The Topology $\tau^*$ on Alexandroff Spaces

Abstract

The generalized closure operator $cl^*$ induces a topology $\tau^*$. In this paper, we study the topology $\tau^*$ on lower bounded Alexandroff spaces. We prove that $(X, \tau^*)$ is a submaximal $T_0 -$Alexandroff space. We get some new results about the relation between $\tau^*$ and $\tau_e$. Then we prove that a subset $A$ in a lower bounded space is $\alpha -$ closed set if and only if $A$ is $\alpha -$ open in the dual space.

1. Introduction:

A topological space $X$ is submaximal [6] if each dense subset is open. $X$ is $T_1 -$space if every singleton set is either open or closed [12]. A subset $A$ of $X$ is said to be $\alpha -$ open [9] (resp. preopen [1], semi-open [8]) if $A \subseteq \overline{A}^o$ (resp. $A \subseteq \overline{A}$, $A \subseteq \overline{A}^o$). A set $F$ is called $j -$ closed for $j \in \{\alpha, semi, pre\}$ if $X \setminus F$ is $j -$ open. The family of all $\alpha -$open (resp. preopen, preclosed, semi-open, semi-closed) is denoted by $\tau_\alpha$ (resp. $PO(X)$, $PC(X)$, $SO(X)$, $SC(X)$). We have the following facts: The collection $\tau_\alpha$ forms a topology on $X$ [9]. $\tau_\alpha = PO(X) \cap SO(X)$ [11]. $\tau \subseteq \tau_\alpha$ (resp. $\tau \subseteq JO(X)$ for $J \in \{S, P\}$). For $j \in \{semi, pre\}$, the union (intersection) of any family of $j -$ open ($j -$ closed) sets is $j -$ open ($j -$ closed).

An Alexandroff space [10] (briefly $A-$space) $X$ is a topological space in which an arbitrary intersection of open sets is open. In this space, each element $x$ possesses a smallest open neighborhood $V(x)$ which is the intersection of all open sets containing $x$. For every $T_0$ $A-$space $(X, \tau)$, there is a corresponding poset $(X, \leq_r)$ in one to one and onto way, where each one of them is completely determined by the other. If $(X, \tau)$ is a $T_0$ $A-$space, we define the corresponding partial order $\leq_r$, called (Alexandroff) specialization order, by: $a \leq_r b$ iff $a \in [b]$ iff $b \in V(a)$. On the other hand, if $(X, \leq)$ is a poset, then the collection $B = \{x : x \in P\}$ forms a base for a $T_0$ $A-$space on $X$, denoted by $\tau_\leq$. 

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Alexandroff spaces, upper and lower bounded space, $\alpha -$ open sets, generalized closed sets, $\tau^*$ spaces.

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So throughout this paper, we consider \((X, \tau(\subseteq))\) to be a closed space \((X, \tau)\) together with its corresponding closed set \((X, \subseteq)\). A space is upper bounded closed space \((X, \subseteq)\) and \((X, \subseteq)\) if every chain of points in the corresponding closed set \((X, \subseteq)\) is bounded above \([2]\). A space is lower bounded \((LB)\) \([2]\) if every chain of points is bounded below. Given a closed set \((X, \subseteq)\), the set of all maximal elements is denoted by \(M(X)\) (or simply by \(M\)) and the set of all minimal elements is denoted by \(m(X)\) (or simply by \(m\)). Moreover, for each \(x \in X\), we define \(\tilde{x} = \{y \in X : x \subseteq y\}\), \(\hat{x} = \{y \in X : x \supseteq y\}\), \(\tilde{x}\) to be the set of all maximal elements greater than or equal to \(x\), and \(\hat{x}\) the set of all minimal elements less than or equal to \(x\) \([4]\). If \(X\) is a \(UB T_0\) space, then \(M \neq \emptyset\) and \(\hat{x} \neq \emptyset\ \forall x \in X\). Similarly in an \(LB T_0\) space, \(m \neq \emptyset\) and \(\tilde{x} \neq \emptyset\).

For a \(T_0\) space, \(V(x) = \hat{x}\), and \(\tilde{x} = \tilde{x}\). Moreover, if \(A \subseteq X\), then \(\bar{A} = \hat{A} = \{x \in X : x \subseteq A\}\) and \(\tilde{A} = \{x \in X : x \supseteq A\}\), and the smallest open set containing \(A\) equals \(\hat{A} = \bigcup_{x \in A} \hat{x}\).

**Theorem 1.1.** \([5]\) Let \((X, \tau(\subseteq))\) be a \(T_0\) space, then all the following are equivalent:

a. \(X\) is \(T_1\) space.

b. \(X\) is submaximal.

c. Each \(\alpha\)-open set is open; that is, \(\tau_a = \tau\).

d. Each element of \(X\) is either maximal or minimal.

**Theorem 1.2.** \([3]\) Let \((X, \tau(\subseteq))\) be a \(UB T_0\) space. Then:

a. \(PO(X) = \tau_a\) and \(PO(X) \subseteq SO(X)\).

b. A subset \(A\) is semi open set if and only if \(\hat{x} \cap A \neq \emptyset\ \forall x \in A\).

c. A subset \(A\) is \(\alpha\)-open set if and only if \(\tilde{x} \subseteq A\ \forall x \in A\).

d. A subset \(A\) is clopen in \(\tau\) if and only if \(A\) is clopen in \(\tau_a\).

e. \((X, \tau_a)\) is a submaximal \(T_0\) space.

### 1. Generalized Closed Sets on \(LB T_0\) \(A\)-Spaces:

**Definition 2.1.** \([7]\) A subset \(A\) of a topological space \((X, \tau)\) is called a generalized closed set (briefly \(g\)-closed), if \(\bar{A} \subseteq U\) whenever \(A \subseteq U\), and \(U\) is open in \(X\). A subset \(A\) of \(X\) is \(g\)-open if \(A^c\) is \(g\)-closed.

**Remark 2.2.** (a) In general, the collection of all \(g\)-open sets need not form a topology on \(X\).

(b) In a \(T_0\) space, since \(\hat{A}\) is the smallest open neighborhood of \(A\), we can use \(\hat{A}\) instead of the open set \(U\) which contains \(A\) in the Definition 2.1.

(c) An alternating definition of \(T_0\) space found in \([7]\) as a space where each \(g\)-closed set is closed.

**Proposition 2.3.** If \((X, \tau(\subseteq))\) is a \(T_0\) space and \(A \subseteq X\), then \(A\) is \(g\)-closed set if and only if \(\forall x \in A, \forall a \in \tilde{x}\) we have \(\bar{a} \cap A \neq \emptyset\).

**Proof.** Let \(x \in A\) and \(a \in \tilde{x}\). As \(A\) \(g\)-closed, \(\bar{A} \subseteq \hat{A}\), and hence \(a \in \hat{A}\). Then there exists \(y \in A\) such that \(a \in \hat{y}\). Equivalently, \(y \in a \subseteq \tilde{a}\).

Conversely, it suffices to prove that \(\bar{A} \subseteq \hat{A}\), so let \(r \in \tilde{A}\). Then there exists \(x \in \hat{A}\) such that \(r \subseteq x\). By given, \(x \subseteq \tilde{A}\). Then \(r \subseteq x\). Hence \(a \subseteq A\). The converse is easy.

**Corollary 2.4.** Let \((X, \tau(\subseteq))\) be an \(LB T_0\) space, and \(A\) a subset of \(X\). Then \(A\) is \(g\)-closed set if and only if \(\forall x \in A, \tilde{x} \subseteq A\).

**Proof.** For \(x \in A\), \(\tilde{x} \subseteq x\). Let \(a \in \tilde{x}\), then \(a \subseteq \tilde{a}\) and \(\bar{a} \cap A \neq \emptyset\). Hence \(a \in A\). The converse is easy.

**Corollary 2.5.** If \((X, \tau(\subseteq))\) is an \(LB T_0\) space, and \(A \subseteq X\), then \(A\) is \(g\)-open if and only if \(\hat{x} \subseteq A\) whenever \(x \in m \cap A\).

**Proof.** Suppose to contrary that \(y \in m \cap A\) such that \(\hat{y} \cap A^c \neq \emptyset\). Then there exists \(r \subseteq y\) such that \(r \in A^c\). By Corollary 2.4, \(\bar{r} \subseteq A^c\). Since \(y \in m\), \(y \in \bar{r} \subseteq A^c\) which is a contradiction. Conversely, Let
$r \in A'$ and let $s \in \bar{r}$. Then $r \uparrow s$. This implies that $s \in A'$, and hence $\bar{r} \subseteq A'$. Therefore, $A'$ is $g$-closed.

**Example 2.6.** Let $X = \{a, b, c, d, e, f, g, h, r\}$, equipped with the partial order as shown in Figure 1 below:

![Figure 1: g-closed and g-open sets](image)

For the set $A = \{a, e, r, g, f\}$, we have that $\bar{a} = \{f, g, r\}$, $\bar{e} = \{r\}$, $\bar{r} = \{r\}$, $\bar{f} = \{f\}$, and $\bar{g} = \{g\}$. That is, $\forall x \in A$, $\bar{x} \subseteq A$. Hence $A$ is $g$-closed. Let $B = \{d, f, c, g, r\}$. Then $B$ is not $g$-closed set, since $\bar{c} \not\subseteq B$. Consider the set $D = \{h, c, f, d, a\}$. We have that $D \cap m = \{h, f\}$, and $\uparrow h \subseteq D$, $\uparrow f \subseteq D$. Hence $D$ is $g$-open.

**2. The Topology $\tau^*$:**

A Kuratowski closure operator $cl^*$ is defined in [13] as the intersection of all g-closed sets. The topology generated by $cl^*$ is denoted as $\tau^*$. For any topological space $(X, \tau)$, the space $(X, \tau^*)$ is always a $T_1$ space. It is worth mentioned that if $A$ is a $g$-closed, then $cl^*(A) = A$ is $g$-closed. But for arbitrary set $A$, $cl^*(A)$ need not be always $g$-closed.

**Theorem 3.1.** If $(X, \tau(\leq))$ is an $LB$ $T_0$ $A$-space and $A \subseteq X$, then $cl^*(A) = A \cup \bigcup_{x \in A} \bar{x}$.

**Proof.** From Corollary 2.4, the set $A \cup \bigcup_{x \in A} \bar{x}$ is a $g$-closed set containing $A$, so $cl^*(A) \subseteq A \cup \bigcup_{x \in A} \bar{x}$. Let $S$ be any $g$-closed set containing $A$. Then $\forall x \in S$, $\bar{x} \subseteq S$. If $y \in A \cup \bigcup_{x \in A} \bar{x}$ and $y \not\in A$, then $y \in \bar{x}$ for some $x \in A$. As $A \subseteq S$, $y \in S$. Hence $A \cup \bigcup_{x \in A} \bar{x}$ is the smallest $g$-closed set containing $A$. Therefore $cl^*(A) = A \cup \bigcup_{x \in A} \bar{x}$.

**Corollary 3.2.** If $X$ is an $LB$ $T_0$ $A$-space, then $\forall A \subseteq X$, $cl^*(A)$ is a $g$-closed.

From this corollary, we deduce that in an $LB$ $T_0$ $A$-space $X$, $\tau^*$ is precisely the collection of all $g$-open sets in $X$.

**Example 3.3.** In Example 2.6, as $A$ $g$-closed, $cl^*(A) = A$. So $A = \{b, c, d, h\} \in \tau^*$. While $cl^*(B) = B \cup \{h\} \neq B$, so $B$ is not $g$-closed.

**Theorem 3.4.** If $(X, \tau(\leq))$ is an $LB$ $T_0$ $A$-space, then $(X, \tau^*)$ is a $T_1$ $A$-space.

**Proof.** Let $\{u_\alpha : \alpha \in \Delta\}$ be a collection of $g$-closed sets in $(X, \tau)$ ($= closed in (X, \tau^*)$), and let $x \in \bigcup_{\alpha \in \Delta} u_\alpha$. Then by Corollary 2.4, $\bar{x} \subseteq u_\alpha$ for some $\alpha \in \Delta$. So, $\bar{x} \subseteq \bigcup_{\alpha \in \Delta} u_\alpha$. Hence $\bigcup_{\alpha \in \Delta} u_\alpha$ is $g$-closed, and $(X, \tau^*)$ is $A$-space.

As $(X, \tau^*)$ is a $T_0$ $A$-space, we denote its specialization order by $\leq^*$. The following theorem describes the partial order $\leq^*$ on $X$.

**Proposition 3.5.** Let $(X, \tau(\leq))$ be an $LB$ $T_0$ $A$-space and $x, y$ two elements in $X$. Then $x \leq^* y$ if and only if $x \in \{y\} \bigcup \bar{y}$.

**Proof.** Direct using the fact $cl^*(y) = \{y\} \bigcup \bar{y}$.

**Remark 3.6.** With respect to the order $\leq^*$, we will use the following notations:

- $M^*$ to be the set of all maximal elements.
- $m^*$ to be the set of all minimal elements.
- $\uparrow x := \{y \in X : x \leq^* y\}$.
- $\downarrow x := \{y \in X : y \leq^* x\}$.
- $\hat{x}^* := \uparrow x \cap M^*$.
- $\check{x}^* := \downarrow x \cap m^*$.
Theorem 3.7. Let \((X, \tau(\leq))\) be an \(LB\) \(T_0\) A-space. Then:

a. each element in \(X\) is either maximal or minimal with respect to \(\leq\).

b. \((X, \tau^*)\) is submaximal space.

c. \(M^* = m\).

d. \(M^* = (X \setminus m) \cup (m \cap M)\).

Proof. The parts (a) and (b) come directly from Theorem 1.1 and the fact that \(\tau^*\) is \(T_1\).

(c) Let \(x \in m^*\), and let \(y \leq x\). Then from part (a), either \(y = x\) or \(y \in \tilde{x}\). So \(y \leq x\), and hence \(y = x\). If \(x \in m\), then \(\tilde{x} = \{x\}\) and \(\tilde{x} \cup \{x\} = \{x\}\). So, if \(y \leq x\), then \(y = x\) and \(x \in m^*\).

