INCOMPLETE FACTORIALS AND SOME COMBINATORIAL IDENTITIES

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Abstract: We prove in the paper some identities related to incomplete factorials. We derive and prove some recurrence relations for the incomplete factorials. Then we prove that the sum of incomplete factorials turns out to be representable in terms of incomplete factorials. Then we use these identities to prove some known as well as new combinatorial identities.

1. INTRODUCTION

Identities involving factorials, binomial coefficients and incomplete factorials can be proved by either algebraic manipulations or combinatorial proofs. The algebraic proofs might be time consuming and involve complex calculations. We introduce the notion of incomplete factorials and develop some of its recurrence relations. We prove a theorem that relates these incomplete factorials with the usual ones. This theorem with other theorems that we also prove in this paper allow us to use a new approach for dealing with known combinatorial identities and binomial coefficients. They also work as tools to generalize and sometimes derive new identities that are easy to visualize and prove using this new approach.

While some of results are known in the paper and well known in the literature, we use different approaches and techniques to prove them. The other results, especially the recurrence relations and some identities are new.

Besides this introduction, this paper contains two sections and the references.

In Section 2, we develop the main properties and relations of the incomplete factorials. We also prove the main theorem that relates the incomplete factorials to their sums.

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In Section 2 we derive and prove some theorems involving combinatorial, factorial, and binomial coefficient identities. We also prove in this section known identities as applications to the properties and identities developed in Section 2.

We will denote by \( \mathbb{N} = \{0, 1, 2, \ldots\} \) the set of all nonnegative integers.

## 2. Sum of Incomplete Factorials

The partial factorials or incomplete factorials are defined as follows:

**Definition 2.1.** For each \( m \geq 0 \), let \( f_n(m) \) be defined by

\[
f_n(m) = \prod_{k=0}^{n-1} (m-k) \quad \forall m \geq 0
\]

It is clear that \( f_n(m) = 1 \) for all \( m \leq n \). One can express \( f_n(m) \) in terms of incomplete factorials as

\[
f_n(m) = (m-n+1) f_n(m-n) \quad m \geq n
\]

where \( m-n \geq 0 \). It follows from (2.2) that

\[
f_{n-1}(m) = (m-n+1) f_{n-1}(m) \quad m \geq n
\]

This function \( f_n(m) \) is closely related to the well-known factorial polynomial \( P(m,n) \) as mentioned in [1]. In fact, they are related via the equation \( f_n(m) = P(m,n+1) \). We will stick to our notations here since \( f_n(m) \) for the sake of simplicity and clarification.

In the following lemma, we prove some recurrence identities for the function \( f_n(m) \).

**Lemma 2.1.** The function \( f_n(m) \) for \( m \geq n \geq 0 \) with \( m \geq n \geq 1 \) and \( m \leq n \) obey the following recurrence identities:

1. \( f_n(m) = \frac{m-n+1}{m} f_n(m-1) \)
2. \( f_n(m) = \frac{m}{m-n} f_n(m-n) \)
3. \( f_n(m) = \frac{m+1}{m} f_{n}(m+1) \)
4. \( f_n(m) = \frac{k}{m} f_{n+k}(m) \) for \( m \geq n \geq 0 \) with \( m \geq n \leq k \leq \min\{m,n\} \).

**Proof.** The first and second terms of Lemma 2.1 follow by direct calculations:

\[
f_n(m) - \frac{m-n+1}{m} f_n(m-1) = \frac{m-n+1}{m} f_n(m-1) - \frac{m-n+1}{m} f_n(m-2) \quad \text{for } m \geq n
\]

\[
- \frac{m-n+1}{m} f_n(m-1) = \frac{m-n}{m} f_n(m-n) - \frac{m-n}{m} f_n(m-2) \quad \text{for } m \geq n
\]
Combining (1) and (2) we get (3) to prove (1) note that
\[
\begin{align*}
  f_n(m) &= \{m(m-1) \cdots (m-k+1)\} \{m-k\} \cdots \{m-2\} \\
  &= f_{k-1}(m) f_{n-k}(m-k).
\end{align*}
\]

**Remark 2.** By using Item 2 of Lemma 2.1 and Item 3 of Lemma 2.1, the equation (3) can be rewritten as
\[
\begin{align*}
  f_n(m) &= m(m-k) f_{k-1}(m) f_{n-k}(m-k).
\end{align*}
\]

We now introduce another function that involves the sum of incomplete factorials.

