Regular-Minimal and Regular-Maximal Continuous Functions

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Abstract
In this paper, we consider new types of continuous functions called regular minimal continuous and regular maximal continuous functions. We investigate their relationship with some other types of continuous functions. Also, we study further properties of the new types of continuous functions.

Keywords:
Regular open sets, Semi-regular space, Regular minimal continuous, Regular maximal continuous.

1. Introduction and preliminaries:
A subset A of a topological space X is said to be a regular open set (Stone, 1937) if A = Int(Cl(A)). It is called a regular closed if Ac is a regular open. In Stone (1937), it was shown that the regularly open sets of a space (X, τ) is a base for a topology τr on X coarser than τ. The space (X, τr) was called the semi-regularization space of (X, τ). A space (X, τ) is semi-regular if and only if the regularly open sets of (X, τ) is a base for τ; that is, τ = τr. For a space (X, τ), the regularly open sets of (X, τ) equal the regularly open sets of (X, τr). Hence, the semi-regularization process generates at most one new topology. Thus (τr)r = τr (Stone, 1937). A space X is connected if X cannot be represented as the union of two or more disjoint nonempty open subsets. If X is not connected, then X is disconnected.

Definition 1.1 A function f : X → Y is called:

a) an almost continuous (Singal and Singal, 1968) if for each x ∈ X and for each regular open set V containing f(x), there exists an open set U containing x such that f(U) ⊆ V.

b) an almost perfectly continuous (Singh, 2010) if f−1(U) is clopen in X, for every regular open set U in Y.

c) an almost strongly θ-continuous (Noiri and Kang, 1984) if for each x ∈ X and for each regular open set V containing f(x), there exists an open set U containing x such that f(Cl(U)) ⊆ V.

d) a δ-continuous (Noiri, 1980) if for each x ∈ X and for each regular open set V containing f(x), there exists a regular open set U containing x such that f(U) ⊆ V.

A proper nonempty open (resp. closed) subset U of X is said to be a minimal open (resp. a minimal closed) set (Nakaoka and Oda, 2001) if any open (resp. closed) set which is contained in U is φ or U. A proper nonempty open (resp. closed) subset M of X is said to be a maximal open (resp. a maximal closed) set (Nakaoka and Oda, 2003) if any open (resp. closed) set which contains M is X or
The collection of all minimal open (resp. maximal open, minimal closed, maximal closed) sets is denoted by $mO(X)$ (resp. $M_oO(X)$, $mC(X)$, $M_oC(X)$).

**Lemma 1.2** (Nakaoka and Oda, 2003) Let $(X, \tau)$ be a topological space.

a) If $U$ is a maximal open set and $W$ is an open set such that $U \cup W \neq X$, then $W \subseteq U$.

b) If $U$ and $V$ are maximal open sets such that $U \cap V \neq X$, then $U = V$.

**Definition 1.3** (Benchalli, Basavaraj, and Wali, 2011) A topological space $(X, \tau)$ is said to be $T_{\text{min}}$ (resp. $T_{\text{max}}$) space if every nonempty proper open subset of $X$ is minimal open (resp. maximal open) set.

**Remark 1.4** (Benchalli et al., 2011) The concepts $T_{\text{min}}$ and $T_{\text{max}}$ spaces are identical. That is, $X$ is $T_{\text{min}}$ if and only if $X$ is $T_{\text{max}}$.

**Definition 1.5** (Benchalli et al., 2011) Let $X$ and $Y$ be topological spaces. A map $f : X \rightarrow Y$ is called:

a) a minimal continuous (briefly, a min-continuous) if $f^{-1}(M)$ is an open set in $X$ for every minimal open set $M$ in $Y$.

b) a maximal continuous (briefly, a max-continuous) if $f^{-1}(M)$ is an open set in $X$ for every maximal open set $M$ in $Y$.

**Remark 1.6** Let $Y$ be a $T_{\text{min}}$ space. Then, $f : X \rightarrow Y$ is a min-continuous iff $f$ is a max-continuous.

**Definition 1.7** A nonempty proper regular open set $A$ of a topological space $(X, \tau)$ is said to be:

a) a minimal regular open set (Jasim and Aziz, 2014) if any regular open set contained in $A$ is $A$ or $\phi$ and a minimal regular closed set (Anuradha and Chacko, 2015) if any regular closed set contained in $A$ is $A$ or $\phi$.

b) a maximal regular open set (Anuradha and Chacko, 2015) if any regular open set contains $A$ is $X$ or $A$ and a maximal regular closed set (Jasim and Aziz, 2014) if any regular closed set contains $A$ is $X$ or $A$.

The collection of all minimal regular open (resp. minimal regular closed, maximal regular open, maximal regular closed) sets in a topological space $(X, \tau)$ is denoted by $mRO(X, \tau)$ (resp. $mRC(X, \tau)$, $M_oRO(X, \tau)$, $M_oRC(X, \tau)$).

**Theorem 1.8** (Mahdi and Nasser, in press) Let $X$ be a topological space and $F \subseteq X$. Then $F$ is a minimal regular open (resp. a minimal regular closed) set if and only if $X \setminus F$ is a maximal regular closed (resp. a maximal regular open) set.

**Lemma 1.9** (Anuradha and Chacko, 2015) Let $X$ be a topological space and $U$ a minimal regular open set.

a) If $W$ is a regular open set such that $U \cap W \neq \phi$, then $U \subseteq W$.

b) If $V$ is a minimal regular open set such that $U \cap V \neq \phi$, then $U = V$.

**Theorem 1.10** (Mahdi and Nasser, in press) If $A$ is a minimal open set in a space $X$ such that $A$ is not dense in $X$, then $\text{Int}(\text{Cl}(A))$ is a minimal regular open.

**Theorem 1.11** (Mahdi and Nasser, in press) Let $X$ be a semi-regular space and $U$ a nonempty regular open set, then the following three conditions are equivalent:

a) $U$ is a minimal regular open.

b) $U \subseteq \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

c) $\text{Cl}(S) = \text{Cl}(U)$ for any nonempty subset $S$ of $U$.

**Theorem 1.12** (Mahdi and Nasser, in press) Let $A$ be a nonempty subspace of $X$ and $U$ a regular open set in $A$ and a regular open in $X$. If $U$ is a minimal regular open in $A$, then $U$ is a minimal regular open in $X$.

**Theorem 1.13** (Mahdi and Nasser, in press) If $A$ is a maximal open set in a topological space $X$, then exactly one of the following holds:

1. $A$ is a maximal regular open set.
2. $A$ is a dense set in $X$.

**Theorem 1.14** (Mahdi and Nasser, in press) Let $X$ be a semi-regular space. Then, $U$ is a minimal regular open set if and only if $U$ is a minimal open set; that is, $mO(X) = mRO(X)$.

**Theorem 1.15** (Mahdi and Nasser, in press) Let $(X, \tau_s)$ be the semi-regularization space of a topological space $(X, \tau)$. Then, $mRO(X, \tau_s) = mRO(X, \tau)$ and $M_oRO(X, \tau_s) = M_oRO(X, \tau)$.

**Definition 1.16** (Anuradha and Chacko, 2015) A topological space $(X, \tau)$ is said to be an $rT_{\text{min}}$ space if every proper nonempty regular open subset of $X$ is minimal regular open.

**Theorem 1.17** (Mahdi and Nasser, in press) Let $X$ be an $rT_{\text{min}}$ space. Then, $mRO(X) = M_oRO(X)$.
2. R-min and r-max continuous functions; definitions and characterizations:

**Definition 2.1** Let $X$ and $Y$ be topological spaces. A map $f: X \to Y$ is called:

a) a regular minimal continuous (briefly, an r-min-continuous) if $f^{-1}(U)$ is open set in $X$, for every minimal regular open set $U$ in $Y$.

b) a regular maximal continuous (briefly, an r-max-continuous) if $f^{-1}(U)$ is open set in $X$, for every maximal regular open set $U$ in $Y$.

**Example 2.2** Let $X = Y = \{1,2,3,4\}$ with the topology

$\tau_X = \{\emptyset, X, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$ and $\tau_Y = \{\emptyset, Y, \{1\}, \{1,2\}, \{1,2,3\} \}$.

Then $RO(Y, \tau_Y) = \{\emptyset, Y, \{1\}, \{1,2\}, \{1,2,3\}\}$.

**Theorem 2.3** For a function $f: X \to Y$, the following are equivalent:

a) $f$ is r-min-continuous (resp. r-max-continuous).

b) The inverse image of any maximal regular closed (resp. minimal regular closed) set is closed set.

c) For any $x \in X$ and any minimal regular open (resp. maximal regular open) set $U$ in $Y$ containing $f(x)$, there exists an open set $W$ in $X$ containing $x$ such that $f(W) \subseteq U$.

**Proof.** (a $\Rightarrow$ b) The complement of minimal regular open (resp. maximal regular open) set is maximal regular closed (resp. minimal regular closed) set. Moreover, $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$.

(b $\Rightarrow$ c) If $U$ is a minimal regular open set in $Y$ such that $f(x) \in U$, then $Y \setminus U$ is a maximal regular closed in $Y$ and $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is closed set in $X$ and so $f^{-1}(U)$ is open set in $X$. Since $f(x) \in U$, $x \in f^{-1}(U)$. Take $W = f^{-1}(U)$, we get $x \in W = f^{-1}(U)$ and $f(W) \subseteq U$.

(c $\Rightarrow$ a) Let $U$ be a minimal regular open set in $Y$. Let $x \in f^{-1}(U)$. Then there exists an open set $W$ in $X$ such that $x \in W$ and $f(W) \subseteq U$. Hence, $x \in W \subseteq f^{-1}(U)$ and so $f^{-1}(U)$ is open in $X$. □

**Theorem 2.4** Let $X$, $Y$ be topological spaces. Consider the following two statements:

1. Every minimal regular open set $U$ in $Y$ is an intersection of finitely many of maximal regular open sets in $Y$.

2. If $f: X \to Y$ is r-max-continuous function, then $f$ is r-min-continuous function.

Then (1) implies (2).

**Proof.** Follows from the fact that $f^{-1}(\bigcap_{i=1}^{n} M_i) = \bigcap_{i=1}^{n} f^{-1}(M_i)$ and the fact that any finite intersection of open sets is open. □

**Theorem 2.5** Let $Y$ be an $rT_{\min}$ space. Then $f: X \to Y$ is r-min-continuous if and only if $f$ is r-max-continuous.

**Proof.** Follows from Corollary 1.17. □

3. Relations between r-min, r-max and other types of continuous functions:

Firstly, there is no relation between r-min-continuous (resp. r-max-continuous) and min-continuous (resp. max-continuous) functions, as shown in the following examples:

**Example 3.1** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be as in Example 2.2. Define $h: Y \to X$ by $h(1) = 2$, $h(2) = 1$, $h(3) = 1$, $h(4) = 4$ and $k: Y \to X$ by $k(1) = 4$, $k(2) = 1$, $k(3) = 2$, $k(4) = 3$. Then, $h$ is both r-min-continuous and r-max-continuous, but:

1. $h$ is not min-continuous, since the set $\{1\}$ is not minimal open in $Y$, but $f^{-1}(\{1\}) = \{2,3\}$ which is not open in $Y$.

2. $h$ is not max-continuous, since the set $\{1,3,4\}$ is minimal open in $X$, but $f^{-1}(\{1,3,4\}) = \{2,3,4\}$ which is not open in $Y$.

Moreover, $k$ is both min-continuous and max-continuous, but $k$ is not r-min-continuous and not r-max-continuous since the set $\{1,2\}$ is both minimal regular open and maximal regular open in $X$, but $k^{-1}(\{1,2\}) = \{2,3\}$ which is not open in $Y$.

**Theorem 3.2** Let $Y$ be a semi-regular space. Then, $f: X \to Y$ is r-min-continuous if and only if $f$ is min-continuous.

**Proof.** Follows from Theorem 1.14. □
Definition 3.3 (Anuradha and Chacko, 2015) Let X and Y be topological spaces. A function \( f : X \rightarrow Y \) is called:

a) a minimal regular continuous (briefly, a min r-continuous) if \( f^{-1}(U) \) is regular open set in \( X \), for every minimal regular open set \( U \) in \( Y \).

b) a maximal regular continuous (briefly, a max r-continuous) if \( f^{-1}(U) \) is regular open set in \( X \), for every maximal regular open set \( U \) in \( Y \).

Remark 3.4 Every min r-continuous (resp. max r-continuous) is r-min-continuous (resp. r-max-continuous), but the converse need not be true as shown in the following example:

Example 3.5 Let \((X, \tau_X)\) and \((Y, \tau_Y)\) be as in Example 2.2. Define \( h : X \rightarrow Y \) by \( h(1) = 1 \), \( h(2) = 1 \), \( h(3) = 3 \), \( h(4) = 1 \). Then \( f \) is r-min-continuous, but not min r-continuous since \{1\} is minimal regular open set in \( Y \), while \( f^{-1}(\{1\}) = \{1,2,4\} \) which is not regular open in \( X \). In Example 2.2, \( g : X \rightarrow Y \) is r-max-continuous, but not max r-continuous since \{1,4\} is maximal regular open in \( Y \), while \( g^{-1}(\{1,4\}) = \{1\} \) which is not regular open in \( X \).

Theorem 3.6 Let \( X, Y \) be topological spaces and \( f : X \rightarrow Y \).

1. If \( f \) is continuous, then \( f \) is both r-min-continuous and r-max-continuous.
2. If \( f \) is almost continuous, then \( f \) is both r-min-continuous and r-max-continuous.
3. If \( f \) is almost perfectly continuous, then \( f \) is both r-min-continuous and r-max-continuous.
4. If \( f \) is almost strongly \( \theta \)-continuous, then \( f \) is both r-min-continuous and r-max-continuous.
5. If \( f \) is \( \delta \)-continuous, then \( f \) is both r-min-continuous and r-max-continuous.

Remark 3.7 The converse of all parts of Theorem 3.6 need not be true as shown in the following example:

Example 3.8 In Example 2.2, \( f \) is r-min-continuous, but:

1. \( f \) is not continuous, since \{1,2,3\} is open in \( Y \), but \( f^{-1}(\{1,2,3\}) = \{1,2,3\} \) which is not open in \( X \).
2. \( f \) is not almost continuous, since \{1,2,3\} is regular open set in \( Y \), but \( f^{-1}(\{1,2,3\}) = \{1,2,3\} \) which is not open in \( X \).
Theorem 4.1 Let $f : X \to Y$ be an $r$-min-continuous and $x \in X$. If $U$ is a minimal open set containing $f(x)$ such that $U$ is not dense in $Y$, then there exists an open set $W$ in $X$ containing $x$ such that $f(W) \subseteq \text{Int}(\text{Cl}(U))$.

Proof. By Theorem 1.10, $\text{Int}(\text{Cl}(U))$ is a minimal regular open set in $Y$ and $f(x) \in \text{Int}(\text{Cl}(U))$. By Theorem 2.3, there exists an open set $W$ containing $x$ such that $f(W) \subseteq \text{Int}(\text{Cl}(U))$.

Theorem 4.2 Let $Y$ be a semi-regular space and $f : X \to Y$ a surjection function. If $f$ is $r$-min-continuous, then for any minimal regular open set $U$ in $Y$ and any nonempty subset $S$ of $U$, there is a nonempty open set $W$ in $X$ such that $W \subseteq f^{-1}(\text{Cl}(S))$.

Proof. Let $U$ be a minimal regular open set in $Y$ and $S$ a nonempty subset of $U$. Then $W = f^{-1}(U)$ is a nonempty open set. By Theorem 1.11, $U \subseteq \text{Cl}(S)$. So, $f(W) \subseteq U \subseteq \text{Cl}(S)$. Therefore, $W \subseteq f^{-1}(\text{Cl}(S))$.

Theorem 4.3 Let $f : X \to Y$ be an $r$-min-continuous (resp. an $r$-max-continuous). Then for any subset $A$ of $X$, $f|A : A \to Y$ is an $r$-min-continuous (resp. an $r$-max-continuous).

Proof. Direct from the fact that $(f|A)^{-1}(U) = f^{-1}(U) \cap A$.

Theorem 4.4 Let $\{A_{\alpha} : \alpha \in \Delta\}$ be an open cover of $X$. Then, $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous) if and only if $f|A_{\alpha} : A_{\alpha} \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), $\forall \alpha \in \Delta$.

Proof. If $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), then by Theorem 4.3, $f|A_{\alpha} : A_{\alpha} \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), $\forall \alpha \in \Delta$. Conversely, assume that $\forall \alpha \in \Delta$, $f|A_{\alpha} : A_{\alpha} \to Y$ is $r$-min-continuous (resp. $r$-max-continuous). Let $U$ be a minimal regular open set in $Y$. Then, $f^{-1}(U) = \bigcup_{\alpha \in \Delta} (f|A_{\alpha})^{-1}(U)$. Since $A_{\alpha}$ is open in $X$, $(f|A_{\alpha})^{-1}(U)$ is open set in $X$ and so, $\bigcup_{\alpha \in \Delta} ((f|A_{\alpha})^{-1}(U)) = f^{-1}(U)$ is open in $X$. Therefore, $f : X \to Y$ is $r$-min-continuous. Similarly, if $f : X \to Y$ is $r$-max-continuous.

Corollary 4.5 Let $X = A \cup B$ where $A$ and $B$ are both open (or both closed) sets in $X$. Then, a function $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous) if and only if $f|A$ and $f|B$ are $r$-min-continuous (resp. $r$-max-continuous).

Theorem 4.6 Let $B$ be a subspace of a space $Y$ and $f : X \to B$. If $\text{RO}(B) \subseteq \text{RO}(Y)$ and $f : X \to Y$ is $r$-min-continuous, then $f : X \to B$ is $r$-min-continuous.

Proof. If $U$ is a minimal regular open set in $B$, then by Theorem 1.12, $U$ is a minimal regular open set in $Y$ and so the result follows.

Remark 4.7 The composition of even two $r$-min-continuous (resp. $r$-max-continuous) functions need not be $r$-min-continuous (resp. $r$-max-continuous).

Example 4.8 Let $(X, \tau_X)$, $(Y, \tau_Y)$ be as in Example 2.2 and $Z = \{a, b, c\}$ with $\tau_Z = \{\phi, Z, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define $f : X \to Y$ as in Example 2.2 and $h : Y \to Z$ by $h(1) = b$, $h(2) = c$, $h(3) = b$, $h(4) = a$. Then $f$ and $h$ are $r$-min-continuous. But $g \circ f : X \to Z$ is not $r$-min-continuous since the set $\{b, c\}$ is minimal regular open in $Z$, while $(g \circ f)^{-1}(\{b, c\}) = f^{-1}(g^{-1}(\{b, c\})) = f^{-1}(\{1, 2, 3\}) = \{1, 2, 3\}$ which is not open in $X$.

Example 4.9 Let $X = Y = Z = \{1, 2, 3, 4\}$ with $\tau_X = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, $\tau_Y = \{\phi, Y, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$, $\tau_Z = \{\phi, Z, \{1\}, \{2\}, \{1, 3\}, \{2, 4\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. Then $\text{RO}(Y, \tau_Y) = \{\phi, Y, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ and $\text{RO}(Z, \tau_Z) = \{\phi, Z, \{4\}, \{1, 2\}\}$. Define $f : X \to Y$ by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 4$ and $g : Y \to Z$ by $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(4) = 3$. Then, both $f$ and $g$ are $r$-max-continuous, but $g \circ f : X \to Z$ is not $r$-max-continuous since $\{1, 2\}$ is maximal regular open set in $Z$, but $(g \circ f)^{-1}(\{1, 2\}) = f^{-1}(g^{-1}(\{1, 2\})) = f^{-1}(\{1, 2\}) = \{1, 3\}$ which is not open set in $X$.

The proofs of the following three theorems are direct and trivial, so they are omitted.

Theorem 4.10 Let $f : X \to Y$ be a continuous function and $g : Y \to Z$ an $r$-min-continuous (resp. an $r$-max-continuous). Then $g \circ f : X \to Z$ is $r$-min-continuous (resp. $r$-max-continuous).
Theorem 4.11 Let $f : X \rightarrow Y$ be an almost continuous and $g : Y \rightarrow Z$ a min r-continuous (resp. a max r-continuous). Then $g \circ f : X \rightarrow Z$ is r-min-continuous (resp. r-max-continuous).

Theorem 4.12 Let $Y$ be a $T_{\infty}$ space. If $f : X \rightarrow Y$ is a min-continuous and $g : Y \rightarrow Z$ is an r-min-continuous (resp. an r-max-continuous), then $g \circ f : X \rightarrow Z$ is r-min-continuous (resp. r-max-continuous).

Theorem 4.13 Let $Z$ be a semi-regular space. If $f : X \rightarrow Y$ is a continuous and $g : Y \rightarrow Z$ is a min-continuous, then $g \circ f : X \rightarrow Z$ is r-min-continuous (resp. r-max-continuous).

Proof. Let $U$ be a minimal regular open set in $Z$, then $(g \circ f)^{-1}(U)$ is open in $X$ and so $f((g \circ f)^{-1}(U)) = f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is open in $Y$. □

Theorem 4.14 Let $f : X \rightarrow Y$ be a surjection open mapping. If $g : Y \rightarrow Z$ is a function such that $g \circ f : X \rightarrow Z$ is r-min-continuous (resp. r-max-continuous), then $g : Y \rightarrow Z$ is r-min-continuous (resp. r-max-continuous).

Proof. Since $A \neq B$, by Lemma 1.9, $A \cap B = \phi$. So, $f^{-1}(A)$ and $f^{-1}(B)$ are two open sets in $X$. Since $A \cap B = \phi$ and $A \cup B = Y$, then $f^{-1}(A) \cap f^{-1}(B) = \phi$ and $f^{-1}(A) \cup f^{-1}(B) = X$. □

Theorem 4.17 Let $M_1$ and $M_2$ be two disjoint maximal open sets in a space $Y$ such that they are not dense. If $f : X \rightarrow Y$ is r-max-continuous, then $X$ is disconnected.

Proof. By Theorem 1.13, $M_1$ and $M_2$ are two maximal regular open sets. As $f : X \rightarrow Y$ is r-max-continuous, $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are two open sets in $X$. Hence, $f^{-1}(M_1) \cup f^{-1}(M_2) = X$ and $f^{-1}(M_1) \cap f^{-1}(M_2) = \phi$. □

References:


