Some Properties of Centralizing in a Unital Complex Banach Algebra

As‘ad Y. As‘ad  Department of Mathematics, Islamic University of Gaza
B.O Box : 108 , E-mail: aasad@mail.iugaza.edu

بعض خواص المركزيَّة في جبر باخ العقدي

ملخص: في هذا البحث تم بحث أنَّ إذا كان J هو مثلي معقل لجبر باخ الوحدات العقدي وأنَّ a هي عضو في J، و b عضو في J فإنَّ ab = ba، و ab = ba عندنا خليلاً على بعضهما البعض إذا وفقط إذا كانت ab = ba.

Abstract In this paper we show that, for a closed ideal J of a unital complexBanach algebra A, if a is a quasi centralizing element of J and b belongs to J, then ab and ba are linearly dependent if and only if ab = ba. Also we get generalizations of some of Rennison's results.

1. Introduction

In this paper we study some properties of centralizing in a unital complex Banach algebra, where we generalize some results related to centrality in a unital complex Banach algebra that was obtained by Rennison in [7] and Sarsour and As‘ad in [8].

Throughout this paper all linear spaces and algebras are assumed to be defined over \( \mathbb{C} \), the field of complex numbers.

Let A be any complex normed algebra, then we denote the center of A by
\[ Z(A) = \{ a \in A : ax = xa \text{ for all } x \in A \}, \]
and the centralizer of a subset B of A by \( C(B) = \{ a \in A : ax = xa \text{ for all } x \in B \} \). For a \( a \in A \), the spectrum in A of \( a \) will be denoted by \( \sigma_a(a) \) and the resolvent set, its complement in \( \mathbb{C} \), will be denoted by \( \rho_A(a) \).

In [6] Rennison defined the set of all quasi central elements in a complex Banach algebra A by \( Q(A) = \bigcup_{k \in \mathbb{C}} Q(k, A) \), where
\[ Q(k, A) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \mathbb{C} \}. \]
Also he defined the set of all \( \sigma \)-quasi central elements in A by \( Q_\sigma(A) = \bigcup_{k \in \mathbb{C}} Q_\sigma(k, A) \), where
\[ Q_\sigma(k, A) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \rho_A(a) \}. \]

In [5] the set of all \( \rho \)-quasi central elements in A was defined by
\[ Q_\rho(A) = \bigcup_{k \in \mathbb{C}} Q_\rho(k, A), \] where \( Q_\rho(k, A) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \text{ for all } x \in A \text{ and all } \lambda \in \sigma_A(a) \}. \]
Similarly in [1] the following three concepts were defined as follows:

1) The quasi centralizer (quasi-commutant) of B is $\text{QC}(B) = \bigcup_{k \geq 1} \text{QC}(k, B)$, where $\text{QC}(k, B) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \}$ for all $x \in B$ and all $\lambda \in \mathcal{E}$. 

2) The $\sigma$-quasi centralizer ($\sigma$-quasi-commutant) of B is $\text{QC}_\sigma(B) = \bigcup_{k \geq 1} \text{QC}_\sigma(k, B)$, where $\text{QC}_\sigma(k, B) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \}$ for all $x \in B$ and all $\lambda \in \rho_\lambda(a)$. 

3) The $\rho$-quasi centralizer ($\rho$-quasi-commutant) of B is $\text{QC}_\rho(B) = \bigcup_{k \geq 1} \text{QC}_\rho(k, B)$, where $\text{QC}_\rho(k, B) = \{ a \in A : \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \}$ for all $x \in B$ and all $\lambda \in \sigma_\lambda(a)$.

**Remark:** In this remark we state [1, Theorem 2.1] that will be used frequently in this paper, the theorem states that:

If $A$ is a complex normed algebra and $D \subseteq B \subseteq A$. Then for $k \geq 1$,

(i) $C(B) \subseteq \text{QC}(k, B) = \text{QC}_\sigma(k, B) \cap \text{QC}_\rho(k, B)$.

(ii) $Q(k, A) = \text{QC}(k, A) \subseteq \text{QC}(k, B) \subseteq \text{QC}(k, D)$.

(iii) $Q_\sigma(k, A) = \text{QC}_\sigma(k, A) \subseteq \text{QC}_\sigma(k, B)$.

(iv) $Q_\rho(k, A) = \text{QC}_\rho(k, A) \subseteq \text{QC}_\rho(k, B) \subseteq \text{QC}_\rho(k, D)$.

2. Some Properties of Centralizing

For a Banach algebra A we well denote $A^{-1}$ for the set of all invertible elements in A. The following propositions generalise some results in [7] and [8] which have similar proofs.

**Proposition 2.1:** Let $J$ be an ideal of a unital complex Banach algebra $A$ and $k \geq 1$.

(i) If $a \in \text{QC}_\sigma(k, J)$ and $p$ is a complex polynomial of degree $d$ then $p(a) \in \text{QC}_\sigma(k^d, J)$.

(ii) If $a \in \text{QC}_\sigma(k, J) \cap A^{-1}$, then $a^{-1} \in \text{QC}_\sigma(k \| a \| \| a^{-1} \|, J)$.

(iii) If $a \in \text{QC}_\sigma(k, J)$ and $b \in A^{-1}$, then $b^{-1}ab \in \text{QC}_\sigma(k \| b \|^2 \| b^{-1} \|^2, J)$.

**Proof:**

(i) Let $x \in J$ and $\lambda \in \rho_\lambda(p(a))$ be arbitrary.

However, for any $\lambda \in \rho_\lambda(p(a))$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$, $\alpha_d$ and $\beta$ such that $\lambda - p(a) = \beta (\alpha_1 - a)(\alpha_2 - a) \ldots \ldots$
( \alpha_{d-1} - a ) ( \alpha_d - a ). [ See 3, page 21]. Then \alpha_1, \alpha_2, \ldots, \alpha_{d-1}, \alpha_d belong to \rho_d(a).

Now, since \( a \in QC_\sigma(k,J) \) and \( J \) is an ideal of \( A \), then
\[
\| x (\lambda - p(a)) \| = \| x \beta (\alpha_1 - a) (\alpha_2 - a) \ldots (\alpha_{d-1} - a) (\alpha_d - a) \| \\
\leq k \| (\alpha_d - a) x (\alpha_1 - a) (\alpha_2 - a) \ldots (\alpha_{d-1} - a) \| \leq k^d
\]
and so on after d-steps we get that
\[
\| x (\lambda - p(a)) \| \leq k^d \| (\alpha - a) (\alpha_2 - a) \ldots (\alpha_{d-1} - a) (\alpha_d - a) \| \beta \|
\]
\[
= k^d \| (\lambda - p(a)) x \|.
\]
Therefore, \( p(a) \in QC_\sigma(k,J) \).

(ii) Let \( a \in QC_\sigma(k,J) \cap A^{-1} \), then \( \| x (\lambda - a) \| \leq k \| (\lambda - a) x \| \)
for all \( x \in J \) and all \( \lambda \in \rho_d(a) \). Since \( a^{-1} \in A \) then \( 0 \in \rho_d(a) \) and \( 0 \in \rho_d(a^{-1}) \).

Note that, for \( \lambda \neq 0 \), \( \lambda \in \rho_d(a^{-1}) \) if and only if \( (\lambda - a^{-1})^{-1} \) exists if and only if \( (\lambda^{-1} a^{-1}) \) \( (\lambda - a^{-1})^{-1} \) exists if and only if \( (\lambda^{-1} a)^{-1} \) exists if and only if \( \lambda^{-1} \in \rho_d(a) \).

Now for any \( \lambda \in \rho_d(a^{-1}) \), \( \lambda \neq 0 \) and all \( x \in J \) we have
\[
\| x (\lambda - a^{-1}) \| = \| x (\lambda^{-1} - a) (\lambda^{-1} a^{-1}) \| \leq \| x (\lambda^{-1} - a) \| \| \lambda^{-1} \| \| a^{-1} \|
\]
\[
\leq k \| (\lambda^{-1} - a) x \| \| \lambda^{-1} \| \| a^{-1} \| = k \| \lambda^{-1} a (\lambda - a^{-1}) x \| \| \lambda^{-1} \| \| a^{-1} \|
\]
\[
\leq k \| (\lambda - a^{-1}) x \| \| a \| \| a^{-1} \|.
\]
However, \( \| x a^{-1} \| = \| a a^{-1} x a^{-1} \| \leq \| a^{-1} \| \| a \| \| a^{-1} \|
\]
\[
\leq k \| a^{-1} \| \| a \| \| a^{-1} \|.
\]

Therefore, \( \| x (\lambda - a^{-1}) \| \leq k \| a \| \| a^{-1} \| \| (\lambda - a^{-1}) x \| \)
for all \( x \in J \) and all \( \lambda \in \rho_d(a^{-1}) \). This means that \( a^{-1} \in QC_\sigma(k \| a \| \| a^{-1} \|, J) \).

(iii) Let \( a \in QC_\sigma(k,J) \) and \( b \in A^{-1} \) then,
\[
\rho_d(b^{-1} a) = \{ \lambda \in \mathbb{C} : (\lambda - b^{-1} a)^{-1} \text{ exists} \} = \{ \lambda \in \mathbb{C} : b^{-1} (\lambda - a)^{-1} b \text{ exists} \}
\]
\[
= \rho_d(a).
\]

Now for any \( \lambda \in \rho_d(b^{-1} a) \) and all \( x \in J \) we have:
\[
\| x (\lambda - b^{-1} a) b \| = \| x b^{-1} (\lambda - a) b \| \leq \| b^{-1} b x b^{-1} (\lambda - a) \| \| b \|
\]
\[
\leq \| b^{-1} \| \| b x b^{-1} (\lambda - a) \| \| b \|
\]
\[
\leq k \| (\lambda - a) b x b^{-1} \| \| b \| \| b^{-1} \| \text{ since } a \in QC_\sigma(k,J) \text{ and } b x b^{-1} \in J.
\]
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\[ = k \| b(\lambda - b^{-1}ab) \times b^{-1}\| \| b \| \| b^{-1}\| \leq k \| b \| ^2 \| b^{-1}\| ^2 \| (\lambda - b^{-1}ab) \times \] 

Therefore, \( b^{-1}ab \in QC_\sigma(k \| b \| ^2 \| b^{-1}\| ^2, J) \) \qed

Similarly one can prove the following two propositions for \( \rho \)-quasi and quasi centralizer. However, Proposition 2.3 (ii) and (iii) can be proved in another way by using Proposition 2.1 and 2.2 and [1, Theorem 2.1(i)].

**Proposition 2.2:-** Let \( J \) be an ideal of a unital complex Banach algebra \( A \) and \( k \geq 1 \).

(i) \( \text{If } a \in QC_\rho(k, J) \cap A^{-1} \text{ then } a^{-1} \in QC_\rho(k \| a \| \| a^{-1}\|, J). \)

(ii) \( \text{If } a \in QC_\rho(k, J) \) and \( b \in A^{-1} \text{ then } b^{-1}ab \in QC_\rho(k \| b \| ^2 \| b^{-1}\| ^2, J). \)

**Proposition 2.3:-** Let \( J \) be an ideal of a unital complex Banach algebra \( A \) and \( k \geq 1 \).

(i) \( \text{If } a \in QC(k, J) \) and \( p \) is a complex polynomial of degree \( d \) then \( p(a) \in QC(k^d, J) \).

(ii) \( \text{If } a \in QC(k, J) \cap A^{-1} \text{ then } a^{-1} \in QC(k \| a \| \| a^{-1}\|, J) \).

(iii) \( \text{If } a \in QC(k, J) \) and \( b \in A^{-1} \text{ then } b^{-1}ab \in QC(k \| b \| ^2 \| b^{-1}\| ^2, J) \).

Now we can remark here that Renneson’s and As’ad’s results in [7] and [8] respectively become corollaries of our propositions.

**Corollary 1 :-** Let \( A \) be a unital complex Banach algebra, \( k \geq 1 \) and \( p \) is a complex polynomial of degree \( d \).

(i) \( \text{If } a \in Q_\sigma(k, A), \text{ then } p(a) \in Q_\sigma(k^d, A). \)

(ii) \( \text{If } a \in Q_\sigma(k, A) \cap A^{-1}, \text{ then } a^{-1} \in Q_\sigma(k \| a \| \| a^{-1}\|, A). \)

(iii) \( \text{If } a \in Q_\sigma(k, A) \) and \( b \in A^{-1}, \text{ then } b^{-1}ab \in Q_\sigma(k \| b \| ^2 \| b^{-1}\| ^2, A) \).

(iv) \( \text{If } a \in Q_\rho(k, A) \cap A^{-1}, \text{ then } a^{-1} \in Q_\rho(k \| a \| \| a^{-1}\|, A). \)

(v) \( \text{If } a \in Q_\rho(k, A) \) and \( b \in A^{-1}, \text{ then } b^{-1}ab \in Q_\rho(k \| b \| ^2 \| b^{-1}\| ^2, A). \)

(vi) \( \text{If } a \in Q(k, A) \text{ then } p(a) \in Q(k^d, A). \)

(vii) \( \text{If } a \in Q(k, A) \cap A^{-1}, \text{ then } a^{-1} \in Q(k \| a \| \| a^{-1}\|, A). \)
(viii) If \( a \in Q(k, A) \) and \( b \in A^{-1} \), then \( b^{-1}ab \in Q(k \| b \|^{2}, b^{-1} \| b^{-1} \|^{2}, A) \).

**Proof:**
Let \( J = A \) in the above proposition, then use [1, Theorem 2.1].

### 3. Linearly Dependence in Quasi Centralizing Sets

If \( A \) is a complex normed algebra and \( a, b \in A \), then the inner derivation corresponding to \( a \) is denoted by \( D_{a} \), which is a bounded linear operator on \( A \) defined by \( D_{a}x = ax - xa \). The bounded linear operators \( I_{a} \) and \( R_{a} \) on \( A \) are defined by \( I_{a}x = ax \) and \( R_{a}x = xa \). \( D_{a} \) is called topologically nilpotent if \( \lim_{n \to \infty} \| D_{a} \|^n = 0 \). The following theorem generalises proposition 3.4 in [7] (which has a similar proof), so the proposition becomes a corollary of our theorem. In the proof of the following theorem we shall use [2, Theorem 2.3] which states that \( \text{Let } A \text{ be a complex Banach algebra with unity, } J \text{ a closed ideal of } A, \text{ and } a \in QC(J) \). Then \( D_{a} \) is topologically nilpotent. Where \( D_{a} \) is restricted to a domain \( J \).

**Theorem 3.1** Let \( J \) be a closed ideal of a unital complex Banach algebra \( A \), and let \( a \in QC(J) \) and \( b \in J \). Then \( ab \) and \( ba \) are linearly dependent if and only if \( ab = ba \).

**Proof:**
Suppose that \( ab \) and \( ba \) are linearly dependent. Then there exist \( \alpha, \beta \in \mathbb{C} \) such that \( \alpha \neq 0 \) or \( \beta \neq 0 \) and \( \alpha ab + \beta ba = 0 \). Then two cases.

**Case 1:** If \( \alpha + \beta = 0 \), it is clear that \( ab = ba \).

**Case 2:** If \( \alpha + \beta \neq 0 \), then
\[
0 = \alpha ab + \beta ba = (\alpha I_{a} + \beta R_{a})b = [(\alpha + \beta)I_{a} - \beta(I_{a} - R_{a})] b = [(\alpha + \beta)I_{a} - \beta D_{a}] b.
\]
Hence \( ab = I_{a}b = \gamma D_{a}b \), where \( \gamma = \frac{\beta}{\alpha + \beta} \in \mathbb{C} \). However, \( I_{a} \) and \( D_{a} \) commute, then by induction we have, \( L_{a}^{n}b = \gamma^{n}D_{a}^{n}b \) for all natural numbers \( n \), and so \( a^{n}b = \gamma^{n}D_{a}^{n}b \). Since \( a \in QC(J) \) then by [2, Theorem 2.3] \( D_{a} \) is topologically nilpotent; that is \( \lim_{n \to \infty} \| D_{a} \|^n = 0 \).

However, \( \| a \cdot b \| = \| \gamma \cdot D_{a} \cdot b \| \leq |\gamma| \| D_{a} \| \cdot \| b \|^{n} \), then for any \( \lambda \in \mathbb{C} \setminus \{0\} \),
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\[
\lim_{n \to \infty} \| \lambda \cdot a \cdot b \|^n = 0. \quad \text{Hence the series } \ f(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} a^n b \text{ is absolutely convergent, then by the completeness of } A \text{ the series converges. But } J \text{ is a closed ideal of } A \text{ and } a^n b \in J \text{ for all natural numbers } n, \text{ then } f(\lambda) \in J \text{ and } (\lambda a) f(\lambda) = b.
\]

However, \( a \in QC(J), \text{ then there exists } k \geq 1 \text{ such that } \| f(\lambda) (\lambda - a) \| \leq k \| \lambda - a \| f(\lambda) \|. \text{ Therefore, } f(\lambda) (\lambda - a) = b + \lambda^{-1} (ab - ba) + \lambda^{-2} (a^2 b - bab) + \ldots. \text{ is a bounded } J\text{-valued function on } \mathfrak{e}\{0\} \text{ which can easily be seen analytic there. But } \{0\} \text{ is a countable compact subset of } \mathfrak{e}, \text{ then by [9] it has zero analytic capacity and so by [4, Theorem 1.10VIII], } f \text{ extends to be analytic on } \mathfrak{e}. \text{ Hence, by Liouville's Theorem } f(\lambda) (\lambda - a) \text{ is constant, which implies that } ab = ba. \text{ Therefore, } ab = ba \text{ whenever } ab \text{ and } ba \text{ are linearly dependent.}

The converse is straightforward \( \Box \)

Corollary:- [ 7, Proposition 3.4 ]

Suppose that \( A \) is a unital complex Banach algebra, \( a \in Q(A) \) and \( b \in A. \) Then \( ab \) and \( ba \) are linearly dependent if and only if \( ab = ba. \)

Proof:

Let \( J = A \) in the above Theorem, then use [ 1, Theorem 2.1 ] \( \Box \)

At the end we shall write the following paragraph which consists of the proofs of Lemma 2.2 and Theorem 2.3 from [2] which are not a part of this paper, but they are written here because [2] is not published until the moment of writing this paper.

2.2 Lemma in [2] Let \( A \) be a complex Banach algebra with unity, \( J \) a closed ideal of \( A, \) and \( k \geq 1 \) such that \( a \in QC(k, J). \) If \( M \) is a closed commutative subalgebra of \( BL(J) \) containing the identity operator \( I, L_a \) and \( R_a \) then \( \| D_a T \| \leq (k + 1) \| T \| (\lambda - L_a) T \| \) for all \( T \in M \) and all \( \lambda \in \mathfrak{e}. \) Where \( L_a \) and \( R_a \) are restricted here to a domain \( J. \)

Proof

Since \( a \in QC(k, J), \) for all \( x \in J \) and \( \lambda \in \mathfrak{e}, \) we have, \( \| x (\lambda - a) \| \leq k \| \lambda - a \| \| x \|. \) However, \( \| (\lambda - a) x \| = \| (\lambda - R_a) x \| \) and \( \| (\lambda - a) x \| = \| (\lambda - L_a) x \|, \) then \( \| (\lambda - a) x \| \leq k \| (\lambda - R_a) x \|. \) So that \( \| D_a x \| = \| (\lambda - R_a) x - (\lambda - L_a) x \| \leq (k + 1) \| (\lambda - L_a) x \|. \) Finally, since \( T \in J \) for all \( x \in J, \) then the result follows by replacing \( x \) by \( T \) in the above inequality and taking the supremum over all \( x \) in \( J \) with \( \| x \| = 1 \) \( \Box \)

In the proof of the following theorem we need [6, Theorem 4.2], which states that "Suppose that \( M \) is a commutative complex Banach algebra with unity, that \( u \) and \( v \) are elements of \( M, \) and that for some \( c \geq 0, \)
\[ \| u \| \leq c \| (\lambda - v) x \| \text{ for all } x \in M \text{ and all } \lambda \in \mathcal{C}. \] Then \( u \in \text{Rad}(M) \), the radical of \( M \).

2.3 **Theorem in [2]** Let \( A \) be a complex Banach algebra with unity, \( J \) a closed ideal of \( A \), and \( a \in \text{QC}(J) \). Then \( D_a \) is topologically nilpotent. Where \( D_a \) is restricted to a domain \( J \).

**Proof**

Since \( J \) is closed in the Banach algebra \( A \), then \( J \) is complete and so \( BL(J) \) is a complex Banach algebra. Since \( a \in \text{QC}(J) \), then there exists \( k \geq 1 \) such that \( a \in \text{QC}(k, J) \). Now, let \( M \) be a closed commutative subalgebra of \( BL(J) \) containing the identity operator \( I \), \( L_a \) and \( R_a \), then by Lemma 2.2, \( \| D_a T \| \leq (k + 1) \| (\lambda - I_a) T \| \) for all \( T \in M \) and all \( \lambda \in \mathcal{C} \). But \( M \) is closed in \( BL(J) \) and \( BL(J) \) is complete, then \( M \) is complete, then by [6, Theorem 4.2] we have \( D_a \in \text{Rad}(M) \) and so \( \lim \| D_a^n \| = 0 \).

That means, \( D_a \) is topologically nilpotent. \( \square \)

**References**


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