**Definition 2.** Let $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, then $g_n(m)$ is defined by
\[
\begin{align*}
  g_n(m) = \sum_{k=1}^{m-1} \binom{n}{k} m(m-k), \quad m \geq 1. \tag{2.3}
\end{align*}
\]

It follows from Definition 2 that $g_n(m) = 1$ for all $m \leq n$. An alternative expression for $g_n(m)$ is
\[
\begin{align*}
  g_n(m) = \begin{cases} 
  \sum_{k=1}^{m} \binom{n}{k} , & \text{if } m \leq n, \\
  1, & \text{if } m > n.
\end{cases} \tag{2.4}
\end{align*}
\]

The next Lemma gives us a recurrence identity that relates the two functions $f_n(m)$ and $g_n(m)$ for each $n \in \mathbb{N}$.

**Lemma 2.2.** Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$ be such that $m \geq n+1$. Then
\[
\begin{align*}
  g_n(m) - g_{n+1}(m) &= \frac{1}{m} f_n(m), \quad m \geq n+1. \tag{2.5}
\end{align*}
\]

**Proof.** By using (2.4) and Item 3 of Lemma 2.1 we may write for $m \geq n+1$
\[
\begin{align*}
  g_n(m) &= \sum_{k=1}^{m} \binom{n}{k} m(m-k) \\
  &= \sum_{k=1}^{m} \binom{n}{k} m - \sum_{k=1}^{m} \binom{n}{k} m(m-k) \\
  &= \sum_{k=1}^{m} \binom{n}{k} m - g_{n+1}(m) \\
  &= \sum_{k=1}^{m} \binom{n}{k} m - f_{n+1}(m) \\
  &= \sum_{k=1}^{m} \binom{n}{k} m - f_{n+1}(m) \\
  &= \frac{1}{m} f_n(m).
\end{align*}
\]
The next theorem gives us an explicit identity that relates the two functions \( f_n(m) \) and \( g_n(m) \) for each \( n \geq 1 \).

**Theorem 2.1** Let \( f_n(m) \) and \( g_n(m) \) for \( m \geq 1 \) and \( n \geq 2 \) be defined by (2.1) and (2.3) respectively. Then \( n + 1 \) \( g_n(m) - f_n(m) \).

**Proof** Let \( m \geq n \) be fixed. Since \( f_n(m) = g_n(m) = 1 \) for all \( m \leq n \), we start our induction at \( m = n + 1 \). It follows from (2.4) that \( g_n(n + 1) = (n + 1) - f_n(n + 1) \). Similarly \( f_n(n + 1) = (n + 1) - f_n(n + 1) \). Using Lemma 2.1, we see that \( f_n(n + 1) - f_n(n + 1) \). Now by Lemma 2.2 we see that

\[
\begin{align*}
\left[1 & \right] g_n(k) - n + 1) g_n(k) = \left[ \frac{1}{k} \right] f_n(k) \\
- f_n(k) &= \left[ \frac{1}{k} \right] f_n(k) \quad \text{by assumption of induction}
\end{align*}
\]

\[
- f_n(k) = \left[ \frac{1}{k} \right] f_n(k)
\]

**Lemma 2.3** Let \( m \geq 1 \) and \( n \geq 1 \) be such that \( m \geq n + 1 \). Then

\[
f_n(m) - f_n(m - 1) = n + 1 \quad f_{n-1}(m - 1) \quad m \geq n + 1
\]

**Proof** The proof follows by multiplying both sides of (2.5) by \( m \). Then by using Theorem 2.1, and noting that \( f_n(m) - f_{n-1}(m - 1) \), the result follows.

In the next lemma we prove some recurrence identities for \( g_n(m) \).

