}$. Hence N is almost 2-absorbing primary sub-module in M.

3. Almost 2-absorbing primary sub-module of multiplication modules:

If M is a multiplication R-module and $N = IM$, $K = JM$ are two sub-modules of M, where I, J are ideals in R, then the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$. In particular, we have $N^2 = ((N:M)M)((N:M)M) = (N:M)^2 M$. Let N be a sub-module of R-module M, N is called idempotent in M if $(N:M)N = N$. Obviously, every idempotent sub-module is almost 2-absorbing primary sub-module in M. Before giving Theorem 3.2 we need the following Lemma.

Lemma 3.1: (Khashan, 2012) Let N be a sub-module of a finitely generated faithful multiplication (and so cancellation) R-module M. Then, we have $(IN:M) = I(N:M)$ for every ideal I of R.

Theorem 3.2: Let M be a finitely generated faithful multiplication R-module and N be a proper sub-module of M. The following are equivalent :

(1) N is almost 2-absorbing primary sub-module in M.
(2) (N : M) is almost 2-absorbing primary ideal in R.
(3) N = QM for some almost 2-absorbing primary ideal Q of R.

Proof: (1 $\rightarrow$ 2) Suppose N is almost 2-absorbing primary sub-module and let $a, b, c \in R$ such that $abc \in (N:M) - (N:M)^2$. Then, $abcM - \{0\} \subseteq N - (N:M)N$. Indeed, if $abcM \subseteq (N:M)N$, then by Lemma (3.1), $abc \in ((N:M)N : M) = (N:M)^2$, a contradiction. Now , since N is almost 2-absorbing primary sub-module then by Proposition 2.7 we have $ab \in \sqrt{(N:M)}$ or $acM \subseteq N$ or $bcM \subseteq N$ (and so $ac \in (N:M)$ or $ab \in (N:M)$). Hence, $(N:M)$ is almost 2-absorbing primary ideal in R.

($2 \rightarrow 1$) Let $a, b \in R$ and $m \in M$, such that $abm \in N - (N:M)N$. Then, $ab((m):M) \subseteq ((abm):M) \subseteq (N:M)$. Moreover, $ab((m):M) \not\subset (N:M)^2$ because otherwise, if $ab((m):M) \subseteq (N:M)^2 \subseteq ((N:M)N:M)$, then, $ab(m) = ab((m):M)M \subseteq (N:M)N$, a contradiction. As $(N:M)$ is almost 2-absorbing primary ideal in R, then, $ab \in (N:M)$ or $a((m):M) \subseteq (N:M)$ or $b((m):M) \subseteq (N:M)$ [by Proposition 2.7]. In the second case, we obtain $(am) \subseteq a(m) = a((m):M)M \subseteq (N:M)M = N$ and so $am \in N$, similarly we have $bm \in N$. Thus N is an almost 2-absorbing primary sub-module in M.

($2 \leftrightarrow 3$) We choose Q = (N : M).

Proposition 3.3: Let M be a local multiplication R-module with a unique maximal sub-module Q and $(Q:M)Q = 0$, then any proper sub-module of M is almost 2-absorbing primary if and only if it is weakly 2-absorbing primary .

Proof: ($\rightarrow$) For any proper sub-module N of M, $N \subseteq Q$, $(N:M)N = 0$, because $(Q:M)Q = 0$. Whenever $a, b \in R$, m ∈ M such that $abm \in N - (N:M)N$ we

have $0 \neq abm \in N$. So if N is almost 2-absorbing primary, then it is weakly 2-absorbing primary in M.
($\leftarrow$) It is trivial, because any weakly 2-absorbing primary sub-module is almost 2-absorbing primary.

Lemma 3.4: (Ali, 2015) Let N be a sub-module of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then,

1) $N = (IN : I)$.
2) If $N \subseteq IM$, then $(JN : I) = J(N : I)$ for any ideal J of R.
3) $(N : I) = ((N : M) : I)M = (N : IM)M$.

Theorem 3.5: Let N be a sub-module of a faithful multiplication R-module M and I be a finitely generated faithful multiplication ideal of R. Then, N is an almost 2-absorbing primary sub-module of IM if and only if $(N : I)$ is almost 2-absorbing primary in M.

Proof: Suppose that N is almost 2-absorbing primary sub-module in IM. Let $a, b \in R$ and $m \in M$, such that $abm \in (N : I) - ((N : I) : M)(N : I)$. Then, $abIm - \{0\} \subseteq N - (N : IM)N$. In fact, if $abIm \subseteq (N : IM)N$, then by Lemma 3.4, $abm \in ((N : IM)N : I) = (N : IM)(N : I) = ((N : I) : M)(N : I)$, a contradiction. As N is almost 2-absorbing primary sub-module in IM, then Proposition 2.7, with $K = \mathrm{Im}$ we have $aIm \subseteq N$ or $bIm \subseteq N$ or $ab \in \sqrt{(N : IM)}$. If $aIm \subseteq N$ or $bIm \subseteq N$, then, $am \in (N : I)$ or $bm \in (N : I)$. Suppose $ab \in \sqrt{(N : IM)}$, then $ab \in \sqrt{((N : I) : M)}$, because $((N : I):M) = (N : IM)$. Therefore, $(\mathrm{N} : \mathrm{I})$ is almost 2-absorbing primary sub-module in M.

Conversely, suppose that $(\mathrm{N} : \mathrm{I})$ is almost 2-absorbing primary sub-module in M. Let $a, b \in R$ and K be a sub-module of IM such that $abK - \{0\} \subseteq N - (N : IM)N$. Then, $ab(K : I) \subseteq (abK : I) \subseteq (N : I)$. Moreover, if $ab(K : I) \subseteq ((N : I) : M)(N : I) = (N : IM)(N : I)$, then,

$abK = ab(IK : I) = ab(K : I)I \subseteq (N : IM)(N : I)I = (N : IM)N$, a contradiction. As $(N : I)$ is almost 2-absorbing primary sub-module in M, then $ab \in \sqrt{((N : I) : M)} = \sqrt{(N : IM)}$ or $a(K : I) \subseteq (N : I)$ or $b(K : I) \subseteq (N : I)$, which implies that $aK = a(K : I)I \subseteq a(N : I)N \subseteq N$ or $bK = b(K : I)I \subseteq b(N : I)N \subseteq N$. Hence, N is almost 2-absorbing primary sub-module in IM.

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حول المقاسات الجزئية الإبتدائية الممتصة من النوع 2 تقريبا

ليكن M مقاسا معرفا على الحلقة الابدالية R ذات المحايد غير الصفري. نقدم في هذا البحث تعريف لمفهوم المقاسات الجزئية الأولية الممتصة من النوع 2 تقريبا. حيث نقول عن المقاس الجزئي التام N بأنه أولي ممتص من النوع 2 تقريبا إذا كان لكل $a,b\in R$ و $m\in M$ بحيث أن $abm\in N-(N:M)N$ فإن $ab\in\sqrt{(N:M)}$ أو $am\in N$ أو $bm\in N$. ندرس خواص هذا النوع من المقاسات و نثبت أن هذا التعريف يعتبر تعميم لتعريفي المقاسات الجزئية الإبتدائية الضعيفة الممتصة من النوع 2 و المقاسات الجزئية الممتصة من النوع 2 تقريبا, كما نبرهن أن المقاس الجزئي الممتص N من النوع 2 تقريبا المعرف على المقاس M يكون مقاسا جزئيا ابتدائيا ممتصا من النوع 2 تقريبا عندما يكون $(N:M)$ مثالي جذري, و نعطي مثال على ذلك المقاسات الجزئية التي يكون بها $(N:M)$ مثالي أولي. ندرس كذلك المقاسات الجزئية الابتدائية الممتصة من النوع 2 تقريبا على المقاسات الضربية و نجد الشرط الكافي لجعل المقاس الجزئي الابتدائي الضعيف الممتصة من النوع 2 مقاسا جزئيا ابتدائيا ممتصا من النوع 2 تقريبا. كذلك ندرس العلاقة بين المقاس الجزئي N و المقاس $(N:I)$, حيث I مثلي على الحلقة R , و المعرفين على المقاس M , حيث نبرهن أن N مقاسا جزئيا ابتدائيا ممتصا من النوع 2 تقريبا إذا و فقط إذا كان $(N:I)$ مقاسا جزئيا ابتدائيا ممتصا من النوع 2 تقريبا عند توفرت شروط معينة على المقاس M و المثالي I .

كلمات مفتاحية:
المقاسات الجزئية الممتصة من النوع**2 ,**
المقاسات الجزئية الضعيفة الممتصة من النوع 2,
المقاسات الجزئية الضعيفة الممتصة من النوع 2 تقريبا,
المقاسات الجزئية الإبتدائية الضعيفة الممتصة من النوع2 تقريبا