On n-primly Ideals

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Abstract

An ideal \( I \) is primal over a commutative ring \( R \) with non zero identity if the set of all elements that are not prime to \( I \), forms an ideal of \( R \). This definition was introduced by Ladislas Fuchs in 1950. In this paper, we define an ideal \( I \) over a commutative ring \( R \) with non zero identity to be n-primly if the set of all elements that are not n-primary to \( I \), forms an ideal of \( R \). But first we introduced the concepts of n-primary elements to an ideal, n-adjoint sets for an ideal, uniformly not n-primary sets for an ideal, n-primly ideals and uniformly n-primly ideals. We study the previous concepts in details illustrated by several examples. We also study the relation between several sets like n-adjoint sets for an ideal, n-adjoint sets for an ideal and the adjoint set for this ideal, sets that are not n-primary for an ideal and uniformly not n-primary sets for this ideal. Also we investigate the relation between some ideals like uniformly n-primly ideals and n-primly ideals, primary ideals and n-primly ideals over a commutative ring with identity

Keywords n-primary elements for an ideal, n-adjoint sets for an ideal, n-primly ideals, Uniformly not n-primary sets for an ideal, Uniformly n-primly ideals.

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للمثال

في بداية البحث نطرد تعريفات بعض المفاهيم الرئيسية في البحث وهي: مفهوم عناصر n-الابتدائية للمثالي، مفهوم مجموعات n-المستهلكة للمثال، مفهوم المجموعات الموحدة n-الابتدائية للمثال، مفهوم متاليات n-بريملي، مفهوم مثاليات n-بريملي الموحدة الشكل. ثم نقوم بدراسة المفاهيم السابقة بالتفصيل ونتناول عليها بالعديد من الأمثلة.

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1. Introduction

The concept of primal ideals over a commutative ring with non-zero identity was introduced and studied by Ladislas Fuchs in [1]. In [2] Dauns generalized this concept to the concept of primal modules. Then many studies have been done based on these concepts, see [3-6]. These previous studies were a motive for us to introduce the concept of n-primly ideals.

We first introduce the concept of n-primary element for an ideal \( I \) over a commutative ring \( R \) with non zero identity. Thus for a positive integer \( n \), we say that an element \( s \) of a ring \( R \) is \( n \)-primary to \( I \) if the set

\[
 s^{-n}I = \{ a \in R : s^n a \in I \} \subseteq \sqrt{I}. 
\]

We call the set of all elements that are not \( n \)-primary to \( I \), the \( n \)-adjoint set for \( I \). Note that the \( n \)-adjoint set for \( I \) is not necessarily an ideal of \( R \). However, if it is an ideal, then we call the ideal \( I \) to be \( n \)-primly ideal.

Let \( n \) be a positive integer. Let \( I \) be an ideal over a commutative ring \( R \) with non zero identity. In Section 2 we introduce the concepts of \( n \)-primary elements to \( I \) illustrated by some examples. Based on this concept, we present sets that are not \( n \)-primary to \( I \) and sets that are uniformly not \( n \)-primary to \( I \). Then we study the relation between these two sets and find the conditions that make these two sets equivalent.

Section 3 deals with \( n \)-adjoint sets for the ideal \( I \). We study the relation between \( n \)-adjoint sets for the ideal \( I \), for different \( n \)'s. We also study the relation between these sets and the adjoint set for the ideal \( I \).

Finally, in Section 4 we introduce the concepts of \( n \)-primly ideals and uniformly \( n \)-primly ideals. We present some properties of \( n \)-primly ideals and study the relation between uniformly \( n \)-primly ideals and \( n \)-primly ideals. We also investigate the relation between primary ideals and \( n \)-primly ideals.

Throughout this paper, all rings are assumed to be commutative with non zero identity.

2. \( n \)-primary elements to an ideal

We start this section by the following definition

**Definition 2.1** Let \( n \) be positive integer. Let \( I \) be an ideal over a ring \( R \). Let \( s \) be an element of \( R \). Define \( s^{-n}I = \{ a \in R : s^n a \in I \} \).

**Remark 2.2** Let \( I \) be an ideal over a ring \( R \). Let \( s \) be an element of \( R \). Then

\[
 I \subseteq s^{-1}I \subseteq s^{-2}I \subseteq s^{-3}I \subseteq ... ,
\]

and the equality does not holds in general.

**Proof** The inclusion is clear and can noticed directly from the definition since \( I \) is an ideal. However, the equality does not hold in general because if we take \( R = \mathbb{Z} \), the
set of integers, and \( I = 8Z \), then \( 2^{-1}I = 4Z \), \( 2^{-2}I = 2Z \) and \( 2^{-3}I = Z \).

**Remark 2.3** Let \( I \) be an ideal over a ring \( R \). Let \( s \) be an element of \( R \). Let \( n \) be a positive integer. Then \( s^nI \) does not mean that \( s^n \) has an inverse in \( R \), however if \( s^n \) has an inverse in \( R \), then \( (s^{-n})I = \{s^{-n}b : b \in I\} \), and the set \( s^{-n}I \) are the same.

**Proof** Let \( r \in (s^{-n})I \), then \( r = s^{-n}b, \ b \in I \).

Now, \( s^n r = s^n (s^{-n}b) = b \in I \). Thus \( r \in s^{-n}I \).

On the other hand, let \( r \in s^{-n}I \), then \( s^n r \in I \).

Hence \( r = s^{-n}(s^n r) \in (s^{-n})I \).

**Definition 2.4** Let \( n \) be a positive integer. Let \( I \) be an ideal over a ring \( R \). Let
\[
s \in R. \text{ If } s^{-n}I \subseteq \sqrt{I},
\]then \( s \) is said to be \( n \)-primary to \( I \).

**Example 2.5** Let \( R = Z \). Let \( I = 4Z \), then \( 2^{-1}I = 2Z \) and \( 2^{-2}I = Z \). Thus \( 2 \) is \( 1 \)-primary to \( I \), but it is not \( 2 \)-primary to \( I \).

**Definition 2.6** Let \( n \) be a positive integer. Let \( I \) be an ideal over a ring \( R \). A subset \( A \) of \( R \) is not \( n \)-primary to \( I \) if for every element \( a \) in the set \( A \), \( a \) is not \( n \)-primary to \( I \).

**Definition 2.7** Let \( n \) be a positive integer. Let \( I \) be an ideal over a ring \( R \). A subset \( A \) of \( R \) is uniformly not \( n \)-primary to \( I \) if \( \exists b \in R - \sqrt{I} \) with \( A^n b = \{a^n b : a \in A\} \subseteq I \).

**Remark 2.8** Let \( n \) be a positive integer. Then every proper ideal is uniformly not \( n \)-primary to itself. Take \( b=1 \), the identity in the previous definition.

**Proposition 2.9** Let \( n \) be a positive integer. Let \( I \) be an ideal over a ring \( R \). If \( A \) is uniformly not \( n \)-primary to \( I \), then \( A \) is not \( n \)-primary to \( I \).

**Proof** Let \( A \) be uniformly not \( n \)-primary to an ideal \( I \). Then there exist an element \( u \) such that \( u \in R - \sqrt{I} \) with \( A^n u \subseteq I \). Let \( a \in A \). Let \( b = u \), then \( a^n b \in I \) and \( b \in R - \sqrt{I} \).

Thus \( a \) is not \( n \)-primary to \( I \). Since \( a \) is an arbitrary element in \( A \), then \( A \) is not \( n \)-primary to \( I \).

**Remark 2.10** The converse of Proposition 2.9 is not true in general. Take \( R = Z \), and let \( I = 6Z \), then \( A = \{2,3\} \) is not \( 1 \)-primary to \( I \). However \( A \) is not uniformly not \( 1 \)-primary to \( I \).

The following Theorem treats the case in which the converse of Proposition 2.9 is true. But first remember that an ideal \( I \) over a ring \( R \) is quasi primary ideal if \( \sqrt{I} \) is prime ideal over \( R \), see [7].

**Theorem 2.11** Let \( n \) be a positive integer. Let \( I \) be a quasi primary ideal over a ring \( R \). If \( A \) is finite subset of \( R \), then \( A \) is uniformly not \( n \)-primary to \( I \) if and only if \( A \) is not \( n \)-primary to \( I \).

**Proof** The necessity is by Proposition 2.9. To prove the sufficiency, suppose that \( A = \{a_1, a_2, \ldots, a_m\} \) is not \( n \)-primary subset to \( I \).

Then \( \exists b_1, b_2, \ldots, b_m \in R - \sqrt{I} \) such that \( a_i^n b_i \in I \),
\[
\forall i \in \{1,2,\ldots,m\}. \text{ Let } b = \prod_{j=1}^{m} b_i.
\]

Since \( I \) is quasi primary ideal over \( R \),
then \( \sqrt{I} \) is prime ideal over \( R \).

Thus \( b \in R - \sqrt{I} \) with \( A^n b \subseteq I \).

Hence \( A \) is uniformly not \( n \)-primary to \( I \).
Since every primary ideal is quasi primary, see[7], then we can conclude the following result.

**Corollary 2.12** Let $n$ be a positive integer. Let $I$ be a primary ideal over a ring $R$ If $A$ is finite subset of $R$, then $A$ is uniformly not $n$-primary to $I$ if and only if $A$ is not $n$-primary to $I$.

**Example 2.13** Let $X$ be a set and let $R$ be the ring $(P(X), \Delta, \cap)$ where $P(X)$ is the power set of $X$ and the operation $\Delta$ is defined by $A \Delta B = (A - B) \cup (B - A)$. Since for every positive integer $n$ and every $A \in P(X)$, $A^n = A$, then $\sqrt{I} = I$ for every proper ideal $I$ of $R$. Thus by Remark 2.8, $\sqrt{I}$ is uniformly not $n$-primary to $I$, for every proper ideal $I$ of $R$.

In fact, we can show that the $n$ radical of a proper ideal of a ring is always uniformly not $n$-primary to this ideal as in the following proposition.

**Proposition 2.14** Let $n$ be a positive integer. Let $I$ be a proper ideal over a ring $R$, then $\sqrt[n]{I} = \{ r \in R : r^n \in I \}$ is uniformly not $n$-primary to $I$.

**Proof** Let $A = \sqrt[n]{I}$. Then $\forall a \in A$, $a^n \in I$.

Let $b = 1$. Since $1 \in R - \sqrt[I]{I}$, then $A^n b \subseteq I$.

Thus $A$ is uniformly not $n$-primary to $I$.

### 3 n-adjoint sets for an ideal.

**Definition 3.1** Let $n$ be a positive integer. Let $I$ be an ideal over a ring $R$. The set of all elements that are not $n$-primary to $I$ is called the $n$-adjoint set for $I$ and is denoted by $n-adj(I)$. That is $n-adj(I) = \{ a \in R : a^n b \in I \}$ for some $b \in R - \sqrt[I]{I}$.

**Remark 3.2** If $I$ is an ideal over a ring $R$, then $n-adj(I) \neq R$, for every positive integer $n$.

**Example 3.6** Since $\mathbb{Z}$ is Noetherian ring, then $1-adj(4\mathbb{Z}) = 4\mathbb{Z}$, $n-adj(4\mathbb{Z}) = 2\mathbb{Z}$ for every positive integer $n \geq 2$.
\[ \bigcup_{n=1}^{\infty} n - \text{adj}(4Z) = 2 - \text{adj}(4Z). \]
\[ \bigcup_{n=1}^{\infty} n - \text{adj}(8Z) = 3 - \text{adj}(8Z). \]
\[ \bigcup_{n=1}^{\infty} n - \text{adj}(9Z) = 2 - \text{adj}(9Z). \]

It is known that for an ideal \( I \) of a ring \( R \), the adjoint set of \( I \), which is denoted as \( \text{adj}(I) = \{ a \in R : ab \in I \text{ for some } b \in R - I \} \), see [4]. The following results give the relation between \( n \)-adjoint sets for the ideal \( I \) and the set \( \text{adj}(I) \).

**Proposition 3.7** For any ideal \( I \) of a ring \( R \), \( 1 - \text{adj}(I) \subseteq \text{adj}(I) \).

**Proof** Let \( a \in 1 - \text{adj}(I) \), then \( \exists b \in R - \sqrt{I} \) such that \( ab \in I \). Hence \( b \in R - I \).

Therefore \( a \in \text{adj}(I) \).

**Remarks 3.8** Since in the ring of integers \( \mathbb{Z} \), \( \text{adj}(4Z) = 4Z \) and \( \text{adj}(4Z) = 2Z \), then the equality in Proposition 3.7 does not hold in general.

**Theorem 3.9** If \( I \) is a prime ideal over the ring \( R \), then \( I - \text{adj}(I) = \text{adj}(I) \).

**Proof** Let \( a \in \text{adj}(I) \), then \( a b \in I \) for some \( b \in R - I \). Since \( I \) is prime, then \( b^m \in R - I \), for any positive integer \( m \).

Hence \( b \in R - \sqrt{I} \) and therefore \( a \in 1 - \text{adj}(I) \).

By Proposition 3.7, the equality holds.

The following results follows immediately from the previous theorem and Remarks 3.4

**Corollary 3.10** If \( I \) is a prime ideal over the ring \( R \), then

\[ \text{adj}(I) = 1 - \text{adj}(I) \subseteq 2 - \text{adj}(I) \subseteq 3 - \text{adj}(I) \subseteq \ldots, \]

that is \( \text{adj}(I) \subseteq n - \text{adj}(I) \), for every positive integer \( n \).

**Proposition 3.11** Let \( n \) be a positive integer. Let \( I \) be a proper ideal of a ring \( R \). Then \( I \subseteq \sqrt[n]{I} \subseteq n - \text{adj}(I) \).

**Proof** It is clear that \( I \subseteq \sqrt[n]{I} \). Now, let \( r \in \sqrt[n]{I} \), then \( r^n \in I \). Thus \( r^n 1 \in I \). Since \( 1 \in R - \sqrt{I} \), then \( r \in n - \text{adj}(I) \).

**Proposition 3.12** Let \( n \) be a positive integer. Let \( I \) be a prime ideal of a ring \( R \). Then \( I \subseteq \sqrt[n]{I} \subseteq n - \text{adj}(I) \).

**Proof** It is clear that \( I \subseteq \sqrt[n]{I} \subseteq I \). Now, let \( a \in \sqrt[n]{I} \), then \( a^m \in I \), for some integer \( m \). Let \( m \) be the smallest such positive integer. Then

i) If \( m = n \), then \( a^m = a^n \in I \) implies \( a \in n - \text{adj}(I) \).

ii) If \( m < n \), then \( a^m \in I \) implies \( a^n 1 = a^{n-m} a^m \in I \).

Thus \( a \in n - \text{adj}(I) \).

iii) If \( m > n \), then \( a^m = a^n a^{m-n} \in I \) with \( a^{m-n} \in R - I \) implies \( a^{m-n} \in R - \sqrt{I} \), because \( I \) is prime ideal of \( R \). Let \( b = a^{m-n} \), then \( b \in R - \sqrt{I} \) with \( a^n b \in I \) implies \( a \in n - \text{adj}(I) \).

**4 n-primly ideals.**

We noticed in the previous section that the \( n \)-adjoint sets of an ideal \( I \) over a ring \( R \) are not necessarily ideals of \( R \). However, in some cases they will be ideals. In this section, we will study the ideals whose \( n \)-adjoint sets are ideals. These kinds of ideals are called \( n \)-primly ideals as in the following definition.

**Definition 4.1** Let \( n \) be a positive integer. An ideal \( I \) over a ring \( R \) is called \( n \)-primly if \( n - \text{adj}(I) \) is an ideal of \( R \).

**Example 4.2** Let \( R = \mathbb{Z} \). Then \( 4Z \), \( 8Z \) and \( 9Z \) are \( n \)-primly ideals of \( \mathbb{Z} \), while \( 6Z \) and \( 12Z \) are
**Proposition 4.3** Let $n$ be a positive integer. Let $I$ be an ideal of a ring $R$. If $n$-$adj(I)$ is closed under addition, then $I$ is an $n$-primly ideal of $R$.

**Proof** We have to show that $n$-$adj(I)$ is an ideal of $R$. Since $n$-$adj(I)$ is closed under addition, it is enough to show that for every $r \in R$ and every $a \in n$-$adj(I)$, $ra \in n$-$adj(I)$. Let $r \in R$ and $a \in n$-$adj(I)$, then $\exists b \in R - \sqrt{I}$ such that $a^n b = I$. Thus $r^n a^n b = (r a)^n b \in I$. Therefore $ra \in n$-$adj(I)$.

**Proposition 4.4** If $I$ is a prime ideal over a ring $R$, then $I$ is $n$-primly ideal of $R$ for every positive integer $n$.

**Proof** Let $n$ be a positive integer. According to Proposition 4.3, it is enough to show that $n$-$adj(I)$ is closed under addition. Since $I$ is prime, then $I$ is primal, see [6]. Thus $adj(I)$ is an ideal of $R$. Let $a_1, a_2 \in n$-$adj(I)$, then $\exists b_1, b_2 \in R - \sqrt{I}$ such that $a_1^n b_1$ and $a_2^n b_2$ are in $I$. Since $a_1^n b_1 \in I$, then $a_1 (a_1^{n-1} b_1) \in I$. If $a_1^{n-1} b_1 \notin I$, then $a_1 \notin adj(I)$. Otherwise, $a_1^{n-1} b_1 \in I$ implies $a_1^{n-1} \in I$ (because $I$ is prime and $b_1 \notin I$). Since $I$ is prime, then $a_1 \in I$. Thus $b = 1 \notin I$ with $a_1 b \in I$ implies $a_1 \in adj(I)$. Hence in each cases $a_1 \in adj(I)$. Similarly, $a_2 \in adj(I)$. Since $adj(I)$ is ideal, then $a_1 + a_2 \in adj(I) \subseteq n$-$adj(I)$, see Corollary 3.10. Thus $I$ is $n$-primly ideal of $R$ for every positive integer $n$.

**Proposition 4.5** Let $n$ be a positive integer. If $I$ is an $n$-primly ideal of a ring $R$, then $n$-$adj(I)$ is a primary ideal of $R$.

**Proposition 4.6** Let $n$ be a positive integer. Let $I$ be an ideal of a ring $R$. If $n$-$adj(I)$ is uniformly not $1$-primary to $I$, then $I$ is an $n$-primly ideal of $R$.

**Proof** According to Proposition 4.3, it is enough to show that $n$-$adj(I)$ is closed under addition. Let $A = n$-$adj(I)$. Let $a, b \in A$. Since $A$ is uniformly not $1$-primary to $I$, then $au \in I$ and $bu \in I$. Hence $a^m u \in I$ and $b^m u \in I$ for every positive integer $m$. Thus $(a + b)^n u = \sum_{k=0}^{n} a^{n-k} b^k u \in I$ with $u \in R - \sqrt{I}$. This implies that $a + b \in n$-$adj(I)$.

**Definition 4.7** Let $n$ be a positive integer. Let $I$ be an ideal over a ring $R$. If $n$-$adj(I)$ is uniformly not $n$-primary to $I$, then $I$ is said to be uniformly $n$-primly ideal of $R$.

**Theorem 4.8** Let $n$ be a positive integer. Let $I$ be a proper ideal of a ring $R$. If $n$-$adj(I)$ is a principal ideal of $R$, then $I$ is a uniformly $n$-primly ideal of $R$.

**Proof** Let $A = n$-$adj(I) = R a$, for some $a \in A$. Then $a^n u \in I$ for some element.
Let $u \in R - \sqrt{I}$. Let $x \in A$, then $x = ra$, for some $r \in R$.
Since $a^n u \in I$, then $x^n u = r^n a^n u \in I$.

Since $x$ is an arbitrary element in $A$, then $A^n u \subseteq I$.
Hence $A$ is uniformly not $n$-primary to $I$. Therefore $I$ is a uniformly $n$-primly ideal of $R$.

The following result follows immediately from the previous theorem.

**Corollary 4.9** Let $n$ be a positive integer. Let $R$ be a principal ideal ring. Then every $n$-primly ideal is uniformly $n$-primly.

**Example 4.10** Let $R = \mathbb{Z}$. Then $2\mathbb{Z}, 4\mathbb{Z}, 8\mathbb{Z}$ and $9\mathbb{Z}$ are uniformly $n$-primly ideals of $\mathbb{Z}$.

**Proposition 4.11** Let $I$ be a proper ideal of a ring $R$. If $n$-adj$(I) = I$, for some positive integer $n$, then $I$ is primary ideal of $R$.

**Proof** Let $ab \in I$ such that $b \notin \sqrt{I}$, then $a \in 1 - \text{adj}(I) \subseteq n - \text{adj}(I) = I$, see Remark 3.4. Thus $I$ is primary.

**Theorem 4.12** Let $I$ be a proper ideal of a ring $R$. If $I$ is primary ideal of $R$, then $n$-adj$(I) = \sqrt[n]{I} = \sqrt{I}$ for every positive integer $n$.

**Proof** Let $I$ be a primary ideal of $R$.
By Proposition 3.12 it is enough to show that $n - \text{adj}(I) \subseteq \sqrt[n]{I}$. Let $a \in n - \text{adj}(I)$, then $a^n b \in I$ for some $b \in R - \sqrt[n]{I}$.
Since $I$ is primary, then $a^n \in I$. Thus $a \in \sqrt{n}I$.

**Corollary 4.13** If $I$ is a primary ideal of a ring $R$, then $I$ is $n$-primly ideal of $R$, for every positive integer $n$.

**Proof** Since $\sqrt{I}$ is an ideal of $R$ and, then by the previous theorem $n - \text{adj}(I) = \sqrt[n]{I}$, for every positive integer $n$, then the result follows immediately.

**Corollary 4.14** If $I$ is a primary ideal of a ring $R$, then $n$-adj$(I)$ is a prime ideal of $R$, for every positive integer $n$.

**Proof** Since $I$ is primary ideal of $R$, then by Theorem 4.12, $n - \text{adj}(I) = \sqrt[n]{I}$, for every positive integer $n$. Now, the result follows immediately because $\sqrt[n]{I}$ is prime ideal of $R$, see [8].

**Remark 4.15** Note that by Corollary 4.14, if $I$ is a primary ideal of a ring $R$, then $1$-adj$(I)$ is a prime ideal of $R$. The following example shows that the converse of the previous statement is not true in general. Take $R = F[X, Y]$, which is the ring of polynomials in $X$ and $Y$ over the field $F$.
Let $I = (X^2, XY)$. As Ladislas Fuchs illustrated in [6], this ideal is not primary. On the other hand,

$$\sqrt{(X^2, XY)} = \sqrt{X(X, Y)} = \sqrt{X} \cap \sqrt{X, Y} = (X) \cap (X, Y) = (X),$$
and $1$-adj$(I) = (X)$. Thus $I$ is $1$-primly ideal, which is not primary.

**Theorem 4.16** Let $R$ be a Boolean ring with unity and let $I$ be an ideal of $R$. Then every $1$-primly ideal of $R$ is primary ideal.

**Proof** According to Proposition 4.11 and Remark 3.4 it is enough to show that $1 - \text{adj}(I) \subseteq I$. Let $a \in 1 - \text{adj}(I)$. Since $I$ is $1$-primly ideal of $R$, then $1 - \text{adj}(I)$ is an ideal of $R$. Note that $1 - a \notin 1 - \text{adj}(I)$, for otherwise, $1 = (1 - a) + a \in 1 - \text{adj}(I)$, which contradicts Remark 3.2. That is $1 - a$ is $1$-primary to $I$.
Therefore, $\forall r \in R$, since $(1 - a)ar = 0 \in I$, then $ar \in \sqrt{I}$.
Thus there exists a positive integer $m$ such that $(ar)m \in I$.
Since $1 \in R$, then $a^m \in I$. Since $R$ is a Boolean ring, then $a = a^m \in I$.
Finally, the following result follows immediately from Theorem 4.16 and Corollary 4.13.

**Corollary 4.17** Let $R$ be a Boolean ring with unity. An ideal $I$ of $R$ is primary if and only if it is $1$-primly ideal of $R$. 

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References