**Lemma 2.4** The function \( g_n(m) \) for \( m \geq 1 \) and \( n \geq 1 \) with \( m \geq n + 1 \) satisfies the following recurrence relations:

1. \( g_n(m) = g_n(m - 1) - \frac{m-n}{m} g_n(m) \)
2. \( g_n(m) = g_n(m - 1) - \frac{m-n}{m} g_n(m) \)
3. \( g_n(m) = g_n(m - 1) - \frac{m-n}{m} g_n(m) \)

**Proof** The proof follows directly from Lemma 2.1 and Theorem 2.1.
Remark 2.7 One may rewrite 2.5 as
\[ q_n(m) = \frac{1}{m} \sum_{k=1}^{m} q_{n-1}(m-1) \]
It can be seen that term 2 in 2.4 may be expressed as
\[ q_n(m-1) \]
for \( m \geq n + 1 \).

Corollary 2.8
\[
\begin{align*}
(1)(2) \cdots (m) &= \\
(2)(3) \cdots (m) &= 1 \\
&= \\
(m)(n+m) &= 1 \\
&= \frac{m!}{(n+m)}
\end{align*}
\]

Proof The left hand side of the above equation has the following form
\[
\sum_{k=1}^{m} k^{n} = \sum_{k=m+1}^{n+m} k^{n}
\]

Example 2.9 Let \( m \in \mathbb{N} \).
\[
(1) + (2) + \cdots + (m) = \binom{n+2}{3} - \binom{n+2}{3}
\]

Example 2.10 Let \( m \in \mathbb{N} \).
\[
(1)(2)(3)(4)(5)(6)(7)(8) = \binom{n+2}{4}
\]

3. COMBINATORIAL IDENTITIES

In this section, we show that both functions \( f_n \) and \( \hat{f}_n \) when summed reproduce themselves. We also prove some theorems that can be used to prove some combinatorial identities like the Pascal's identity, the hexagon property, and the hypergeometric identity. We also generalize some of the known identities and derive new ones.

Although \( f_n(m) \) and \( \hat{f}_n(m) \) seem to be unrelated, it turns out that they related in a very interesting way. This equation that relates them has many applications in probability like moments of integer-valued random variables and the birthday problem.
Theorem 3.1 Let \( f_n(m) \) and \( g_n(m) \) for \( m \) and \( n \) in \( \mathbb{N} \) be defined by 2.1 and 2.3 respectively. Then

1. \( g_n(m) = \sum_{k=0}^{n} k \cdot f_n(k) = \sum_{k=1}^{n+1} f_n(k) - f_n(1) \)
2. \( f_n(m) = \sum_{k=1}^{n+1} g_n(k) - \sum_{k=m+1}^{n+1} g_n(k) \)
3. \( g_n(m) = \prod_{k=1}^{n} g_n(k) \)

Proof To prove the first item note that \( f_n(m) \). Then we may write \( g_n(m) = \sum_{k=0}^{n} k \cdot f_n(k) = \sum_{k=1}^{n+1} f_n(k) - f_n(1) \). Next by using item 2 and Theorem 2.1 we get item 3.

Next we will use Theorem 3.1 to prove some factorial identities.

Theorem 3.2 Let \( m \) and \( n \) be in \( \mathbb{N} \) then

\[
\sum_{k=0}^{m} k \cdot n! = \sum_{m}^{n} \prod_{k=1}^{m} (n+1-k) \quad \text{3.1}
\]

Proof First note that \( g_n(m) = \sum_{k=0}^{n} k \cdot f_n(k) \). This term \( \sum_{k=0}^{n} k \cdot f_n(k) \) for each \( n \). Therefore

\[
g_n(m+n+1) = \sum_{k=1}^{m+1} (k \cdot f_n(k)) - \sum_{k=m+1}^{n+1} (k \cdot f_n(k))
\]

Therefore we start the sum at \( k = n+1 \) Since the preceding terms are zeroes. Now let \( k = n+1 \) in the last term of 3.2 to get

\[
g_n(m+n+1) = \sum_{k=1}^{m+1} (n+1)!
\]

Using item 1 of 2.1 with \( m \) replaced by \( m+1 \) \( \Rightarrow \) iteratively \( m+1 \) times, we get

\[
f_n(n+m+1) = \prod_{k=1}^{m+1} f_n(k)
\]

But note that \( f_n(n+1) = f_n(n+1) \). Then 3.4 becomes

\[
f_n(n+m+1) = \prod_{k=1}^{n+1} f_n(k)
\]

Combining 3.3 and 3.5 and using Theorem 2.1 we get 3.1.
Corollary 3.1. Let \( m \) and \( n \) be in \( \mathbb{N} \). Then
\[
\sum_{i=m}^{n} \binom{n+1}{i} - \binom{m+m}{m} = 3.6
\]
In particular, we have
\[
\sum_{i=m}^{n} \binom{n+1}{i} - \binom{2n+1}{n} = 3.7
\]

Proof. It follows from (3.1) that
\[
\sum_{i=m}^{n} \binom{n+1}{i} - \sum_{i=1}^{m} \binom{n+i}{i} = 1 - \frac{n+m+1}{(n+1)!} - \binom{m+m}{m}
\]
This gives us (3.6). To get (3.7), we simply let \( m = n \) in (3.6).

Theorem 3.8. Let \( m, n \) be in \( \mathbb{N} \) such that \( 1 \leq \min\{m, n\} \). Then
\[
f_{r-1}(m+n) = \sum_{k=1}^{n} \binom{r}{k} f_{k-1}(m)f_{r-k-1}(n). \quad 3.8
\]

Proof. First, we note that
\[
\frac{d^r}{dt^r} f(j) = f_{r-1}(j) t^{r-1}
\]
in other words, evaluating the \( r \)-th derivative of both sides at \( t = 1 \) yields
\[
\frac{d^r}{dt^r} f|_{t=1} = f_{r-1}(1). \quad 3.9
\]
Using (3.9), when taking the \( r \)-th derivative of both sides of the equation
\[
\sum_{m}^{n} f_{r-1}(m) = \sum_{n}^{m} f_{r-1}(n)
\]
using the rule of differentiating the product of two functions \( f(r) \) and \( g(t) \) and then letting \( t = 1 \) we get
\[
f_{r-1}(m+n) = \sum_{k=1}^{n} \binom{r}{k} f_{k-1}(m)f_{r-k-1}(n)
\]
as required.

The following corollary gives an easy proof of the fact that the sum of the hypergeometric probability mass function over all possible values is equal to 1.

Corollary 3.10. Let \( m, n \) be in \( \mathbb{N} \), \( m, n \neq 0 \) such that \( n \leq \min\{m, n\} \). Then
\[
\binom{m+n}{r} - \sum_{k=1}^{n} \binom{m}{k} \binom{n}{r-k} = 3.10
\]
Proof. The proof follows by using (3.8) and the fact that
\[ f_n(m) = \begin{cases} \binom{n}{1} \binom{m}{n+1} & \text{if } m < n \\ 0 & \text{if } m \geq n \end{cases} \quad 3.11 \]

By dividing both sides of (3.10) by \(\binom{m+n}{r}\) we get
\[ 1 = \sum_{k=0}^{m} \binom{m}{k} \binom{k}{r} \binom{m+n}{n+k} \]

Corollary 3.3. If \(a\) and \(m\) in \(k\) such that \(m \geq k\) then
\[ \sum_{k=1}^{n} \binom{n}{k} = \binom{2n}{n} \]

Proof. The proof follows from (3.10) by letting \(m = r\) and \(a = n\).

Theorem 3.1. Let \(m\) and \(n\) be in \(\mathbb{N}\) then
\[ f_k(n) = a_{k-1} f_{k-1}(n) + \ldots + a_k f_1(n) = 2^m f_0(n) - n^{k+1} \quad 3.12 \]

Here the coefficients \(a_0, a_1, \ldots, a_k\) are obtained for \(1 \leq k \leq m\) from the equation
\[ (-1)^{j+k+1} a_j \beta_{j-k}, \text{ where } \quad 3.13 \]

\[ \beta_k \geq \exists \mathbf{l} \quad \forall k < n \quad 3.14 \]

Proof. Note that the \(f_k(n) = 2(n) - \ldots - n\) is a polynomial of degree \(k\). The coefficient \(a_j\) in the expansion of \(f_k(n)\) is the sum of the product of the numbers \(\{1, 2, \ldots, k\}\) taken \(j\) at a time. This means that we can write the expansion of \(f_k(n)\) as
\[ f_k(n) = \sum_{i=1}^{k} (-1)^i \beta_{k-i+1} n^{k-i}, \text{ where } \quad 3.15 \]

\[ \beta_k \geq \exists \mathbf{l} \quad \forall k < n \quad 3.16 \]

We need to find constants \(a_0, a_1, \ldots, a_m\) such that
\[ f_k(n) = a_{k-1} f_{k-1}(n) + \ldots + a_0 f_0(n) - n^{k+1} \]
Theorem 3.11 The coefficients for each odd power \( \alpha \) must vanish for \( 1 \leq \alpha \leq 2k-2 \). Therefore we should have

\[
\begin{align*}
\alpha &= -\beta_k, \\
\alpha &= -\beta_{k-1}, \\
\alpha &= -\beta_{k-2}, \\
\alpha &= -\beta_{k-3}, \\
&\vdots
\end{align*}
\]

In general, we have

\[
\alpha = \sum_{i=1}^{k} (-1)^{i+1} a_i \beta_{j-i}
\]

as required. Incidentally, you can check that

\[
\alpha = a_1 \beta_{1,1} = a_2 \beta_{2,2} = \cdots = a_k \beta_k = 0
\]

since \( \beta_k = 0 \).

Example 3.12 We have the following equations

\[
\begin{align*}
f_0(n) &= n, \\
f_1(n) &= f_0(n) - \frac{n}{2}, \\
f_2(n) &= 3f_1(n) = f_0(n) - \frac{n}{2}, \\
f_3(n) &= 5f_2(n) = f_0(n) - \frac{n}{2}, \\
f_4(n) &= 10f_3(n) = 25f_2(n) = 15f_1(n) = f_0(n) - \frac{n}{2}, \\
f_5(n) &= 15f_4(n) = 85f_3(n) = 90f_2(n) = 31f_1(n) = f_0(n) - \frac{n}{2}
\end{align*}
\]

Corollary 3.13 Let \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \). Then

\[
\sum_{k=1}^{m} k! \binom{n}{k} a_{k-1} = n^{m+1}
\]

where the \( \alpha \) are defined as in Theorem 3.11.

Proof The proof follows by using 3.11 and Theorem 3.1.

Example 3.14 In we let \( n \) take the values 1, 2, and 3 in Theorem 3.3, we get respectively...
\[
\binom{n}{1} = 2 \binom{n}{2} - n \\
\binom{n}{1} = 6 \binom{n}{2} = 6 \binom{n}{3} - n \\
\binom{n}{1} = 12 \binom{n}{2} = 12 \binom{n}{3} = 24 \binom{n}{4} - n^2
\]

**Theorem 3.16** Let \(m \leq n \) and \( m, n \) be \( \in \mathbb{N}^+ \) such that \( m \leq n \). Then

\[
2^n f_{n-1}(m) - \sum_{k=1}^{n} \binom{n}{k} f_{k-1}(m) f_{n-k-1}(m - k)
\]

**Proof** It follows from Item 1 of Lemma 2.1 that

\[
f_{n-1}(m) - f_{k-1}(m) f_{n-k-1}(m - k)
\]

Multiplying both sides of 3.17 by \( \binom{n}{k} \) and take the sum as \( k \) ranges from 1 to \( n \), we get

\[
\sum_{k=1}^{n} \binom{n}{k} f_{n-1}(m) - \sum_{k=1}^{n} \binom{n}{k} f_{k-1}(m) f_{n-k-1}(m - k)
\]

