ON PROPERTIES OF GEOMETRY OF TYPE D_{n,k}

Mohammed Suliman El-Atrash
Professor of mathematics Science College, Mathematics Department
Islamic University of Gaza, Gaza, Palestine
E-mail: matrash@mail.iugaza.edu

Abdelsalam Abou zayda
Assistant professor of mathematics
Math. Department, Al-Aqsa University, Gaza, Palestine
E-mail: zayda8@hotmail.com

Abstract: In this paper we present a general case of the class of geometries D_{n,2}, D_{n,3} and D_{n,4} that is a point-line geometry of type D_{n,k}(F) where k \geq 2, n \geq k+3 and F is a field. We define the varieties of the geometry and we prove that the geometry is a parapolar with diameter k+1. The properties related to the relations between the varieties will be investigated.

1- Introduction
In [5] and [7] it has been proved that the class of geometries D_{n,2}, D_{n,3} and D_{n,4} are weak parapolar geometries with diameters 3, 4 and 5 respectively. We prove in this paper that the general case D_{n,k}(F) is a parapolar geometry with diameter k+1 for k \geq 2 and n \geq k+3 and we get the same results with respect to the relations between symplecta themselves and the relations between points and symplecta.

x^t means the set of all points in P collinear with x, including x itself.

A subspace of a point-line geometry \Gamma=(P, L) is a subset X \subseteq P such that any line which has at least two of its incident points in X has all of its incident points in X. \langle X \rangle means the intersection over all subspaces containing X, where X \subseteq P. Lines incident with more than two points are called thick lines, those incident with exactly two points are called thin lines.

The singular rank of a space \Gamma is the maximal number n (possibly \infty) for which there exist a chain of distinct subspaces \emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n such that X_i is singular for each i, X_i \neq X_j, i \neq j, for example rank(\emptyset)=-1, rank(\{p\})=0 where p is a point and rank(L)=1 where L a line.

In a point-line geometry \Gamma=(P, L), a path of length n is a sequence of n+1 (x_0, x_1, \ldots, x_n) where, (x_i, x_{i+1}) are collinear, x_0 is called the initial point and x_n is called the end point. A geodesic from a point x to a point y is a path of minimal possible length with initial point x and end point y. We denote this length by d_{\Gamma}(x, y).
A geometry $\Gamma$ is called \textit{connected} if and only if for any two of its points there is a path connecting them. A subset $X$ of $P$ is said to be \textit{convex} if $X$ contains all points of all geodesics connecting two points of $X$.

\textbf{A polar space} is a point-line geometry $\Gamma=(P, L)$ satisfying the Buekenhout-Shult axiom [1]:

For each point-line pair $(p, l)$ with $p$ not incident with $l$, $p$ is collinear with one or all points of $l$, that is $|p \cap \overline{l}|=1$ or else $p \supset \overline{l}$. Clearly this axiom is equivalent to saying that $p^{\perp}$ is a geometric hyperplane of $\Gamma$ for every point $p \in P$.

A point-line geometry $\Gamma=(P, L)$ is called \textit{a projective plane} if and only if it satisfies the following conditions [2]:

(i) $\Gamma$ is a linear space; every two distinct points $x, y$ in $P$ lie exactly on one line,

(ii) every two lines intersect in one point,

(iii) there are four points no three of them are on a line.

A point-line geometry $\Gamma=(P, L)$ is called \textit{a projective space} if the following conditions are satisfied:

(i) every two points lie exactly on one line,

(ii) if $l_1, l_2$ are two lines $l_1 \cap l_2 \neq \emptyset$, then $\langle l_1, l_2 \rangle$ is a projective plane.

($\langle l_1, l_2 \rangle$ means the smallest subspace of $\Gamma$ containing $l_1$ and $l_2$.)

A point-line geometry $\Gamma=(P, L)$ is called \textit{a parapolar space} if and only if it satisfies the following properties:

(i) $\Gamma$ is a connected gamma space,

(ii) for every line $l$, $l^{\perp}$'s not a singular subspace,

(iii) for every pair of non-collinear points $x, y$; $x^{\perp} \cap y^{\perp}$ is either empty, a single point, or a non-degenerate polar space of rank at least 2.

If $x, y$ are distinct points in $P$, and if $|x^{\perp} \cap y^{\perp}|=1$, then $(x, y)$ is called \textit{a special pair}, and if $x^{\perp} \cap y^{\perp}$ is a polar space, then $(x, y)$ is called \textit{a polar pair} (or \textit{a symplectic pair}). A parapolar space is called \textit{a strong parapolar} space if it has no special pairs [4].

\textbf{2. Construction of D_{n,k}(F)}

Consider the polar space $\Delta=\Omega^+(2n, F)$ that comes from a vector space $V$ of dimension $2n$ over a finite field $F=GF(k)$ with a symmetric hyperbolic bilinear form $B$. $S_i$ is the set of all totally isotropic $i$-dimensional subspaces of $V$; $1 \leq i \leq n-2$. The two classes $M_1$, $M_2$ consist of maximal totally isotropic $n$-dimensional subspaces. Two $n$-subspaces fall in the same class if their intersection is of odd dimension.
The geometry of type \( D_{n,k}(F) \) is the point-line geometry \((P, L)\), whose set of points \( P \) is the collection of all \( k \)-dimensional subspaces of the vector space \( V \), and whose lines are the pairs \((A, B)\) where \( A \) is \( k-1 \)-dimensional subspace of \( k+1 \)-subspace \( B \) – that is, the set of \((k-1, k+1)\)-subspace of \( V \). A point \( C \) is incident with a line \((A, B)\) if and only if \( A \subset C \subset B \) as a subspaces of \( V \).

To define the collinearity, let \( C_1 \) and \( C_2 \) be two point (the points are the T.I \( k \)-spaces), then \( C_1 \) is collinear to \( C_2 \) if and only if the intersection of \( C_1 \cap C_2 = k-1 \)-space and \( \langle C_1, C_2 \rangle = k+1 \)-space.

The elements of the classes \( G_1 \) and \( G_2 \) are Grassmannian geometries of type \( A_{n-1,k} \) for more details about these kinds of geometries see [6] and [3].

There are two kinds of symplecta (1) The first kind is the convex polar spaces \( A_{3,2} \) that represent the \((k-2, k+2)\) subspaces of \( V \). Then symplecton \( S \) of kind \( A_{3,2} \) is the set of TI \( k \)-dimensional spaces that contain the TI \( k-2 \)-dimensional space and contained in the TI \( k+2 \)-dimensional space. (2) The second kind of symplecta is the convex polar spaces of type \( D_{n,k+1,1} \), that represent the collection of all TI \( k-1 \)-subspaces of \( V \). Thus this kind of symplecta is defined as the collection of all TI \( k \)-subspaces of \( V \) that contain such TI \( k-1 \)-subspaces.

**Notation.** Let the map \( \Psi : P \to V \) defined above, i.e., \( \Psi(p) \) is the T.I. \( k \)-dimensional subspace corresponding to the point \( p \). We will use \( \Psi \) for the
rest of the varieties of the geometry; for example $\Psi(D_{n-1,k-1})$ is the T.I. 1-dimensional subspace corresponding to a geometry of type $D_{n-1,k-1}(F)$. The inverse map $\Psi^{-1}$ will be used for the inverse.

3-The main result.

The following theorem represents the first part of the main result of this paper that is: $D_{n,k}(F)$ is weak parapolar geometry of diameter $k+1$.

3.1 Theorem. Let $\Gamma=(P, L)$ be the geometry of type $D_{n,k}(F)$. Thus:

i- $\Gamma$ is of diameter $k+1$.

ii- $\Gamma$ is parapolar geometry.

Proof: i We shall use the mathematical induction for $k \geq 2$. If $k=2$, then we have geometry of type $D_{n,2}(F)$ with diameter 3 [5]. Now for all the cases $k-1, k-2, \ldots, 4, 3, 2$ the geometries $D_{n,k-1}, D_{n,k-2}, \ldots, D_{n,4}, D_{n,3}, D_{n,2}$ have the diameters $k, k-1, \ldots, 5, 4, 3$ respectively. Now we shall prove that the diameter of $D_{n,k}$ is $k+1$. Suppose that $p$ and $q$ are two arbitrary points of $D_{n,k}$, that represent TI $k$-spaces $\Psi(p) = C_1 = \langle x_1, x_2, \ldots, x_k \rangle$ and $\Psi(q) = C_2 = \langle y_1, y_2, \ldots, y_k \rangle$. If $\langle x_1, x_2, \ldots, x_k \rangle \cap \langle y_1, y_2, \ldots, y_k \rangle = 1$-space, 2-space, \ldots or $(k-2)$-space, then the two points lie in the geometries $D_{n,k-1}, D_{n,k-2}, \ldots$ and $D_{n,2}$ respectively. Thus by the hypotheses the diameter are $k, k-1, \ldots, 3$ respectively.

If $\langle x_1, x_2, \ldots, x_k \rangle \cap \langle y_1, y_2, \ldots, y_k \rangle = (k-1)$-space, without loss of generality we take $z_1 = x_2 = y_2, z_2 = x_3 = y_3, \ldots, z_{k-1} = x_k = y_k$, then $\langle x_1, x_2, \ldots, x_k \rangle \cap \langle y_1, y_2, \ldots, y_k \rangle = \langle z_1, z_2, \ldots, z_{k-1} \rangle$. Thus we have two cases:

1- $x_1 \perp C_2 = C_2$

2- $x_1 \cap C_2 = \langle z_1, z_2, \ldots, z_{k-1} \rangle$.

In case 1, $\langle x_1, y_1, z_1, z_2, \ldots, z_{k-1} \rangle$ forms a $(k+1)$-space and $(\langle z_1, z_2, \ldots, z_{k-1} \rangle, \langle x_1, y_1, z_1, z_2, \ldots, z_{k-1} \rangle)$ is the line that contains the two points. Thus $d(p,q) = 1$.

In case 2, $B(x_1, y_1) \neq 0$ and $C_2$ is contained in a maximal TI $n$-space $K$, say $K = \langle y_1, z_1, z_2, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_n \rangle$. Then $x_1 \cap K$ is a hyperplane of $K$, say $H = \langle z_1, z_2, \ldots, z_{k-1}, z_k, z_{k+1}, \ldots, z_n \rangle$. We have two $(k+1)$-spaces $\langle x_1, z_1, z_2, \ldots, z_{k-1}, z_k \rangle$ and $\langle y_1, z_1, z_2, \ldots, z_{k-1}, z_k \rangle$, then $\langle z_1, z_2, \ldots, z_{k-1}, z_k \rangle$ is a point that is collinear to each of $C_1$ and $C_2$. Thus $d(p,q) = 2$.

If $\langle x_1, x_2, \ldots, x_k \rangle \cap \langle y_1, y_2, \ldots, y_k \rangle = 0$-space, then we the following cases:

1- $x_i \cap C_2 = C_2, i = 1, 2, \ldots, k$.

2- $x_i \cap C_2 = \langle y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k \rangle$ and $x_j \cap C_2 = C_2$ where $i \neq j$ and $i, j = 1, 2, \ldots, k$.

3- $x_i \cap C_2 = \langle y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k \rangle, i = 1, 2, \ldots, k$.
In the first two cases we have always $d(p,q) \leq k$ because at least one of the vectors $x_i$ we have $x_i \perp C_2 = C_2$.

In case 3, the least number of vertices that can make a geodesic between $p$ and $q$ is equal $k$. Since $C_2$ is contained in a maximal TI $n$-space, then without loss of generality we can choose the points $r_1, r_2, \ldots, r_k$ such that

\[
\Psi(r_1) = \langle z, x_2, x_3, \ldots, x_k \rangle, \quad \Psi(r_2) = \langle z, y_1, x_3, \ldots, x_k \rangle, \quad \Psi(r_3) = \langle z, y_1, y_2, x_4, \ldots, x_k \rangle, \quad \ldots, \\
\Psi(r_k) = \langle z, y_1, y_2, \ldots, y_{k-1} \rangle.
\]

where $z$ is a vector in the $n$-space in which $C_2$ is contained. Thus $p \sim r_1, r_1 \sim r_2, \ldots, r_k \sim q$ i.e., $d(p, q) = k+1$ and diameter of $D_{n,k}$ is $k+1$.

**ii** - To prove that the geometry is parapolar, we first prove that $D_{n,k}$ is gamma space. Assume that $l$ is a line containing two points $p$ and $q$ such that $\psi(p) = \langle x_1, x_2, \ldots, x_k \rangle$ and $\psi(q) = \langle y_1, y_2, \ldots, y_k \rangle$. Then by the definition of collinearity, the two points form a TI $(k+1)$-space $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle$, where

\[
z_1 = x_2 = y_2, \quad z_2 = x_3 = y_3, \quad \ldots, \quad z_{k-1} = x_k = y_k.
\]

Let $s$ be a point not in $l$ such that $s$ is collinear to $p$ and $q$, we shall prove that $s$ is collinear to every point incident to $l$. Since $s$ is collinear to $p$ and $q$, then $s$ does not contain the TI $(k-1)$-space $\langle z_1, z_2, \ldots, z_{k-1} \rangle$ and form $(k+1)$-spaces with $p$ and $q$. Now assume that $r$ is arbitrary point in $l$ such that $r \neq p \neq q$, then $\psi(r), \psi(s)$ contain the same $(k-1)$-space $\langle z_1, z_2, \ldots, z_{k-1} \rangle$ and they are contained in $(k+1)$-space $\langle z_1, z_2, \ldots, z_k, x_1, y_1 \rangle$. Thus $s$ is collinear to $r$.

Now for the completion of the proof we need to show that for any line $l; l^\perp$ is non-singular. For this purpose we take the same above line i.e.,

\[
l = \langle \langle z_1, z_2, \ldots, z_{k-1}, z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle, \langle z_1, z_2, \ldots, z_{k-1}, u \rangle \rangle
\]

Thus the points $r, s$ can be chosen to be non-collinear and each of them is collinear to $p$ and $q$. Since $\psi(s)$ is contained in a maximal $n$-space, then we take $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle, \langle z_1, z_2, \ldots, z_{k-1}, u \rangle$ to the points $r, s$ where $u$ is a vector in the $n$-space not in $\psi(s)$. Now $\langle z_1, z_2, \ldots, z_{k-1}, y_1 \rangle \cap \langle z_1, z_2, \ldots, z_{k-1}, u \rangle$ does not contain a $(k-1)$-space, this means that $r$ is not collinear to $s$ but $\langle z_1, z_2, \ldots, z_{k-1}, x_1, y_1 \rangle$ and $\langle z_1, z_2, \ldots, z_{k-1}, y_1, u \rangle$ form $(k+1)$-spaces, then $r, s$ are collinear to each of $p$ and $q$. Thus $l^\perp$ is non-singular. ♦

**3.2 Corollary.** $D_{n,k}$ is a non-degenerate weak geometry.

**Proof.** By the definition of collinearity and By Theorem 3.1, $D_{n,k}$ is a non-degenerate. Since the geometry of type $D_{n,2}$ is a weak sub-geometry of $D_{n,k}$ [5], then $D_{n,k}$ is a weak. ♦

**4- Properties of $D_{n,k}$**

It has been proved that for any pair of distinct sympleta $(S_1, S_2)$ of the class of geometries $D_{n,2}, D_{n,3}, D_{n,4}$, rank$(S_1 \cap S_2) = -1, 0, 2$. Moreover for any pair
of non-incident point and symplecton \((p, S)\), we have \(\text{rank}(p^\perp \cap S) = -1, 1, 2\) see [5] and [7]. In this Paper we prove the same result is satisfied for the general case \(D_{n,k}\).

**Remark.** Each geometry of the class \(G_1\) or \(G_2\) of \(D_{n,k}\) is denoted by \(A_T\), where \(T\) is the TI \(n\)-space that corresponds to such geometry and \(A_D\) denotes to the symplecton of type \(A_{3,2}\), where \(D\) is the TI \((k+2)\)-space that corresponds to such symplecton.

**4.1 Theorem.** Let \(\Gamma\) be a geometry of type \(D_{n,k}\), then:

1- Let \(A_{D1}, A_{D2}\) be two distinct symplecta of type \(A_{3,2}\) with then \(\text{rank}(A_{D1} \cap A_{D2}) = -1, 0, 2\).

2- If \(S_1\) and \(S_2\) are symplecta of type \(D_{n-1,1}\), then \(\text{rank}(S_1 \cap S_2) = -1, 0\).

3- If \(S\) is a symplecton of type \(D_{n-1,1}\) and \(A_D\) is a symplecton of type \(A_{3,2}\), then \(\text{rank}(S \cap A_D) = -1, 2\).

4- If \((p, S)\) is a non-incidence pair of point and symplecton, then \(\text{rank}(p^\perp \cap S) = -1, 1, 2\).

**Proof.**

1- \(A_{T1}\) and \(A_{T2}\) are located in the same class \((G_1)\), so \(A_{D1}\) and \(A_{D2}\) are located in the located in \(G_1\), then we have two cases:

a- \(A_{D1}, A_{D2}\) are symplecta of the same geometry \(A_T\), then it has three cases:

a1- If \(D_1 \cap D_2 \leq (k-1)\)-space, then \(\text{rank}(A_{D1} \cap A_{D2}) = -1\).

a2- if \(D_1 \cap D_2 = k\)-space, then \(\text{rank}(A_{D1} \cap A_{D2}) = 0\).

a3- if \(D_1 \cap D_2 = (k+1)\)-space, then \(\text{rank}(A_{D1} \cap A_{D2}) = 2\).

b- \(A_{D1}, A_{D2}\) are symplecta of different geometries \(A_{T1}\) and \(A_{T2}\) respectively, then \(T_1 \cap T_2\) is a space of odd dimension and we have three cases:

b1- For \(\text{dim}(T_1 \cap T_2) \leq k-1\), \(\text{dim}(D_1 \cap D_2) \leq k-1\), so \(\text{rank}(A_{D1} \cap A_{D2}) = -1\).

b2- For \(\text{dim}(T_1 \cap T_2) = k\), \(\text{dim}(D_1 \cap D_2) \leq k\). Then \(\text{rank}(A_{D1} \cap A_{D2}) = -1, 0\).

b3- For \(\text{dim}(T_1 \cap T_2) = k\), then either \(\text{dim}(D_1 \cap D_2) \leq k\) and \(\text{rank}(A_{D1} \cap A_{D2}) = -1, 0\) or \(\text{dim}(D_1 \cap D_2) = k+1\) and \(\text{rank}(A_{D1} \cap A_{D2}) = 2\). Then \(\text{rank}(A_{D1} \cap A_{D2}) = -1, 0, 2\).

Now the case in which \(A_{T1}\) and \(A_{T2}\) are located in different classes is similar to those cases above i.e., \(\text{rank}(A_{D1} \cap A_{D2}) = -1, 0, 2\).

2- In this case \(\psi(S_1)\) and \(\psi(S_2)\) correspond to \((k-1)\)-spaces. We have two cases:

i- If \(\langle \psi(S_1), \psi(S_2) \rangle < k-2\), then we cannot find a \(k\)-space that contains \(\langle \psi(S_1), \psi(S_2) \rangle\), i.e., \(S_1 \cap S_2 = \emptyset\). Thus \(\text{rank}(S_1 \cap S_2) = -1\).

ii- If \(\psi(S_1) \cap \psi(S_2) = k-2\) and \(\langle \psi(S_1), \psi(S_2) \rangle = k\)-space, then \(\text{rank}(S_1 \cap S_2) = 0\).
3- If $S$ is a symplecton of type $D_{n-1,1}$ and $A_D$ is a symplecton of type $A_{3,2}$, then we have two cases:

i- $\psi(S) \cap D \leq (k-2)$-space, then we there is no a k-space that contains $\psi(S)$ and contained in $D$, i.e., rank$(S \cap A_D) = -1$.

ii- $\psi(S) \subseteq D$, then the number of different k-spaces in $D$ that contains $\psi(S)$ form a space of rank 3 i.e., rank$(S \cap A_D) = 2$.

4- a- For any pair $(p, A_D)$ of a point $p$ and a symplecton $A_D$ of type $A_{3,2}$, there are two cases:

ai- If dim$(\psi(p) \cap D) < k-1$, then there is no any k-space contained in $D$ and meets $\psi(p)$ in a $(k-1)$-space i.e., $\psi(p) \cap D = \emptyset$ and rank$(p^\perp \cap S) = -1$.

aii- If dim$(\psi(p) \cap D) = k-1$, then set of k-spaces in $D$ that contain the $(k-1)$-space $\psi(p) \cap D$ form a projective plane i.e., rank$(p^\perp \cap S) = 2$.

b- For any pair $(p, S)$ of a point $p$ and a symplecton $S$ of type $D_{n-k+1,1}$, there are two cases:

bi- If dim$(\psi(p) \cap \psi(S)) < k-2$, then we cannot find any k-space contains $\psi(S)$ and meets $\psi(p)$ in a $(k-1)$-space i.e., rank$(p^\perp \cap S) = -1$.

bii- If dim$(\psi(p) \cap \psi(S)) = k-2$, and $(\psi(p), \psi(S)) = (k+1)$-space, then there are two k-spaces meet $\psi(p)$ in a $(k-1)$-space and contain $\psi(S)$. Since the two k-space form a $(k+1)$-space, then $p^\perp \cap S$ is a line i.e., rank$(p^\perp \cap S) = 1$.

References


2. Cameron, P. J. “Projective and polar spaces” Published by the School of Mathematical Sciences, QMW Univ. of London, U.K., (1990).


5. El-Atrash, Mohammed and Zayda, Abdelsalam “Properties of point-line geometry of type $D_{n,2}(F)$” Al-Aqsa University Journal Vol. 5, No. 1, 35-44 , (2001)
