On Fredholm Theory In a Banach

Algebra of Operators

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Introduction
Let \( B(X) \) be the Banach algebra of all bounded linear operators on the infinite dimensional Banach space \( X \).
We say that two infinite dimensional Banach spaces \( X \) and \( Y \) form a dual system, denoted \( (X,Y) \), if there is defined on \( X \times Y \) a nondegenerated bounded bilinear form \( < \, , \, > \) [6, P. 6].
Let \( T \in B(x) \), if there exists \( T^* \in B(Y) \) such that
\[
<Tx, y> = <x, T^* y>
\]
for all \( x \in X \) and \( y \in Y \), then \( T^* \) is called a conjugate operator to \( T \) relative to the dual system \((X,Y)\). [4, p. 44]
Let \( A(X,Y) \) be the algebra of all \( T \in B(X) \) that have, with respect to the dual system \((X,Y)\), a conjugate \( T^* \in B(Y) \). \( A(X,Y) \) is a Banach algebra with respect to the normal given by
\[
\|T\|_4 = \max \{\|T\|,\|T^*\|\}, \quad [4, \text{P. 45}]
\]
An operator \( T \) in \( B(X) \) is called a Fredholm operator if both \( \alpha(T) \) and \( \beta(T) \) are finite, Where \( \alpha(T) \) denotes the dimension of the null space of \( T \), \( \dim N(T) \), and \( \beta(T) \) denotes the codimension of the image of \( T \), \( \text{codim } R(T) \) [2, P. 3]
In this paper we study the connection between some properties of operators \( T \in A(X,Y) \) and their conjugates \( T^* \). For example we show that if \( T \) is an operator in \( A(X,Y) \), such that the dimension of the null space of \( T \) is finite or the codimension of the image of \( T^* \) is finite, then \( R(T^*) = N(T)^\perp \) and \( \beta(T^*) = \alpha(T) \) are equivalent. We also prove that if the codimension of the image of \( T \) is finite, then the dimension of the null space of the conjugate operator \( T^* \) is also finite and in fact, is less or equal it.
Now let \( \Phi(X) \) be the set of all Fredholm operators in \( B(X) \), and for any \( T \) in \( \Phi(X) \), the index \( \text{ind}(T) \), is defined by the formula
\[
\text{ind}(T) = \alpha(T) - \beta(T)
\]
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let $\mathcal{D}_\theta(X)$ be the set of all Fredholm operators of index zero and $F(X)$ be the set of all finite rank operators.

Also let $\mathcal{M}$ be the subspace generated or spanned by a non-empty subset $M$ of $X$.

§ 1. Fredholm Operators in $B(X)$.

In this section we study the connection between the properties of operators $T$ and their conjugates $T^*.$

First we give the following general proposition which appeared in [7, P. 99] for the particular case of the dual system $(X,X')$ where $X'$ is the dual space of $X$ and $(X,X')$ the dual system with bilinear form defined by $<x,x'> = x'(x)$.

**Proposition (1.1)**

Let $T \in A(X,Y)$, then we have

1. $R(T)^\bot = N(T^*)$
2. $N(T^*) = N(T^*)^\bot\bot$
3. $R(T^*)^\bot = N(T)$
4. $N(T) = N(T)^\bot\bot$

**Proof:**

1. $y \in N(T^*) \iff T^*y = 0 \iff <x,T^*y> = 0$ for all $x \in X$
2. $\iff <Tx,y> = 0$ for all $x \in X$

$\iff y \in R(T)^\bot$

(2) From part (1), it follows that

$N(T^*)^\bot\bot = (R(T)^\bot)^\bot = R(T)^\bot = N(T^*)$

In similar way, we can prove (3) and (4). □

Form the previous proposition, we Conclude the following:

**Corollary (1.2)**

Let $T \in A(X,Y)$, if $T$ (or $T^*$) is surjective, then $T^*$ (or $T$) is injective.

Also we need the following two lemmas which are known.

**Lemma (1.3)**

Assume $T \in A(X,Y)$

1. If $x_1,x_2,...,x_n$ are elements in $N(T)$ and $y_1,y_2,...,y_n$ are elements in $Y$ such that $<x_i,y_k> = \delta_{ik}$ for $i, k = 1,2,...,n$ then $\{y_1,y_2,...,y_n\} \cap R(T^*) = \{0\}$
2. If $w_1,...,w_n$ are elements in $N(T^*)$ and if $z_1,...,z_n$ are elements in $X$ such that $<z_i,w_k> = \delta_{ik}$ for $i, k = 1,...,n$ then $\{z_1,...,z_n\} \cap R(T) = \{0\}$

lemma (1.4)
If E and F are subspaces of a vector space V whose intersection is trivial, then E has a complementary space that contains F.

Now we can state one of our main results.

**Theorem (1.5)**

Let $T \in A(X,Y)$

1. If $\alpha(T) < \infty$ then $R(T^*) = N(T)^\perp$ iff $\beta(T^*) = \alpha(T)$
2. If $\alpha(T^*) < \infty$ then $R(T) = N(T^*)$ iff $\beta(T) = \alpha(T^*)$

**proof**

$(1) \Rightarrow$ If $\alpha(T) = 0$, then it is clear that $\beta(T^*) = \alpha(T)$. Now, assume that $\alpha(T) = n > 0$ and $\{x_1, \ldots, x_n\}$ is a basis of $N(T)$, there exist $n$ linearly independent elements $y_1, y_2, \ldots, y_n$ in $Y$ such that $<x_i, y_k> = \delta_{ik}$ for $i, k = 1, \ldots, n$ [5, P. 18], [6, P. 63]. So $\{y_1, \ldots, y_n\} \cap R(T^*) = \{0\}$ [lemma 1.3].

For every $y$ in $Y$, we define

$$w_y = y - \sum_{i=1}^{n} <x_i, y> y_i$$

so

$$<x_k, w_y> = <x_k, y> - \sum_{i=1}^{n} <x_i, y> <x_k, y_i> = 0$$

for all $k, 1 \leq k \leq n$

this implies that $w_y \in N(T)^\perp$ so $w_y \in R(T^*)$

Hence, every $y \in Y$ can be represented in the form

$$y = \alpha_1 y_1 + \ldots + \alpha_n y_n + w_y$$

so $w_y \in R(T^*)$ and $\alpha_i = <x_i, y>$

Therefore, $\{y_1, \ldots, y_n\} \oplus R(T^*) = Y$ and so

$\text{codim } R(T^*) = n = \text{dim } N(T)$ that is $\beta(T^*) = \alpha(T)$

$(\Leftarrow)$ Clearly, $R(T^*) \subset N(T)^\perp$ [3, P. 90, P. 134]. To prove that $N(T)^\perp \subset R(T^*)$, if $\alpha(T) = 0$, then $N(T) = 0$ implies that $N(T)^\perp = Y$ and hence $R(T^*) = N(T)^\perp$. Assume that $\alpha(T) = n > 0$ and $\{x_1, \ldots, x_n\}$ is a basis for $N(T)$ [5, P. 18] [6, P. 63].

There exist $n$ linearly independent elements $y_1, \ldots, y_n$ in $Y$ such that $<x_i, y_k> = \delta_{ik}$ for $i, k = 1, \ldots, n$

so $\{y_1, \ldots, y_n\} \cap R(T^*) = \{0\}$ [lemma 1.3] and $R(T^*)$ has a complementary space that contains $\{y_1, \ldots, y_n\}$ since
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\[ \text{dim} \{ y_1, \ldots, y_n \} = n = \alpha(T) = \beta(T^+) \] 

hence 

\[ Y = \{ y_1, \ldots, y_n \} \oplus R(T^+) \] 

Now, if \( y \in N(T) \perp \subseteq Y \), then 

\[ y = \alpha_1 y_1 + \ldots + \alpha_n y_n + T^* w \quad \alpha_i \in \mathbb{C}, w \in Y \quad \text{since for all } k, \]

\[ 1 \leq k \leq n, \text{ we have} \]

\[ \theta = \langle x_k, y \rangle = \sum \alpha_i \langle x_k, y_i \rangle + \langle x_k, T^* w \rangle = \alpha_k + \langle T x_k, w \rangle = \alpha_k \]

hence \( y = T^* w \), i. e. \( y \in R(T^+) \), thus \( N(T) \perp \subseteq R(T^+) \)

therefore \( R(T^+) = N(T) \perp \).

(2) Assume \( T \in A(X, Y) \) and \( \alpha(T^+) < \infty \), then \( T^* \in A(Y, X) \) and 

\( T \) is the conjugate operator to \( T^* \) and since \( \alpha(T^+) < \infty \), then by part one 

\( R(T) = N(T^+) \) and \( \beta(T) = \alpha(T^+) \) are equivalent. \( \square \)

Now, suppose \( T \in A(X, Y) \). In the following theorem we show that if the codimension of the image of \( T \) is finite then the dimension of the null space of the conjugate operator \( T^* \) to \( T \) is also finite.

**Theorem (1.6)**

Let \( T \in A(X, Y) \)

(1) If \( \beta(T) < \infty \), then \( \alpha(T^+) < \infty \). In particular \( \alpha(T^+) \leq \beta(T) \).

(2) If \( \beta(T^+) < \infty \), then \( \alpha(T) < \infty \). In particular \( \alpha(T) \leq \beta(T^+) \).

**Proof:**

(1) If \( \beta(T) = 0 \), then \( R(T) = X \) this implies that the conjugate operator 

\( T^* \) to \( T \) is injective [ Corollary 1.2] so \( \alpha(T^+) = 0 \).

Now we may assume that \( \beta(T) = m > 0 \), and assume the contrary, that 

\( N(T^+) \) is infinite dimensional, therefore one can choose \( n \) linearly 

independent elements \( y_1, \ldots, y_n \) in \( N(T^+) \) such that \( n > m \).

There exist \( n \) linearly independent elements \( x_1, \ldots, x_n \) in \( X \) such that 

\[ \langle x_i, y_k \rangle = \delta_{ik} \quad i, k = 1, 2, \ldots, n \]

then \( \{ x_1, \ldots, x_n \} \cap R(T) = \{ 0 \} \) (lemma 1.3).

Since \( R(T) \) has a complementary space which contains \( \{ x_1, x_2, \ldots, x_n \} \), then 

\[ n = \text{dim} \{ x_1, \ldots, x_n \} \leq \text{codim} R(T) = \beta(T) = m \]

and this is contradiction.
Thus $\alpha(T^*)$ must be finite and also $\alpha(T^*) \leq \beta(T)$.

2. Since $T \in A(X,Y)$ then $T^+ \in A(Y,X)$ and since $\beta(T^+) < \infty$, therefore by part (1) $\alpha(T) \leq \beta(T^+)$. \(\square\)

Using this theorem we have the following Corollary

**Corollary (1.7)**

Let $T \in A(X,Y)$, if $\beta(T)$ and $\beta(T^+)$ are finite, then $T \in \Phi(X)$ and $T^+ \in \Phi(Y)$.

Form our preceding theorems (1.5) and (1.6), we obtain directly the following result.

**Corollary (1.8)**

Let $T \in A(X,Y)$, then

1. If $\beta(T) < \infty$, then $R(T) = N(T^+)\perp$ if and only if $\beta(T) = \alpha(T^+)$.
2. If $\beta(T^+) < \infty$, then $R(T^+) = N(T)\perp$ if and only if $\beta(T^+) = \alpha(T)$.

It was proved in [4, P. 111] that if $T \in \Phi(X), T^+ \in \Phi(Y)$ and $\text{ind}(T) = -\text{ind}(T^+)$ then $R(T) = N(T^+)\perp$, $R(T^+) = N(T)\perp$.

\begin{align*}
\beta(T) &= \alpha(T^+) \quad \text{and} \quad \beta(T^+) = \alpha(T)
\end{align*}

We prove a stronger result as follows

**Theorem (1.9)**

Let $T \in A(X,Y), T^+ \in \Phi(Y)$, then the following statements are equivalent:

1. $R(T) = N(T^+) \quad$ and $\quad R(T^+) = N(T)$
2. $\beta(T) = \alpha(T^+) \quad$ and $\quad \beta(T^+) = \alpha(T)$
3. $\text{ind}(T) = -\text{ind}(T^+)$

**Proof:**

1. $(1) \iff (2)$ follows from Theorem (1.4)

2. $(2) \implies (3)$ trivial.

3. $(3) \implies (2)$ since $\beta(T)$ and $\beta(T^+)$ are finite, by Theorem (1.6),

\begin{align*}
\alpha(T^+) \leq \beta(T) \quad\text{and}\quad \alpha(T) \leq \beta(T^+)
\end{align*}

then

\begin{align*}
\text{ind}(T) &= \alpha(T) - \beta(T) \leq \beta(T^+) - \alpha(T^+) = -\text{ind}(T^+) = \text{ind}(T)
\end{align*}

Therefore, $\alpha(T) - \beta(T) = \beta(T^+) - \alpha(T^+)$. Thus $\alpha(T) = \beta(T^+)$, and $\alpha(T^+) = \beta(T)$.

From the previous theorem, we have the following corollary.

**Corollary (1.10)**
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Assume \( T \in A(X,Y) \), \( T \in \Phi^0(X), T^* \in \Phi^0(Y) \), then \( T \) and \( T^* \) are bijective if and only if one of them is injective or surjective. Let \( T \in A(X,Y) \), Jorgens in \([4, P. 111]\) proved that if \( T \in \Phi(X), T^* \in \Phi(Y) \), and \( \text{ind}(T) = -\text{ind}(T^*) \), then there exist projections \( P \) and \( Q \) in \( F(A) = A(X,Y) \cap F(X) \) such that \( R(P) = N(T) \) \( N(P^*) = R(T^*) \) \( R(Q) = R(T) \) \( R(Q^*) = N(T^*) \) In fact, we can say more than this 

**Theorem (1.11)** 

Let \( T \in A(X,Y) \). Then the following are equivalent: 

1. \( T \in \Phi(X), T^* \in \Phi(Y) \), and \( \text{ind}(T) = -\text{ind}(T^*) \) 
2. There exist projections \( P \) and \( Q \) in \( F(A) \) such that \( R(P) = N(T) \) \( N(P^*) = R(T^*) \) \( R(Q) = R(T) \) \( R(Q^*) = N(T^*) \) 

**Proof** 

(1) \( \Rightarrow \) (2) 

Let \( \{x_1, \ldots, x_n\} \) be a basis of \( N(T) \), then there exist \( n \) linearly independent elements \( y_1, \ldots, y_n \) in \( y \) such that \( <x_i, y_k> = \delta_{ik} \) \( i, k = 1, \ldots, n \) \([5, P. 18]\) \([6, P. 63]\) 

Define \( P \) by \( P = \sum_{i=1}^{n} <x_i, y_k> x_i \). \([4, p. 47]\) 

So \( P \in F(A) \), and since for all \( k \), \( 1 \leq k \leq n, Px_k = x_k \), \( P \) is a projection and \( R(P) = \{x_1, \ldots, x_n\} = N(T) \). Clearly, the conjugate operator \( P^* \) to \( P \) is given by \( P^* y = \sum_{i=1}^{n} <x_i, y> y_i \), thus \( N(P^*) = \{y \in Y : <x_i, y> = 0 \} \) for all \( i, 1 \leq i \leq n \) \( = \{ y \in Y : <x_i, y> = 0 \} \) for all \( x \in N(T) \) \( = N(T)^\perp \) but \( R(T^+) = N(T^+)^\perp \) [Theorem 1.9], hence \( N(P^*) = R(T^+) \). 

We can define the projection \( Q \) in \( F(A) \) by \( Q = \sum_{i=1}^{n} <x_i, w_i> z_i \) 

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With \( w_1, \ldots, w_n \) form a basis of \( N(T^*) \) and \( z_1, \ldots, z_n \) linearly independent elements in \( X \) such that \( < z_i, w_k > = \delta_{ik} \).

In a similar way and with the aid of \( R(T) = N(T^*)^\perp \) we can show that
\[
N(Q) = R(T) \quad R(Q^*) = N(T^*)
\]

\((2) \Rightarrow (1)\)

To prove that \( T \in \Phi(X) \), it follows from \( P \in F(X) \) that \( \dim R(P) < \infty \) and since \( R(P) = N(T) \), so \( \alpha(T) < \infty \). Since \( Q \) is a projection and \( N(Q) = R(T) \), this implies that \( R(Q) \) is a Complementary space to \( R(T) \) and since \( Q \in F(X) \), then \( \dim R(Q) < \infty \) and so \( \beta(T) < \infty \). Now, we show that \( T^* \in \Phi(Y) \).

Since \( P \) and \( Q \) are projections of finite rank, we can easily show that \( P^* \) and \( Q^* \) are also projections of finite rank, therefore,
\[
\alpha(T^*) = \dim N(T^*) = \dim R(Q^*) < \infty \quad \text{and by} \quad N(P^*) = R(T^*)
\]
\[
\beta(T^*) = \text{codim } R(T^*) = \dim R(P^*) < \infty.
\]

It remains to prove that \( \text{ind}(T) = -\text{ind}(T^*) \). For this, we show that \( \alpha(T^*) = \beta(T) \). By hypothesis, there exists a projection \( Q \) in \( F(A) \) such that \( N(Q) = R(T) \) and \( R(Q^*) = N(T^*) \), by [4, P.47] \( Q \) can be represented in the form \( Qx = \sum_{i=1}^{n} < x, y_i > x_i \),

where \( x_1, \ldots, x_n \) are linearly independent in \( X \) and \( y_1, y_2, \ldots, y_n \) are linearly independent in \( Y \). By [3, P. 125 \( R(Q) = \{ x_1, \ldots, x_n \} \). and since \( N(Q) = R(T) \), therefore \( X = \{ x_1, \ldots, x_n \} \oplus R(T) \) and so \( \beta(T) = n \).

Now, since \( R(T) = N(Q) = \{ z \in X : < z, y_k > = 0, k = 1, \ldots, n \} \), then for all \( k, l \leq k \leq n, 0 = < T_x, y_k > = < x, T^* y_k > \) for all \( x \in X \) and since \( (X, Y) \) is a dual system, then \( T^* y_k = 0 \) for all \( k, l \leq k \leq n \), that is \( y_k \in N(T^*) \) which are linearly independent in \( N(T^*) \), therefore \( n \leq \dim(N^*) \) and so \( \beta(T) \leq \alpha(T^*) \). Since \( \beta(T) \leq \infty \) then \( \alpha(T^*) \leq \beta(T) \) [Theorem 1.6]. Hence \( \alpha(T^*) = \beta(T) \). Similarly, with the aid of the projection \( P, \alpha(T) \leq \beta(T^*) \). Thus
\[
\text{ind}(T) = \alpha(T) - \beta(T) = \beta(T^*) - \alpha(T^*) = -\text{ind}(T^*)
\]
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References