But the left hand side is equal to

\[
f_{n-1}(m) \sum_{k=1}^{n} \binom{n}{k} - 2^n f_{n-1}(m)
\]

Therefore we have

\[
2^n f_{n-1}(m) - \sum_{k=1}^{n} \binom{n}{k} f_{k-1}(m) f_{n-k-1}(m - k)
\]

**Corollary 3.17** Let \( m \leq n \) and \( m, n \) be \( \in \mathbb{N}^+ \) such that \( m \leq n \). Then

\[
\binom{n}{k} \binom{m}{n} - \binom{m-k}{k} \binom{n-k}{n-k}
\]

**Proof** The proof follows by using Theorem 3.16 and 3.11:

\[
m! \binom{m}{n} - k! \binom{m}{k} = m! \binom{m-k}{n-k} \quad \text{implies}
\]

\[
\binom{n}{k} \binom{m}{n} - \binom{m-k}{k} \binom{n-k}{n-k}
\]
Remark 3.3. Corollary 3.1 has a very interesting meaning in probability. It says that the number of ways to choose \(m\) out of \(n\) objects and then \(k\) out of the chosen \(m\) objects is equal to the number of ways of choosing \(k\) objects out of \(m\) objects and then \(m-k\) objects out of the remaining \(n-m\) objects.

Corollary 3.6

\[
\sum_{n=0}^{m} \binom{m}{k} \binom{m-k}{n-k} = \binom{n}{n}
\]

Proof. The proof follows from (3.16) and (3.11). \(\Box\)

Corollary 3.7

\[
\sum_{k=0}^{m} \binom{n}{k} \binom{m}{n-k} - 2^{m-k} \binom{m}{k}
\]

Proof. By (3.19) we see that

\[
\sum_{k=0}^{m} \binom{n}{k} \binom{m}{n-k} - \sum_{k=0}^{m} \binom{m}{k} \binom{m-k}{n-k} = \binom{m}{m} \sum_{k=0}^{m} \binom{m-k}{n-k} - \binom{m}{k} \sum_{j=0}^{m-k} \binom{m-k}{j} - 2^{m-k} \binom{m}{k}
\]

\(\Box\)

Corollary 3.8

\[
\sum_{n=0}^{m} \sum_{k=0}^{m} 2^{-n} \binom{m}{k} \binom{m-k}{n-k} = \binom{m}{n}
\]

Proof. The proof follows by taking the sum in (3.20) \(\forall n \in \mathbb{N}\) and noting that \(\sum_{m=0}^{\infty} \binom{m}{n} = \binom{m}{n}\). \(\Box\)

In the following Theorem we prove the hexagon property which appears in Pascal's triangle.

Theorem 3.6. Let \(m\) and \(k\) be \(m\) and \(k\) be \(m\) such that \(1 \leq k \leq m\). Then

\[
\binom{n}{k} \binom{n}{k+1} \binom{n}{k+2} - \binom{n-1}{k-1} \binom{n+1}{k} \binom{n+1}{k+1}
\]
Proof: By using Lemma 2.1 we see that
\[
\begin{align*}
    f_{k-1}(n) &= \frac{1}{2} f_{k-2}(n) \\
    f_k(n+1) &= \frac{1}{2} f_k(n) \quad \text{if } n = 1 \quad f_k(n+1) = f_k(n) \\
    f_{k-2}(n) &= \frac{1}{k} f_{k-1}(n+1) \\
    f_k(n) &= \frac{1}{k} f_{k-1}(n+1) - \frac{1}{k} f_{k-2}(n) \\
\end{align*}
\]
Multiplying the left hand sides of these equations yields
\[
f_{k-1}(n) f_k(n+1) f_{k-2}(n) = f_{k-2}(n+1) f_{k-1}(n+1) f_k(n).
\]
Using (3.11) and (3.21) we get
\[
\left(\begin{array}{c} n \cr k-1 \end{array}\right) \left(\begin{array}{c} n-1 \cr k \end{array}\right) = \left(\begin{array}{c} n+1 \cr k \end{array}\right) \left(\begin{array}{c} n \cr k \end{array}\right).
\]

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