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On Certain Classes of Multivalent Functions Involving a Generalized Differential Operator Defined by a Convolution

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Abstract
Making use of a generalized differential operator we introduce some new subclasses of multivalent analytic functions in the open unit disk and investigate their inclusion relationships. Some integral preserving properties of these subclasses are also discussed. Furthermore, we generalize the integral operator F_p studied by Frasin [11] and investigate sufficient conditions for the operator to be $p-$valentlystarlike, $p-$valently close-to-convex and uniformly $p-$valent close-to-convex.

Keywords:
Analytic function,
Convex univalent function,
Hadamard Product,
Integral operator,
Multivalent function,

1. Introduction:

Let A_p denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p}, \quad (p\in\mathbb{N}=\{1,2,3,\ldots\}), \tag{1}$$

which are analytic and $p-$valent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.

For functions f given by (1) and g given by

$$g(z)=z^p+\sum_{k=1}^{\infty}b_{k+p}z^{k+p} \quad (p\in\mathbb{N};\ z\in U), \tag{2}$$

the Hadamard product (or convolution)of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}b_{k+p}z^{k+p}. \tag{3}$$

Given two functions f and g, which are analytic in U, f is said to be subordinate to g in U, written as $f\prec g$, if there exists a Schwarz function w analytic in U satisfying $w(0)=0$ and $|w(z)|<1$ such that $f(z)=g(w(z))$, $z\in U$.

In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

A function $f\in A_p$ is said to be $p-$valently starlike of order $\alpha\,(0\le\alpha<p)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha, \quad (z\in U).$$

We denote by $S_p^*(\alpha)$, the class of all such functions. On the other hand, a function $f\in A_p$

is said to be $p-$valently convex of order $\alpha(0\le\alpha<p)$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (z\in U).$$

We denote by $K_p(\alpha)$, the class of all such functions. Furthermore, a function $f\in A_p$ is said to be in the subclass $C_p(\alpha)$ of $p-$valently close-to-convex of order $\alpha(0\le\alpha<p)$ if and only if

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\}>\alpha \quad (z\in U).$$

Note that $S_p^*(0)=S_p^*$, $K_p(0)=K_p$ and $C_p(0)=C_p$ are, respectively, $p-$valently starlike, $p-$valently convex and $p-$valently close-to-convex functions in U .

A function $f(z)\in A_p$ is said to be in the class $US_p(\alpha)$ of uniformly $p-$valent starlike functions of order α $(-p\le\alpha<p)$ in U if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\ge\left|\frac{zf'(z)}{f(z)}-p\right| .$$

Also, a function $f(z)\in A_p$ is said to be in the class $UC_p(\alpha)$ of uniformly $p-$valent close-to-convex functions of order α $(0\le\alpha<p)$ in U if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}-\alpha\right\}\ge\left|\frac{zf'(z)}{g(z)}-p\right| \quad (z\in U).$$

The class $US_p(\alpha)$ was first introduced and studied by Goodman [14].

The following definition of fractional derivative given by Owa [18] will be required in our investigation.

The fractional derivative of order γ for a function f is defined by

$$D_z^{\gamma}f(z)=\frac{1}{\Gamma(1-\gamma)}\frac{d}{dz}\int_0^z\frac{f(\zeta)}{(z-\zeta)^{\gamma}}d\zeta \quad (0\le\gamma<1), \tag{4}$$

where the function f is analytic in a simply connected region of the complex $z-$plane containing the origin and the multiplicity of $(z-\zeta)^{-\gamma}$ is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta)>0$.

It readily follows from (4) that

$$D_z^{\gamma}z^k=\frac{\Gamma(k+1)}{\Gamma(k+1-\gamma)}z^{k-\gamma} \tag{5}$$

$$\left(0\le\gamma<1,\ k\in\mathbb{N}=\{1,2,\dots\}\right).$$

Using $D_z^{\gamma}f$, the operator $\Omega_p^{\lambda}:A_p\to A_p$, which is known as an extension of fractional derivative and fractional integral, is defined by

$$\Omega_p^{\gamma}f(z)=\frac{\Gamma(p-\gamma+1)}{\Gamma(p+1)}z^{\gamma}D_z^{\gamma}f(z),$$

$$\gamma\ne p+1,p+2,\dots$$

$$=z^p+\sum_{k=1}^{\infty}\frac{(p+1)_k}{(p-\gamma+1)_k}a_{k+p}z^{k+p} \tag{6}$$

Note that $\Omega_p^0f(z)=f(z)$.

Let P denote the class of analytic functions $h(z)$ with $h(0)=1$, which are convex and univalent in U and for which $\operatorname{Re}\{h(z)\}>0\ (z\in U)$.

For a fixed function $g\in A_p$ given by (2), Bulut [7] define the differential operator $U_{\lambda,p,l}^{n,\delta}g(z):A_p\to A_p$ by

$$U_{\lambda,p,l}^{0,\delta}g(z)=g(z)$$

$$U_{\lambda,p,l}^{1,\delta}g(z)=U_{\lambda,p,l}^{\delta}g(z)$$

$$=\left(\delta-\frac{\lambda p}{p+l}\right)g(z)+\left(\frac{\lambda}{p+l}-\frac{\delta-1}{p}\right)z\,g'(z)$$

$$,\ \lambda,\delta,l\ge0 \tag{7}$$

$U^{2,\delta}_{\lambda,p,l}g(z)=U^{\delta}_{\lambda,p,l}\left[U^{1,\delta}_{\lambda,p,l}g(z)\right],$

$\vdots$

$U^{n,\delta}_{\lambda,p,l}g(z)=U^{\delta}_{\lambda,p,l}\left[U^{n-1,\delta}_{\lambda,p,l}g(z)\right],\ n\in\mathbb{N}.$ (8)

For $g(z)$ given by (2), then by (7) and (8), we see that For $n\in\mathbb{N}_0$.

$$U^{n,\delta}_{\lambda,p,l}g(z)=z^p+\sum_{k=1}^{\infty}\left(1+k\left(\frac{\lambda}{p+l}+\frac{1-\delta}{p}\right)\right)^n b_{k+p}z^{k+p},\quad n\in\mathbb{N}_0. \quad (9)$$

Remark 1.

i. $U^{n,1}_{\lambda,p,l}=I_p(n,\lambda,l)$ defined by Cata's [8]
ii. $U^{n,\delta}_{\lambda,1,0}=D^n_{\delta,\lambda}$ defined and studied by Darus and Ibrahim [9]
iii. $U^{n,1}_{\lambda,1,0}=D^n_{\lambda}$ which is Al-Oboudi (generalized Salagean) differential operator [2].
iv. $U^{n,1}_{1,1,0}=D^n$ which is Salagean differential operator [22].

Using (6) and (9), we define $D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z):A_p\to A_p$ by

$$D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)=\Omega^{\gamma}_p f(z)*U^{n,\delta}_{\lambda,p,l}g(z)\quad n\in\mathbb{N}_0.$$

We can write for $n\in\mathbb{N}_0$

$$D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)=z^p+\sum_{k=1}^{\infty}\frac{(p+1)_k}{(p-\gamma+1)_k}\left(1+k\left(\frac{\lambda}{p+l}+\frac{1-\delta}{p}\right)\right)^n a_{k+p}b_{k+p}z^{k+p}. \quad (10)$$

where $\lambda,\delta,l\geq 0$ *and* $0\leq\gamma<1$.

It is easily follows from (10) that

$$\begin{aligned}p(p+l)D^{n+1,\delta}_{\lambda,p,l,\gamma,g}f(z)&=p(\delta(p+l)-\lambda p)D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\\&+(\lambda p+(1-\delta)(p+l))z\left(D^{n+1,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'.\end{aligned} \quad (11)$$

We assume that $p,m\in\mathbb{N}$, $\varepsilon_m=\exp\left(\frac{2\pi i}{m}\right)$, and

$$f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)=\frac{1}{m}\sum_{j=0}^{m-1}\varepsilon_m^{-jp}\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(\varepsilon_m^j z)\right)=z^p+\ldots,\ (f\in A_p). \quad (12)$$

Clearly, for $m=1$, we have $f^n_{p,1}(l,\delta,\lambda,\gamma;g;z)=D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)$.

Making use of the operator $D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)$, we now introduce the following subclasses of A_p of $p-$ valent analytic functions.

Definition 1.1. A function $f\in A_p$ is said to be in the class $M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$ if it satisfies

$$\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'}{(p+l)f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}\prec h(z),\ (z\in U), \quad (13)$$

where $h\in P$ and $f^n_{p,m}(l,\lambda,\gamma,\delta;g;z)\neq 0$.

Remark 2.

i. For $\gamma=l=0$ and $\delta=1$ in (13) then $M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$ reduces to the function class $S^n_{p,m}(\lambda;g;h)$ studied by Selvaraj and Selvakumaran [23].
ii. In (13) if we let $n=\gamma=l=0$, $\delta=1$ and $g(z)=z^p\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$, then we obtain the function class $S^{q,s}_{p,m}(\alpha_1;h)$ introduced by Wang, Jiang and Srivastava [24].
iii. If we let $n=l=\gamma=0$, $\delta=1$ and $g(z)=z^p+\sum_{k=1}^{\infty}\frac{(a)_n}{(c)_n}z^{p+n}$ in (13) then $M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$ reduces to the function class $T_{p,m}(a,c;h)$ introduced by Xu and Yang[25].
iv. Let $l=\gamma=0$, $\delta=1$and $g(z)=h(z)=\frac{1+z}{1-z}$, then $M^0_{1,2}(0,1,\lambda,0;g;h)=S^*_s$. The class S^*_s of functions starlike with respect to symmetric points has been studied with several authors ([19],[21],[26]).

Definition 1.2. A function $f\in A_p$ is said to be in the class $N^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$ if it satisfies

$$\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'}{(p+l)\varphi^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)} \prec h(z), \quad (z\in U), \quad (14)$$

for some $\varphi(z)\in M^{n}_{p,m}(l,\delta,\lambda,\gamma;g;h)$, where $h\in P$ and $\varphi^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)\neq 0$ is defined as (12).

Remark 3. For $\gamma=l=0$ and $\delta=1$ then $N^{n}_{p,m}(l,\delta,\lambda,\gamma;g;h)$ reduces to the function class $K^{n}_{p,m}(\lambda;g;h)$ studied by Selvaraj and Selvakumaran [23].

Definition 1.3. A function $f\in A_p$ is said to be in the class $O^{n}_{p,m}(\alpha,l,\delta,\lambda,\gamma;g;h)$ if it satisfies

$$(1-\alpha)\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'}{(p+l)\varphi^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)} +\alpha\frac{\left(z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'\right)'}{(p+l)\left(\varphi^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'} \prec h(z). \quad (15)$$

for some $\alpha\geq 0$ and $\varphi(z)\in M^{n}_{p,m}(l,\delta,\lambda,\gamma;g;h)$, where $h\in P$ and $\left(\varphi^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'\neq 0$.

Remark 4. For $\gamma=l=0$ and $\delta=1$ then $O^{n}_{p,m}(\alpha,l,\delta,\lambda,\gamma;g;h)$ reduces to the function class $C^{n}_{p,m}(\alpha,\lambda;g;h)$ studied by Selvaraj and Selvakumaran [23].

Definition 1.4 Let $s\in\mathbb{N}$, $n\in\mathbb{N}_0$, $\alpha_i>0$ and $f_i\in A_p$, we define the following general integral operator

$$\bar{F}_p(z)=\int_0^z p\,t^{p-1}\prod_{i=1}^{s}\left(\frac{D^{n,\delta}_{\lambda,p,l,\gamma,g}f_i(t)}{t^p}\right)^{\alpha_i}dt. \quad (16)$$

Remark 5.

i. In (16) if we take $b_{k+p}=1$, $\forall k,p\in\mathbb{N}$, $n=\gamma=0$, we obtain the general integral operator F_p studied by Frasin [11].

ii. In (16) if we take $b_{k+p}=1$, $\forall k,p\in\mathbb{N}$, $n=\gamma=0$ and $p=1$, we obtain of the general integral operator $F_1(z) = F_s(z)$ introduced and studied by Breaz and Breaz [3] and Breaz et al. [6] (see also [4,5, 12, 13]).

iii. Also for $b_{k+p}=1$, $\forall k,p\in\mathbb{N}$, $n=\gamma=0$, $p=s=1$ and $\alpha_1=\alpha\in[0,1]$ in (16), we obtain the integral operator $\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt$ studied in [16].

Lemma 1.5. (Eenigenburg et al. [13]) Let $\beta(\beta\neq 0)$ and γ be complex numbers and let $h(z)$ be analytic and convex univalent in U with $\mathrm{Re}\{\beta h(z)+\gamma\}>0$ $(z\in U)$. If $q(z)$ is analytic in U with $q(0)=h(0)$, then the subordination

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\prec h(z) \quad (z\in U)$$

implies that

$$q(z)\prec h(z) \quad (z\in U).$$

Lemma 1.6. (Miller and Mocanue [15]) Let $h(z)$ be analytic and convex univalent in U and let $w(z)$ be analytic in U with $\mathrm{Re}\{w(z)\}\geq 0$ $(z\in U)$. If $q(z)$ is analytic in U with $q(0)=h(0)$, then the subordination

$$q(z)+w(z)zq'(z)\prec h(z) \quad (z\in U)$$

implies that

$$q(z)\prec h(z) \quad (z\in U).$$

Lemma 1.7.

Let $f(z)\in M^{n}_{p,m}(l,\delta,\lambda,\gamma;g;h)$. Then

$$\frac{z\left(f^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'}{(p+l)f^{n}_{p,m}(l,\delta,\lambda,\gamma;g;z)}\prec h(z) \quad (z\in U). \quad (17)$$

Proof. For $f(z)\in A_p$, we have from (12) that

$$f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;\varepsilon_m^j z\right)$$
$$=\frac{1}{m}\sum_{d=0}^{m-1}\varepsilon_m^{-dp}\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{d+j}z\right)\right)$$
$$=\frac{\varepsilon_m^{jp}}{m}\sum_{d=0}^{m-1}\varepsilon_m^{-(d+j)p}\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{d+j}z\right)\right)$$
$$=\varepsilon_m^{jp}f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right),\left(j\in\{0,1,\dots,m-1\}\right)$$

and

$$\left(f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)\right)'$$
$$=\frac{1}{m}\sum_{j=0}^{m-1}\varepsilon_m^{j(1-p)}\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{j}z\right)\right)'.$$

Hence

$$\frac{z\left(f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)\right)'}{(p+l)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)}$$
$$=\frac{1}{m}\sum_{j=0}^{m-1}\frac{\varepsilon_m^{j(1-p)}z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{j}z\right)\right)'}{(p+l)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)}$$
$$=\frac{1}{m}\sum_{j=0}^{m-1}\frac{\varepsilon_m^{j}z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{j}z\right)\right)'}{(p+l)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;\varepsilon_m^j z\right)}\quad(z\in U).\ (18)$$

Since $f(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$, we have

$$\frac{\varepsilon_m^{j}z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f\left(\varepsilon_m^{j}z\right)\right)'}{(p+l)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;\varepsilon_m^j z\right)}\prec h(z)\ \text{for}\quad j\in\{0,1,\dots,m-1\}.\qquad(19)$$

Noting that $h(z)$ is convex univalent in U, from (18) and (19) we conclude that (17) holds true.

Lemma 1.8 [17]. If $f\in A_p$ satisfies

$$\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]<p+\frac{1}{4}\qquad(z\in U),$$

then f is $p-$valently starlike in U.

Lemma 1.9 [20]. If $f\in A_p$ satisfies

$$\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]<p+\frac{a+b}{(1+a)(1-b)}\qquad(z\in U),$$

where $a>0,\ b\geq 0$ and $a+2b\leq 1$, then f is $p-$valently close to convex in U.

Lemma 1.10. [1]. If $f\in A_p$ satisfies

$$\mathrm{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]<p+\frac{1}{3}\qquad(z\in U),$$

then f is uniformly $p-$valent close-to-convex in U.

2. A set of inclusion relationships:

Theorem 2.1. Let $h(z)\in P$ with

$$\mathrm{Re}\{h(z)\}>\frac{p\left(\lambda p-\delta(p+l)\right)}{(p+l)\left(\lambda p+(1-\delta)(p+l)\right)}\quad(20)$$
$$\left(z\in U;\ p\in N;\ l,\delta\geq 0,\ \lambda>1\right).$$

If $f(z)\in M_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$, then $f(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ provided $f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\neq 0\ (z\in U)$.

Proof. Using (11) and (12), we have

$$p\left(\delta(p+l)-\lambda p\right)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)$$
$$+\left(\lambda p+(1-\delta)(p+l)\right)z\left(f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)\right)'$$
$$=\frac{1}{m}\sum_{j=0}^{m-1}\varepsilon_m^{-jp}\left[p(p+l)\left(D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f\left(\varepsilon_m^{j}z\right)\right)\right]$$
$$=p(p+l)f_{p,m}^{n+1}\left(l,\delta,\lambda,\gamma;g;z\right)\quad(21)$$

Let $f(z)\in M_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$ and

$$w(z)=\frac{z\left(f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)\right)'}{(p+l)f_{p,m}^{n}\left(l,\delta,\lambda,\gamma;g;z\right)}\qquad(22)$$

Then $w(z)$ is analytic in U, with $w(0)=1$, and from (21) and (22) we have

$$\frac{\delta(p+l)-\lambda p}{(p+l)}+\frac{\lambda p+(1-\delta)(p+l)}{p}w(z)=\frac{f_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)}{f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)} \quad (23)$$

Differentiating (23) with respect to z and using (22), we get

$$w(z)+\frac{zw'(z)}{\frac{p(\delta(p+l)-\lambda p)}{(\lambda p+(1-\delta)(p+l))}+(p+l)w(z)}=\frac{z\left(f_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)\right)'}{(p+l)\left(f_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)\right)} \quad (24)$$

From (24) and Lemma 1.7 we note that

$$w(z)+\frac{zw'(z)}{\frac{p(\delta(p+l)-\lambda p)}{\lambda p+(1-\delta)(p+l)}+(p+l)w(z)}\prec h(z), \quad (z\in U). \quad (25)$$

In view of (20) and (25), we deduce from Lemma 1.5 that

$w(z)\prec h(z)$, $z\in U$. (26) Now, Suppose that

$$q(z)=\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}. \quad (27)$$

Then $q(z)$ is analytic in U, with $q(0)=1$, and we obtain that

$$f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)q(z)=\frac{p}{(\lambda p+(1-\delta)(p+l))}D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)-\frac{p(\delta(p+l)-\lambda p)}{(p+l)(\lambda p+(1-\delta)(p+l))}D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z). \quad (28)$$

Differentiating both sides of (28) with respect to z and multiply by z, we get

$$zq'(z)+\left[\frac{p(\delta(p+l)-\lambda p)}{\lambda p+(1-\delta)(p+l)}+(p+l)w(z)\right]q(z)=\frac{pz}{\lambda p+(1-\delta)(p+l)}\frac{\left(D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)\right)'}{f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}. \quad (29)$$

From (21), (23) and (29), we find that

$$q(z)+\frac{zq'(z)}{\left[\frac{p(\delta(p+l)-\lambda p)}{\lambda p+(1-\delta)(p+l)}+(p+l)w(z)\right]}=\frac{z}{p+l}\frac{\left(D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)\right)'}{f_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)}, \quad (30)$$

which is subordinate to $h(z)$ since $f(z)\in M_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$.

From (20) and (26), we observe that

$$\operatorname{Re}\left\{\frac{p(\delta(p+l)-\lambda p)}{(\lambda p+(1-\delta)(p+l))}+(p+l)w(z)\right\}>0$$

Therefore, from (30) and Lemma 1.6 we conclude that $q(z)\prec h(z)$, $(z\in U)$, that means $f(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$.

Theorem 2.2.

If $f(z)\in N_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$, with respect to $\varphi(z)\in M_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$, then $f(z)\in N_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ provided $\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\neq 0$ $(z\in U)$ where (20) holds.

Proof. Let $f(z)\in N_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$. Then for $z\in U$, there exists a function $\varphi(z)\in M_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;h)$ such that

$$\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)\right)'}{(p+l)\varphi_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)}\prec h(z). \quad (31)$$

So $\varphi(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ by Theorem 2.1. And Lemma 1.7 leads to

$$\psi(z)=\frac{z\left(\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right)'}{(p+l)\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}\prec h(z). \qquad (32)$$

Now, let

$$q(z)=\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}. \qquad (33)$$

Using (11), $q(z)$ can be written as

$$\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)q(z)$$
$$=\frac{p}{\lambda p+(1-\delta)(p+l)}D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)$$
$$-\frac{p(\delta(p+l)-\lambda p)}{(p+l)(\lambda p+(1-\delta)(p+l))}D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z). \qquad (34)$$

Differentiating both sides of (37) with respect to z, and using (21) (with f replaced by φ), we get

$$q(z)+\frac{z\,q'(z)}{\frac{p(\delta(p+l)-\lambda p)}{\lambda p+(1-\delta)(p+l)}+(p+l)\psi(z)}$$
$$=\frac{z}{p+l}\frac{\left(D_{\lambda,p,l,\gamma,g}^{n+1,\delta}f(z)\right)'}{\varphi_{p,m}^{n+1}(l,\delta,\lambda,\gamma;g;z)}.(35)$$

From (31) and (35), we find that

$$q(z)+\frac{z\,q'(z)}{\left[\frac{p(\delta(p+l)-\lambda p)}{\lambda p+(1-\delta)(p+l)}+(p+l)\psi(z)\right]}\prec h(z).(36)$$

Therefore, from (20),(32) and (36), we deduce from Lemma 1.6 that

$$q(z)\prec h(z)\quad (z\in U), \qquad (37)$$

which shows that $f(z)\in N_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ with respect to $\varphi(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$.

Theorem 2.3. Let $0\le\alpha_1<\alpha_2$ such that (20) holds. Then

$$O_{p,m}^{n}(\alpha_2,l,\delta,\lambda,\gamma;g;h)\subset O_{p,m}^{n}(\alpha_1,l,\delta,\lambda,\gamma;g;h).(38)$$

Proof.
Let $f(z)\in O_{p,m}^{n}(\alpha_2,l,\delta,\lambda,\gamma;g;h)$. Then for $z\in U$ there exists a function $\varphi(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ such that

$$(1-\alpha_2)\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}+\alpha_2\frac{\left(z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'\right)'}{(p+l)\left(\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right)'}\prec h(z) \qquad (39)$$

provided that $\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\neq 0$.

Suppose that

$$q(z)=\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}. \qquad (40)$$

Then $q(z)$ is analytic in U, with $q(0)=1$. Differentiating both sides of (40) we get

$$q(z)+\frac{\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}{\left[\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right]'}q'(z)$$

$$=\frac{\left(z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'\right)'}{(p+l)\left[\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right]'} \qquad (41)$$

Now, using (39), (40) and (41) we have

$$q(z)+w(z)z\,q'(z)\prec h(z), \qquad (42)$$

where

$$w(z)=\alpha_2\left(\frac{z\left(\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right)'}{\varphi_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}\right)^{-1}. \qquad (43)$$

In view of Lemma 1.7 and $\alpha_2>0$, we observe that $w(z)$ is analytic in U and $\operatorname{Re}\{w(z)\}>0$.

Consequently, in view of (42), we deduce from Lemma 1.6 that

$q(z)\prec h(z)$. (44) Since $0\le\frac{\alpha_1}{\alpha_2}<1$ and since $h(z)$ is convex univalent in U , we deduce from (39) and (44) that

$$(1-\alpha_1)\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'}{(p+l)\varphi^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}+\alpha_1\frac{\left(z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'\right)'}{(p+l)\left(\varphi^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'}$$

$$=\frac{\alpha_1}{\alpha_2}\left[(1-\alpha_2)\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'}{(p+l)\varphi^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}+\alpha_2\frac{\left(z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)\right)'\right)'}{(p+l)\left(\varphi^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'}\right]+\left(1-\frac{\alpha_1}{\alpha_2}\right)q(z)\prec h(z).$$

Thus $f(z)\in O^n_{p,m}(\alpha_1,l,\delta,\lambda,\gamma;g;h)$ which completes the proof of Theorem 2.3.

3. Integral operator:

Theorem 3.1. Let $h(z)\in P$ and

$\mathrm{Re}\{h(z)\}>\max\left\{0,-\frac{\mathrm{Re}\{c\}}{p+l}\right\}\quad(z\in U)$, (45) where c is a complex number such that $\mathrm{Re}\{c\}>-p$. If $f(z)\in M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$, then the function

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt\,, \tag{46}$$

is also in the class $M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$, provided that $F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\neq 0$, where $F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)$ is defined as in (12)

Proof. Let $f(z)\in M^n_{p,m}(l,\delta,\lambda,\gamma;g;h)$. Then for $\mathrm{Re}\{c\}>-p$, we note that $F(z)\in A_p$ and

$$F(z)=z^p+\sum_{k=1}^{\infty}\left[\frac{c+p}{c+p+k}\right]a_{k+p}z^{k+p}\ .$$

We can deduce that

$$(c+p)D^{n,\delta}_{\lambda,p,l,\gamma,g}f(z)=cD^{n,\delta}_{\lambda,p,l,\gamma,g}F(z)+z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}F(z)\right)' \tag{47}$$

Also we have from above

$$(c+p)f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)=\frac{1}{m}\sum_{j=0}^{m-1}\varepsilon_m^{-jp}\left(cD^{n,\delta}_{\lambda,p,l,\gamma,g}F(\varepsilon_m^jz)+\varepsilon_m^jz\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}F(\varepsilon_m^jz)\right)'\right)$$
$$=cF^n_{p,m}(l,\delta,\lambda,\gamma;g;z)+z\left(F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)' \tag{48}$$

Let

$$w(z)=\frac{z\left(F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'}{(p+l)F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)} \tag{49}$$

Then $w(z)$ is analytic in U, with $w(0)=1$, and from (48) we observe that

$$(p+l)w(z)+c=(c+p)\frac{f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}{F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)} \tag{50}$$

Differentiating both sides of (50) with respect to z and applying Lemma 1.7, we obtain

$$w(z)+\frac{zw'(z)}{(p+l)w(z)+c}\cdot=\frac{z\left(f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)\right)'}{(p+l)f^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}\prec h(z) \tag{51}$$

In view of (51), Lemma 1.5 leads to $w(z)\prec h(z)$. If we let

$$q(z)=\frac{z\left(D^{n,\delta}_{\lambda,p,l,\gamma,g}F(z)\right)'}{(p+l)F^n_{p,m}(l,\delta,\lambda,\gamma;g;z)}\ ,$$

then $q(z)$ is analytic in U, with $q(0)=1$, and it follows from (47) that

$$F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\,q(z).$$
$$=\frac{c+p}{p+l}D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)-\frac{c}{p+l}D_{\lambda,p,l,\gamma,g}^{n,\delta}F(z) \qquad (52)$$

Differentiating both sides of (52) with respect to z and multiply by z, we get

$$zq'(z)+\frac{z\left(F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\right)'}{F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}q(z)$$
$$=(c+p)\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}$$
$$-c\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}F(z)\right)'}{(p+l)F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)},$$

or

$$zq'(z)+\left((p+l)w(z)+c\right)q(z).$$
$$=(c+p)\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)} \qquad (53)$$

Now, from (52) and (53) we deduce that

$$q(z)+\frac{zq'(z)}{(p+l)w(z)+c}$$
$$=\frac{c+p}{(p+l)w(z)+c}\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)F_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}$$
$$=\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f(z)\right)'}{(p+l)f_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)}$$
$$\prec h(z), \qquad (54)$$

as $f(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$.

Combining (45) and $w(z)\prec h(z)$ we have $\mathrm{Re}\{(p+l)w(z)+c\}>0 \quad (z\in U)$.

Therefore, from (54) and Lemma 1.6 we find that $q(z)\prec h(z)$, which shows that $F(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$.

By applying similar method as in Theorem 3.1, we have:

Theorem 3.2.

If $f(z)\in N_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$, with respect to $\varphi(z)\in M_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$, then the function

$$F(z)=\frac{c+p}{z^{c}}\int_0^z t^{c-1}f(t)dt, \qquad (55)$$

is also in the class $N_{p,m}^{n}(l,\delta,\lambda,\gamma;g;h)$ with respect to

$$G(z)=\frac{c+p}{z^{c}}\int_0^z t^{c-1}\varphi(t)dt, \qquad (56)$$

provided that $G_{p,m}^{n}(l,\delta,\lambda,\gamma;g;z)\neq 0$ where (45) holds.

4. Sufficient Conditions for The Operator $\bar{F}_p$:

We begin by establishing sufficient conditions for the operator $\bar{F}_p$ to be in S_p^*.

Theorem 4.1. Let $\alpha_i>0$ be real numbers for all $i=1,2,\dots,s$. If $f_i\in A_p$ for all $i=1,2,\dots,s$ satisfies

$$\mathrm{Re}\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right)<p+\frac{1}{4\sum_{i=1}^{s}\alpha_i}\quad (z\in U), \qquad (57)$$

then $\bar{F}_p$ is p-valently starlike in U.

Proof. From Definition 1.4, we observe that $\bar{F}_p(z)\in A_p$. Also we see that

$$\bar{F}_p'(z)=p\,z^{p-1}\prod_{i=1}^{s}\left(\frac{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}{z^{p}}\right)^{\alpha_i}. \qquad (58)$$

Differentiating the equation logarithmically and multiplying by z, we obtain

$$\frac{z\bar{F}_p''(z)}{\bar{F}_p'(z)}=(p-1)+\sum_{i=1}^{s}\alpha_i\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}-p\right)$$

Thus we have

$$1+\frac{z\bar{F}_p''(z)}{\bar{F}_p'(z)} = p\left(1-\sum_{i=1}^{s}\alpha_i\right)+\sum_{i=1}^{s}\alpha_i\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right). \quad (59)$$

Taking the real part of both sides, we have

$$\operatorname{Re}\left(1+\frac{z\bar{F}_p''(z)}{\bar{F}_p'(z)}\right) = p\left(1-\sum_{i=1}^{s}\alpha_i\right)+\sum_{i=1}^{s}\alpha_i \operatorname{Re}\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right). \quad (60)$$

From (60) and (57), we have

$$\operatorname{Re}\left(1+\frac{z\bar{F}_p''(z)}{\bar{F}_p'(z)}\right) < p\left(1-\sum_{i=1}^{s}\alpha_i\right)+\sum_{i=1}^{s}\alpha_i\left(p+\frac{1}{4\sum_{i=1}^{s}\alpha_i}\right)=p+\frac{1}{4}. \quad (61)$$

Hence by Lemma 1.8, we get $\bar{F}_p$ is $p-$valently starlike in U.

Corollary 4.2 [31].Let $\alpha_i>0$ be real numbers for all $i=1,2,\dots,s$. If $f_i\in A_p$ for all $i=1,2,\dots,s$ satisfies

$$\operatorname{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)<p+\frac{1}{4\sum_{i=1}^{n}\alpha_i} \quad (z\in U).$$

Then the function $F_p(z)=\int_0^z pt^{p-1}\prod_{i=1}^{s}\left(\frac{f_i(t)}{t^p}\right)^{\alpha_i}dt$ is $p-$valently starlike in U.

Proof. Take $b_{k+p}=1\ \forall k,p\in\mathbb{N}$, $n=0$ and $\gamma=0$ in Theorem 4.1.

Corollary 4.3. If $f\in A$ satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)<1+\frac{1}{4\alpha} \quad (z\in U),$$

where $\alpha>0$, then $\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt$ is starlike in U.

Proof. Take $s=p=1,\alpha_1=\alpha$ and $f_1=f$ in Corollary 4.2.

Now, we obtain the following sufficient conditions for $\bar{F}_p$ to be $p-$valently close-to-convex and uniformly $p-$valent close-to-convex using Lemmas 1.9 and 1.10.

Theorem 4.4. Let $\alpha_i>0$ be real numbers for all $i=1,2,\dots,s$. If $f_i\in A_p$ for all $i=1,2,\dots,s$ satisfies

$$\operatorname{Re}\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right)<p+\frac{(a+b)}{(1+a)(1-b)\sum_{i=1}^{s}\alpha_i} \quad (z\in U), (62)$$

where $a>0, b\geq 0$ and $a+2b\leq 1$, then $\bar{F}_p$ is $p-$valently close-to-convex in U.

Proof. From (60) and (62), we have

$$\operatorname{Re}\left\{1+\frac{z\bar{F}_p''(z)}{\bar{F}_p'(z)}\right\} = p\left(1-\sum_{i=1}^{s}\alpha_i\right)+\sum_{i=1}^{s}\alpha_i \operatorname{Re}\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right)$$

$$< p\left(1-\sum_{i=1}^{s}\alpha_i\right)+\sum_{i=1}^{s}\alpha_i\left(p+\frac{(a+b)}{(1+a)(1-b)\sum_{i=1}^{s}\alpha_i}\right)$$

$$< p+\frac{(a+b)}{(1+a)(1-b)}.$$

Using Lemma 1.9, we have $\bar{F}_p$ is $p-$valently close-to-convex in U .

Theorem 4.5 . Let $\alpha_i>0$ be real numbers for all $i=1,2,\dots,s$. If $f_i\in A_p$ for all $i=1,2,\dots,s$ satisfies

$$\operatorname{Re}\left(\frac{z\left(D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)\right)'}{D_{\lambda,p,l,\gamma,g}^{n,\delta}f_i(z)}\right)<p+\frac{1}{3\sum_{i=1}^{s}\alpha_i}\quad (z\in U)\ (63)$$

then $\bar{F}_p$ is uniformly p-valent close-to-convex in U.

Proof. It follows that by applying Lemma 1.10 and using (60) and (63) to get the result.

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فصول جزئية معينة لاقترانات تحليلية متعددة القطبية متضمنة معامل تفاضلي معمم ومعرف بصورة الضرب

في هذا البحث تم دراسة مجموعه من الفصول الجزئية المعينة للدوال التحليلية والمتعددة القطبية باستخدام بعض المعاملات المعممة ومن ثم دراسة بعض خصائص هذه الفصول متمثلة بنظرية معامل التقدير, نظرية الاحتواء, ونظريات أخرى. أيضا تم دراسة معامل تفاضلي معمم والتحقق من بعض الشروط الكافية لهذا المعامل التي تجعله دالة نجمية متعددة القطبية، محدبة متعددة القطبية أو منتظمة متعددة القطبية.

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