(d) Comes directly from parts (a) and (c).

Proposition 3.8. If \((X, \tau(\leq))\) is an \(LB\) \(T_0\) A-space and \(A \subset X\), then \(\text{int}^*(A) = (A \setminus m) \cup \{x \in m:\uparrow x \subset A\}\).

Proof. \(x \in \text{int}^*(A)\) iff \(\uparrow^* x \subset A\).

Example 3.9. For the \(LB\) \(T_0\) A-space described in Example 2.6, the induced order \(\leq^*\) on \(X\) is given in Figure 2 below:

![Figure 2: Induced poset \((X, \leq^*)\)](image)

\section{3. The Relation Between \(\tau^*\) and \(\tau_\alpha\)}

If \((X, \tau)\) is an \(A\) – space with a collection \(F\) of closed sets, then \(F\) is itself an Alexandrov topology on \(X\), called the dual Alexandrov topology of \(\tau\) on \(X\), and usually denoted by \(\tau_\alpha\). If \((X, \tau(\leq))\) is a \(T_0\) A-space, then the Alexandroff dual is also \(T_0\) -space. The induced order \(\leq_\varnothing\) is the reverse order of the order \(\leq\); that is, \(x \leq_\varnothing y\) iff \(y \leq x\). So, \(x \in X\), we have \(V(x) = cl_\varnothing(x)\) and \(V(x) = cl(x)\). Moreover, if \((X, \tau(\leq))\) is a \(UB\) (resp. an \(LB\)) \(T_0\) A-space, then the dual space \((X, \tau_\alpha(\leq_\varnothing))\) is an \(UB\) (resp. a \(UB\)) \(T_0\) A-space. The set of maximal elements \(M^\varnothing\) in the corresponding poset \((X, \leq_\varnothing)\) equals the set of minimal points \(m\) in \((X, \leq)\), and similarly \(m^\varnothing = M\). For \(A \subset X\), the up (resp. down) set \(\uparrow_\varnothing A = \downarrow_\varnothing A\) (resp. \(\downarrow_\varnothing A = \uparrow_\varnothing A\)). For \(x \in X\), \(\tilde{x}_\varnothing = \uparrow_\varnothing x \cap M^\varnothing\) and \(\bar{x}_\varnothing = \downarrow_\varnothing x \cap M^\varnothing\). It is clearly that \(\tilde{x}_\varnothing = \tilde{x}\) and \(\bar{x}_\varnothing = \bar{x}\).

Theorem 4.1. Let \((X, \tau(\leq))\) be an \(LB\) \(T_0\) A-space, and \(A\) is a subset of \(X\). Then \(A\) is g-closed in \((X, \tau)\) if and only if \(A\) is \(\alpha\) – open in the dual space \((X, \tau_\alpha(\leq_\varnothing))\).

Proof. \(A\) is g-closed set in \((X, \tau(\leq))\) if and only if \(\tilde{x} \subset A\ \forall x \in A\) (Theorem 2.4) if and only if \(\tilde{x}_\varnothing \subset A\ \forall x \in A\) and if only if \(A\) is \(\alpha\) – open in \((X, \tau_\alpha)\) (Theorem 1.2 part (c)).

Proposition 4.2. Let \((X, \tau(\leq))\) be a \(UB\) \(T_0\) A-space, then \((\tau_\alpha)^* = \tau_\alpha\).

Proof. Let \((X, \tau(\leq))\) be a \(UB\) \(T_0\) A-space. Then by Theorem 1.1 and Theorem 1.2 (e), \((X, \tau_\alpha)\) is \(T_\frac{1}{2}\). So any g-closed set in \(\tau_\alpha\) is closed set. Equivalently, any g-open is open in \(\tau_\alpha\). Therefore \((X, (\tau_\alpha)^*) = (X, \tau_\alpha)\).

Proposition 4.3. Let \((X, \tau(\leq))\) be an \(LB\) \(T_0\) A-space then \((\tau_\alpha) = (\tau^*)_{\varnothing}\).
Proof. $U$ is $\alpha$-open in $(X, \tau_X)$ if and only if $U^c$ is open in $X$ (Theorem 4.1) if and only if $U^c$ is open in $(\tau^*_X)$ if and only if $U \in (\tau^*_X)$.

Proposition 4.4. Let $(X, \tau(\leq))$ be an $LB$-space and $A \subseteq X$. Then $A$ is clopen if and only if $A$ is $g$-closed in $(\tau_X)$.

Proof. $A$ is clopen in $\tau$ if and only if $A$ is clopen in $(\tau^*_X)$ (Theorem 4.3) if and only if $A$ is clopen in $\tau^*_X$.

References:


