Neighborhoods and Partial Sums of Certain Class of Analytic and Multivalent Functions Involving the Cho–Kwon-Srivastava Operator Defined by Convolution

Abstract
In this present paper, we derive several subordination results for analytic function defined by convolution, and we prove several inclusion relations associated with the $\delta$-neighborhood of certain subclasses of analytic functions with negative coefficients defined by the Cho-Kwon-Srivastava operator by making use of familiar concepts of neighborhood of analytic function. The results of partial sums and subordinating results and result of integral means inequalities and also obtained.

1. Introduction:
Let $S_p$ bethe class of function $f(z)$ of the form
$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (p \in \mathbb{N}) \quad (1.1)$$
which are analytic and $P$-valent in the unit disk $U = \{z : z \in \mathbb{D}, |z| < 1\}$. Also let $T_p$ denote the subclass of $S_p$ consisting of analytic and $P$-valent functions which can be expressed in the form
$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (a_{k+p} \geq 0) \quad (p \in \mathbb{N}) \quad (1.2)$$
Let
$$g(z) = z^p - \sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \quad (b_{k+p} \geq 0) \quad (1.3)$$
Then the Hadamard product (or convolution) $f \ast g$ of the analytic functions $f$ and $g$ is defined by
$$(f \ast g)(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p}, \quad (a_{k+p}, b_{k+p} \geq 0) \quad (1.4)$$

Definition 1.1 ([3] see also [7,9]), we define the $\delta$-neighborhood of function $f(z) \in T_p$ is defined by
$$N_\delta(f) = \left\{ k : k \in T_p, k(z) = z^p - \sum_{k=1}^{\infty} c_{k+p} z^{p+k} \right\}$$
and
$$\left| \sum_{k=1}^{\infty} (k+p) c_{k+p} \right| \leq \delta \quad (1.5)$$
In particular, for the identity function $h(z) = z^p$, we immediately have
$$N_\delta(h) = \left\{ k : k \in T_p, k(z) = z^p - \sum_{k=1}^{\infty} c_{k+p} z^{p+k} \right\}$$
and
$$\left| \sum_{k=1}^{\infty} (k+p) c_{k+p} \right| \leq \delta \quad (1.6)$$

Definition 1.2 [10]
The linear operator $L_p(a,c): S_p \to S_p$, is defined by
$$L_p(a,c)f(z) = \phi_p(a,c;z) \ast f(z) \quad (z \in U) \quad (1.7)$$
where
$$\phi_p(a,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad (1.8)$$
$(a)_k$ is the Pochhammer symbol.
Definition 1.3 [2] introduced the linear operator \( L^\lambda_p (a,c) : S_p \to S_p \) analogs to \( L_p (a,c) \), defined by
\[
L^\lambda_p (a,c)f (z)=\phi^\lambda_p (a,c;z)f (z) \quad (p\in\mathbb{N}; a,c\in \mathbb{R}/Z^*_0 ; \lambda>-p),
\]
where \( \phi^\lambda_p (a,c;z) \) is the function defined in terms of the Hadamard product (or Convolution) by the following condition
\[
\phi^\lambda_p (a,c;z)\phi^\lambda_p (a,c;z)=\frac{z^p}{(1-z)^p+\lambda}, \quad (1.10)
\]
We can easily find from (1.7), (1.8) and (1.9) and for the function \( f (z) \in T_p \) that
\[
L^\lambda_p (a,c)=z^p+\sum_{k=1}^{\infty} \frac{(p+\lambda)_k}{k!a_k}z^{p+k} - a_{k+p} \geq 0. \quad (1.11)
\]
It's easily verified from (1.11) that
\[
z(L^\lambda_p (a,c+1)f (z))=(\lambda+p)L^\lambda_p (a,c)f (z)-(a_p)L^\lambda_p (a+1,c)f (z), \quad (1.12)
\]
And
\[
z(L^\lambda_p (a,c)f (z))=(\lambda+p)L^\lambda_p (a,c)f (z)-\lambda L^\lambda_p (a,c)f (z). \quad (1.13)
\]
Also by specializing the parameters \( a,c,\lambda \) we obtain
Remark 1.4 If we put \( a=p+1, c=\lambda=1 \) in (1.11) then we get
\[
L^1_p (p+1,f (z))=f (z). \quad (1.14)
\]
Remark 1.5 If we put \( a=p, c=\lambda=1 \) in (1.11) then we get
\[
L^1_p (p,1)f (z)=\frac{zf'(z)}{p}. \quad (1.15)
\]
Remark 1.6 If we put \( c=a, \lambda=n \) in (1.12) then we get
\[
L^n_p (a,a)f (z)=D^{n+p-1}f (z) \quad (n>-p). \quad (1.16)
\]
where \( D^{n+p-1}f (z) \) is the well-known Ruchewey derivative of order \( n+p-1 \).

Now making use of Cho-Kwon-Srivastava operator \( L^\lambda_p (a,c) \) defined by (1.11), we introduced the following subclass \( S_p (f,g,a,b,c,\lambda,\beta) \) of \( p \)-valent analytic function.
then \( f(z) \in S_p(f, g, a, b, c, \lambda, \beta) \).

\[
\beta \left| \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right| \leq \left| \left( \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right) \right| \leq |b|.
\]

Let (2.2) hold. Then we have

\[
\beta \left| \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right| \leq \left| \left( \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right) \right| \leq (1 + \beta) \sum_{k=1}^{\infty} k \phi_k^i(a,c)b_{p+k} |a_{p+k}| \leq \left| b \right|.
\]

This completes the proof of Lemma 1.

3. Inclusion relationships involving \( N_\beta(h) \) of the class \( S_p(f, g, a, b, c, \lambda, \beta) \):

In our investigation of the inclusion relation involving \( N_\beta(h) \), we shall require the following theorem.

**Theorem 3.1A** A function \( f(z) \in T_p \) defined by (1.2) belongs to the class \( S_p(f, g, a, b, c, \lambda, \beta) \) if and only if

\[
\sum_{k=1}^{\infty} \left[ (k+1) + |\beta| \right] \phi_k^i(a,c)a_{p+k} b_{p+k} \leq |b| \quad (3.1)
\]

\(( p \in \mathbb{N}; a,c \in \mathbb{R}; |k| > p; 0 < \beta \leq 1; z \in U)\),

where \( \phi_k^i(a,c) \) defined by (2.2). The result is sharp.

**Proof.** We only need to prove the "only if" part of Theorem 3.1.

Let \( f(z) \) defined by (1.1) belongs to the class \( S_p(f, g, a, b, c, \lambda, \beta) \) we can write

\[
\beta \left| \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right| \leq \left| \left( \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right) \right| \leq (1 + \beta) \sum_{k=1}^{\infty} k \phi_k^i(a,c)b_{p+k} a_{p+k} |z|^k
\]

or equivalently,

\[
\beta \left| \frac{z \left[ L_p^k(a,c)(f \ast g)(z) \right]'}{L_p^k(a,c)(f \ast g)(z)} - p \right| \leq (1 + \beta) \sum_{k=1}^{\infty} k \phi_k^i(a,c)b_{p+k} a_{p+k} |z|^k
\]

\[
\leq \left| b \right| \left( 1 + \beta \right) \sum_{k=1}^{\infty} \phi_k^i(a,c)b_{p+k} a_{p+k} |z|^k
\]

We now choose values of \( z \) on the real axis and let \( z \to 1 \) through the real values, then the inequality (3.2) immediately yields the desired condition (3.1).

**Theorem 2** Let \( \delta = \frac{a(p+1)\beta}{c(1+\beta) + |\beta|} \) \( (p > |\beta|) \),

then \( S_p(f, g, a, b, c, \lambda, \beta) \subset N_\beta(h) \). (3.4)

**Proof.** Let \( f(z) \in S_p(f, g, a, b, c, \lambda, \beta) \), then in view of Theorem 1 and under the condition we have

\[
\left[ (k+1) + |\beta| \right] \phi_k^i(a,c) a_{p+k} b_{p+k} \leq \left[ \left( k \phi_k^i(a,c) a_{p+k} b_{p+k} \right) \right]
\]

Which yields

\[
\sum_{k=1}^{\infty} a_{p+k} \leq \frac{a}{\left[ (k+1) + |\beta| \right]} c(p+\lambda) b_{p+k} \quad (3.5)
\]

On the other hand, we also find from (3.1) and (3.5)

\[
\sum_{k=1}^{\infty} \left[ (k+1) + |\beta| \right] \phi_k^i(a,c) a_{p+k} b_{p+k} \leq \left| b \right|
\]

\[
+ \sum_{k=1}^{\infty} \left[ p - (k+1) \right] \phi_k^i(a,c) a_{p+k} b_{p+k} \leq \left| b \right|
\]

\[
\left. \left( (k+1) + |\beta| \right) \phi_k^i(a,c) a_{p+k} b_{p+k} \leq \left| b \right| \right|
\]

Or,

\[
\frac{c(p+\lambda) b_{p+k} \sum_{k=1}^{\infty} (k+1) + |\beta| a_{p+k}}{a} \leq \left| b \right| + \sum_{k=1}^{\infty} \left[ p - (k+1) \right] \phi_k^i(a,c) a_{p+k} b_{p+k}
\]

Hence

\[
\left| b \right| + \sum_{k=1}^{\infty} \left[ p - (k+1) \right] \phi_k^i(a,c) a_{p+k} b_{p+k}
\]

\[
\leq \left( p + 1 \right) \left| b \right| \left( (k+1) + |\beta| \right)
\]

\[
\left| b \right| \left( 1 + \beta \right)
\]

\[
\left| b \right|
\]

\[
Hence\]
\[ \sum_{k=1}^{\infty} (p+k)a_{p+k} \leq \frac{a(p+1)b}{c((\beta+1)+b)(\beta+\lambda)b_{p+1}} = \delta \quad (p \geq b), \quad (3.6) \]
then \( f(z) \in N_{\delta}(h) \). This completes the proof of Theorem 3.2

**Corollary 3.2** If \( \frac{(p+1)b}{c((\beta+1)+b)b_{p+1}} = \delta \quad (p \geq b), \quad (3.7) \)
and assume that \( k \geq 1 \), then \( S_{p}(f,g,p,b,\beta) \subseteq N_{\delta}(h) \) \( (3.8) \)

**Proof.** Putting \( a = p+1, c = \lambda = 1 \) in the Theorem 3.2.

**Corollary 3.3** If \( \delta = \frac{1-\gamma}{2+\gamma}, \quad 0 \leq \gamma \leq 1, (3.9) \)
and assume that \( k \geq 1 \). Then \( \text{UST}(\gamma,\beta) \subseteq N_{\delta}(h) \) \( (3.10) \)

**Proof.** Putting \( a = p+1, p = c = \lambda = 1, b = 1-\gamma \) \( (0 \leq \gamma < 1) \), and \( g(z) = \frac{z}{1-z} \) in the Theorem 3.2.

**Corollary 3.4** If \( \delta = \frac{1-\gamma}{3-\gamma}, \quad 0 \leq \gamma < 1, (3.11) \)
and assume that \( k \geq 1 \). Then \( S^{*}(\gamma) \subseteq N_{\delta}(h) \) \( (3.12) \)
Where \( S^{*}(\gamma) \) the class of startlike functions of order \( \gamma \).

**Proof.** Putting \( a = p+1, p = c = \lambda = \beta = 1, b = 1-\gamma \) \( (0 \leq \gamma < 1) \), and \( g(z) = \frac{z}{1-z} \) in the Theorem 3.2.

### 4. \( \delta \)-neighborhood for the class \( S^{(\alpha)}(f,g,a,b,c,\lambda,\beta) \).

In this section, we determine the neighborhood for the class \( S^{(\alpha)}(f,g,a,b,c,\lambda,\beta) \), which is define as follows.

**Definition 4.1** A function \( f(z) \in T_{p} \) is said to be in the class \( S^{(\alpha)}(f,g,a,b,c,\lambda,\beta) \) if there exist a function \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) such that,
\[
\left| \frac{f(z)}{k(z)} - 1 \right| < p - \alpha, (z \in U, 0 \leq \alpha < p). \quad (4.1)
\]

**Theorem 4.1** Let \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) and \( \alpha = p - \frac{\delta b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]}{(p+1)b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]} - |b| \), \( (4.2) \)
then \( N_{\delta}(k) \subseteq S^{(\alpha)}(f,g,a,b,c,\lambda,\beta). \quad (4.3) \)

**Proof.** Suppose that \( f(z) \in N_{\delta}(k) \). we find that from (1.4)
\[
\sum_{k=1}^{\infty} (p+k)a_{k+p} \leq c[p+1]b_{k+p+1} = \delta, \quad (4.4)
\]
which immediately have
\[
\sum_{k=1}^{\infty} a_{k+p} - c_{k+p} \leq \frac{\delta}{(1+p)} \quad (p \in N), \quad (4.5)
\]
as \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \), then we have from Theorem 3.1
\[
\sum_{k=1}^{\infty} c_{p+k} \leq \frac{|b|}{\phi^{\alpha}(a,c)b_{p+1}[\beta+1+b]} \quad (4.6)
\]
so that
\[
\left| \frac{f(z)}{k(z)} - 1 \right| < \frac{\delta b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]}{(p+1)b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]} - |b| = p - \alpha
\]
provided that \( \alpha \) is given by (3.2), thus \( f(z) \in S^{(\alpha)}(f,g,a,b,c,\lambda,\beta) \). This completes the proof of Theorem 4.1.

**Corollary 4.1** let \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) and \( \alpha = p - \frac{\delta b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]}{(p+1)b_{p+1}\phi^{\alpha}(a,c)[\beta+1+b]} - |b| \), \( (4.7) \)
then \( N_{\delta}(k) \subseteq S^{(\alpha)}(f,g,a,b,c,\lambda,\beta). \quad (4.8) \)

**Proof.** Putting \( a = p+1, c = \lambda = 1 \) in the Theorem 3.2.

**Corollary 4.2** let \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) and \( \alpha = 1 - \frac{\delta(2+\beta-\gamma)}{2(2+\beta)(1-\gamma)+2\gamma^{2}}, \quad (4.9) \)
then \( N_{\delta}(k) \subseteq \text{UST}^{(\alpha)}(\gamma,\beta). \quad (4.10) \)

**Proof.** Putting \( a = 2, p = c = \lambda = 1, b = 1-\gamma \) \( (0 \leq \gamma < 1) \), and \( g(z) = \frac{z}{1-z} \) in the Theorem 4.1.

**Corollary 4.3** let \( k(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) and \( \alpha = 1 - \frac{\delta(3-\gamma)}{6(1-\gamma)+2\gamma^{2}}, \quad (4.11) \)
then \( N_{\delta}(k) \subseteq \text{ST}^{(\alpha)}(\gamma). \quad (4.12) \)
Proof. Putting \( a=2, p=c=\lambda =\beta =1, b=1-\gamma \) (\( 0<\gamma <1 \)), and \( g(z)=\frac{z}{1-z} \) in the Theorem 4.1.

5. Subordination Results:

Definition 5.1 A sequence \( \{b_{p+k}\}_{k=0}^{\infty} \) of complex is called subordination factor sequence if for any regular and convex function

\[
k(z) = \sum_{k=0}^{\infty} c_{p+k}z^{p+k}, \quad \text{with} \quad c_{p} = 1, z \in U,
\]

\[
\sum_{k=0}^{\infty} b_{p+k}c_{p+k}z^{p+k} < g(z) \quad (z \in U).
\]

Lemma 5.1 [11] the sequence \( \{b_{p+k}\}_{k=0}^{\infty} \) is subordination factor if and only if

\[
\Re\left\{1+2\sum_{k=0}^{\infty} b_{p+k}z^{p+k}\right\} > 0 \quad (z \in U).
\]

Now, we obtain the subordination result for the class \( S_{p}(f,g,a,b,c,\lambda,\beta) \).

Theorem 5.1 let \( f(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) of the form (1.2) and

\[
m(z) = \sum_{k=0}^{\infty} c_{p+k}z^{p+k}, \quad c_{p} = 1, \text{be regular and convex function in } U, \text{then}
\]

\[
b_{p+k}f_{i}^{+}(a,c)(\beta+1+|p|)\left[\frac{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)+|p|}{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)}\right](f* m)(z) < m(z),
\]

moreover,

\[
\Re\{f(z)\} > \frac{2\left[ b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)+|p| \right]}{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)} ,
\]

and the subordination result (5.3) is sharp for the maximum factor

\[
b_{p+k}f_{i}^{+}(a,c)(\beta+1+|p|)\left[\frac{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)+|p|}{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)}\right] .
\]

Proof. Let \( f(z) \in S_{p}(f,g,a,b,c,\lambda,\beta) \) of the form (1.1) and

\[
m(z) = \sum_{k=0}^{\infty} c_{p+k}z^{p+k}, \quad c_{p} = 1,
\]

be regular and convex function in \( U \) to show subordinating result (5.3), we need to show that

\[
\left\{ \frac{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)}{2[b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)+|p|]} \right\} \geq 0, \quad (z \in U), (5.6)
\]

isa subordinating factor sequence with \( a_{p} = 1 \) which in view of lemma 5.1 is true if

\[
\Re\left\{1+\sum_{k=0}^{\infty} b_{p+k}f_{i}^{+}(a,c)(\beta+1+|p|)z^{p+k}\right\} > 0 \quad (z \in U).
\]

Since

\[
k[(\beta+1)+|p|]b_{p+i}f_{i}^{+}(a,c) \geq [(\beta+1)+|p|]b_{p+i}f_{i}^{+}(a,c) > 0 \quad (k \geq 1),
\]
on using Theorem 3.1, we have for \( |z| = r < 1 \),

\[
\Re\left\{1+\sum_{k=0}^{\infty} b_{p+k}f_{i}^{+}(a,c)(\beta+1+|p|)z^{p+k}\right\} = 1
\]

which evidently prove of (5.7), and hence the subordination result (5.3), we easily get the result (5.4), and for the function

\[
f(z) = z^{\alpha} - \frac{b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)}{2[b_{p+i}f_{i}^{+}(a,c)(\beta+1+|p|)+|p|]} z^{p+i} \in S_{p}(f,g,a,b,c,\lambda,\beta),
\]
it can be verified
\[ \frac{b_{p+1}(a, c)}{2b_{p+1}(a, c)} (\beta + 1 + |b|) = \text{is the maximum factor for the subordination result in (5.3).} \]

6. Partial Sum:
In this section, we determine inequalities involving partial sums of \( f(z) \in T_p \) where the Partial sum of \( f(z) \in T_p \) defined by (1.2) is defined as follows
\[ f_0(z) = z^p \quad \text{and} \quad f_n(z) = z^n - \sum_{i=1}^{n-1} a_{i, n} z^{i+1} \quad (a_{i, n} \geq 0; n \geq 1). \quad (6.1) \]

**Theorem 6.1** Let \( f(z) \in T_p \) be defined by (1.2) belong to \( S_p \), then from Theorem 3.1 and using \( H_{n+1}(f, g, a, b, c, \lambda, \beta) > H_n(f, g, a, b, c, \lambda, \beta) > 1 \), we get
\[ \sum_{k=1}^{n} a_{p+k} z^{k} + H_{n+1}(f, g, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^{k} \leq 1. \quad (6.6) \]

**Proof.** Let \( f(z) \in T_p \) be defined by (1.2) belong to \( S_p \), then from Theorem 3.1 and using \( H_{n+1}(f, g, a, b, c, \lambda, \beta) > H_n(f, g, a, b, c, \lambda, \beta) > 1 \), we get
\[ H_{n+1}(f, g, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^{k} \leq 1. \]

Set
\[ \frac{(f*g)(z)}{(f*g)_{n}(z)} \geq \left[ 1 - \frac{1}{H_{n+1}(f, g, a, b, c, \lambda, \beta)} \right] \quad (6.7) \]

so that
\[ \frac{H_{n+1}(f, g, a, b, c, \lambda, \beta) \sum_{k=n+1}^{\infty} a_{p+k} z^{k}}{1 - \sum_{k=1}^{n} a_{p+k} b_{p+k} z^{k}} = 1 \]

which is analytic in \( U \) and \( g_1(0) = 0 \). If (6.7) holds we find that
\[ \frac{g_1(z) - 1}{g_1(z) + 1} \]

which show the \( \Re \{ g(z) \} > 0 \), and from (6.7), we obtain the inequality (6.2).

Similarly, if we put
\[ g_1(z) = 1 + \frac{H_{n+1}(f, g, a, b, c, \lambda, \beta)}{1 - \sum_{k=1}^{n} a_{p+k} b_{p+k} z^{k}} \]

and making use (6.5), we find that
\[ \frac{g_1(z) - 1}{g_1(z) + 1} \]

which proves the inequality (6.3).
7. Integral means inequalities:

**Lemma 7.1** If the functions \( f \) and \( g \) are analytic in \( U \) with \( g \prec f \), then for
\[
\eta > 0 \text{ and } 0 < r < 1, \\
\int_0^{2\pi} \left| g(re^{i\theta}) \right|^\eta \ d\theta \leq \int_0^{2\pi} \left| f(re^{i\theta}) \right|^\eta \ d\theta.
\]


**Theorem 7.1** Let \( f(z) \in S_p(f, g, a, b, c, \lambda, \beta) \), and suppose that
\[
\sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^k \leq \left\| \frac{|\beta|}{j(\beta+1)+|\beta|} \right\| (j = 1, 2, \ldots), (7.1)
\]

If there exist analytic function \( w(z) \) given by
\[
\{w(z)\}_j = \left[ \frac{j(\beta+1)+|\beta|}{|\beta|} \right] \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^k, (7.2)
\]
then for \( z = re^{i\theta} \) \((0 < r < 1), \)
\[
\int_0^{2\pi} \left| \phi_k^k(a, c) (f \ast g)(z) \right|^i d\theta \leq \int_0^{2\pi} \left| \phi_k^k(a, c) (f \ast g)(z) \right|^i d\theta (\sigma > 0), (7.3)
\]
where \((f \ast g)_{p+j}(z)\) is given by
\[
(f \ast g)_{p+j}(z) = z^{p+j} - \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^{k+p+j} (7.4)
\]

**Proof.** By virtue of (1.11) and (7.4) we have
\[
L_p^f(a, c)(f \ast g)(z) = z^p - \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^{k+p} (7.5)
\]
and
\[
L_p^f(a, c)(f \ast g)_{p+j}(z) = z^p - \frac{|\beta|}{j(\beta+1)+|\beta|} z^{p+j} (7.6)
\]
then we must show that
\[
\int_0^{2\pi} \left| \phi_k^k(a, c) a_{p+k}b_{p+k} z^k \right|^i d\theta \leq \int_0^{2\pi} \left| \phi_k^k(a, c) a_{p+k}b_{p+k} z^k \right|^i d\theta (\sigma > 0), (7.7)
\]
by lemma 7.1, it is sufficient to show that
\[
1 - \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^k \leq 1 - \frac{|\beta|}{j(\beta+1)+|\beta|} z^i (7.7)
\]
Setting
\[
1 - \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^k = 1 - \frac{|\beta|}{j(\beta+1)+|\beta|} \{w(z)\}_j, (7.7)
\]
which readily yields (7.2) and \( w(0) = 0 \). Now, we prove that the analytic function \( w(z) \) satisfies \(|w(z)| < 1, z \in U \). Using (7.1), we obtain
\[
\left| \{w(z)\}_j \right| = \left[ \frac{j(\beta+1)+|\beta|}{|\beta|} \right] \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} z^k < \left[ \frac{j(\beta+1)+|\beta|}{|\beta|} \right] \sum_{k=1}^{\infty} \phi_k^k(a, c) a_{p+k}b_{p+k} < 1.
\]
This completes the proof of the Theorem.

**Corollary 3.8** Let \( f(z) \in S_p(f, g, a, b, c, \lambda, \beta) \) and suppose that
\[
\sum_{k=1}^{\infty} a_{p+k}b_{p+k} \leq \left[ \frac{|\beta|}{j(\beta+1)+|\beta|} \right] (j = 1, 2, \ldots), (7.8)
\]
then for \( z = re^{i\theta} \) \((0 < r < 1), \)
\[
\int_0^{2\pi} \left| (f \ast g)(z) \right|^i d\theta \leq \int_0^{2\pi} \left| (f \ast g)_{p+j}(z) \right|^i d\theta (\sigma > 0), (7.9)
\]
where \((f \ast g)_{p+j}(z)\) is given by
\[
(f \ast g)_{p+j}(z) = z^{p+j} - \sum_{k=1}^{\infty} a_{p+k}b_{p+k} z^{k+p+j} (7.10)
\]
**Proof.** Putting \( a = p+1, c = \lambda = 1 \) in Theorem 7.1.

References: